A New Feedback Technique for Synchronizing Chaotic Chua's Circuits with $x|x|$ Nonlinearity

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Abstract: – This paper presents a new output feedback technique for synchronizing chaotic circuits. In particular, it is shown that synchronization can be systematically achieved via a scalar signal for a large class of chaotic systems. The approach is successfully applied to the recently proposed Chua's circuit with $x|x|$ nonlinearity.

Key-words: – Chaotic circuits, Synchronization, Feedback techniques, Nonlinear dynamics.

1 Introduction

In their seminal paper, Carroll and Pecora have shown that the dynamics of two chaotic systems (designed as a master-slave system) synchronize if all the Lyapunov exponents of the slave are less than zero, assuming that both the systems have initial conditions in the same basin of attraction [1]. Successively, different synchronization methods have been proposed. In particular, linear combinations of the transmitted state variables [2], parameter control methods [3] and observer concept [4] have been developed with the aim of synchronizing complex chaotic dynamics via a scalar signal. Additionally, a theoretic approach based on state feedback technique has been recently proposed in [5]. However, such interesting method does not guarantee chaos synchronization via a scalar transmitted signal.

Based on these considerations, this paper aims to give a further contribution in the field of synchronization techniques. Namely, an output feedback method for synchronizing chaotic circuits via a scalar signal is illustrated. The proposed approach has the following features:

a) it enables synchronization to be achieved in a systematic way;
b) it can be successfully applied to several well-known chaotic and hyperchaotic systems;
c) it enables the scalar transmitted signal to be easily designed.

The paper is organized as follows. Section 2 illustrates the proposed output feedback approach. In particular, Section 2 shows how chaos synchronization can be achieved by exploiting linear control theory [6]. Section 3 illustrates the application of the proposed technique to the novel Chua's circuit with $x|x|$ nonlinearity [5]. Finally, Section 4 shows that the proposed approach can be successfully applied to a large class of hyperchaotic circuits.

2 Synchronization via Output Feedback

We start by considering a chaotic system for which state and output equations can be written, respectively, as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{g}(\boldsymbol{x}(t)) \qquad (1)$$

$$y(t) = h(\boldsymbol{x}(t)) \qquad (2)$$

where $\boldsymbol{A} \in \Re^{n \times n}$, $\boldsymbol{g}(\boldsymbol{x}) \in \Re^{n}$ is a nonlinear function, h is a scalar function and y is a generic scalar output. By exploiting output feedback approach, the response system can be designed as follows:

$$\dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{g}(\tilde{\boldsymbol{x}}(t)) + \boldsymbol{F}\,(y(t) - \tilde{y}(t)) \qquad (3)$$

$$\tilde{y}(t) = h(\tilde{\boldsymbol{x}}(t)) \qquad (4)$$

where $\boldsymbol{u} = \boldsymbol{F}\,(y(t) - \tilde{y}(t)) \in \Re^{n}$ is an output feedback control input vector and $\boldsymbol{F}$ is a nonlinear function. In order to achieve synchronization, $\boldsymbol{F}$ has to be constructed so that the synchronization error system

$$\dot{\boldsymbol{e}}(t) = (\dot{\boldsymbol{x}}(t) - \dot{\tilde{\boldsymbol{x}}}(t)) \quad (5)$$

possesses an asymptotically stable equilibrium point at the origin. This objective can be achieved in two steps.

Step 1

It consists in making the error system (5) linear time-invariant when the output feedback input vector $\boldsymbol{u}$ is properly chosen. To this purpose a proposition is given.

Proposition 1: Given the response system (3)-(4), let

$$\boldsymbol{g}(\tilde{\boldsymbol{x}}) = \boldsymbol{b} f(\tilde{\boldsymbol{x}}) \quad (6)$$

be the nonlinear function in (3), where $\boldsymbol{b} \in \Re^n$ is a constant vector and $f(\tilde{\boldsymbol{x}})$ is a scalar nonlinearity. Moreover, let

$$\tilde{y} = h(\tilde{\boldsymbol{x}}) = f(\tilde{\boldsymbol{x}}) + \boldsymbol{k}\tilde{\boldsymbol{x}} \quad (7)$$

be the scalar output (4) with $\boldsymbol{k} = [k_1, k_2, \ldots k_n] \in \Re^{1\times n}$, and let

$$\boldsymbol{u} = \boldsymbol{F}(y(t) - \tilde{y}(t)) = \boldsymbol{b}(y - \tilde{y}) \quad (8)$$

be the output feedback input vector in (3). Then the error system (5) becomes linear time-invariant and can be written as:

$$\dot{\boldsymbol{e}} = (\boldsymbol{A} - \boldsymbol{bk})\boldsymbol{e}\,. \quad (9)$$

Proof: By taking into account (1) and (6), it follows that $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{b} f(\boldsymbol{x})$. Moreover, from (2) and (7) it follows that $y = h(\boldsymbol{x}) = f(\boldsymbol{x}) + \boldsymbol{kx}$. Therefore the error system (5) becomes:

$$\dot{\boldsymbol{e}} = \dot{\boldsymbol{x}} - \dot{\tilde{\boldsymbol{x}}} = [\boldsymbol{Ax} + \boldsymbol{b}f(\boldsymbol{x})] - [\boldsymbol{A}\tilde{\boldsymbol{x}} + \boldsymbol{b}f(\tilde{\boldsymbol{x}}) + \boldsymbol{b}(y - \tilde{y})] =$$
$$= \boldsymbol{Ae} - \boldsymbol{bke} = (\boldsymbol{A} - \boldsymbol{bk})\boldsymbol{e}$$

This completes the proof.

Step 2

It consists in stabilizing the error system (5) at the origin. To this purpose a proposition is given.

Proposition 2: If Proposition 1 holds, the drive and response systems described by (1)-(4) are globally asymptotically synchronized by a suitable feedback gain $\boldsymbol{k}$, provided that the pair ($\boldsymbol{A}$, $\boldsymbol{b}$) is controllable.

The proof of *Proposition 2* follows from linear control theory [6]. Namely, the error system (9) is globally asymptotically stabilized at the origin if its controllability matrix is full rank.

3 Case study: Chua's circuit with $x|x|$

Herein the proposed technique is applied to a recently introduced version of Chua's circuit, which includes the novel $x|x|$ nonlinearity [5]. The state and output equations of the Chua's circuit with $x|x|$ can be written in form (1)-(2) as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -a\, f(x_1) \\ 0 \\ 0 \end{bmatrix} \quad (10)$$

$$y(t) = h(\boldsymbol{x}(t)) \quad (11)$$

where $f(x_1) = ax_1 + bx_1|x_1|$ is the circuit nonlinearity. The parameters reported in [5] enable to obtain the chaotic attractor in Fig.1.

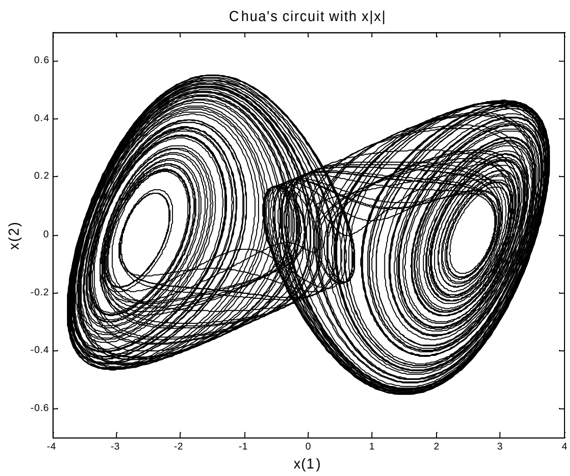


Fig. 1. Chaotic attractor in Chua's circuit with $x|x|$.

The state and output equations of the response system can be written in the form (3)-(4) as:

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \end{bmatrix} = \begin{bmatrix} 0 & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} + \begin{bmatrix} -a\, f(\tilde{x}_1) \\ 0 \\ 0 \end{bmatrix} + \boldsymbol{F}(y - \tilde{y}) \quad (12)$$

$$\tilde{y}(t) = h(\tilde{\boldsymbol{x}}(t)) \,. \tag{13}$$

According to *Proposition 1*, let

$$\boldsymbol{g}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} -a\, f(\tilde{x}_1) \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{b} f(\tilde{\boldsymbol{x}}) = \begin{bmatrix} -a \\ 0 \\ 0 \end{bmatrix} f(\tilde{x}_1) \tag{14}$$

be the nonlinearity in (12). Moreover, let

$$\begin{aligned} \tilde{y} = h(\tilde{\boldsymbol{x}}) = f(\tilde{\boldsymbol{x}}) + \boldsymbol{k}\tilde{\boldsymbol{x}} = \\ = f(\tilde{x}_1) + k_1\tilde{x}_1 + k_2\tilde{x}_2 + k_3\tilde{x}_3 \end{aligned} \tag{15}$$

be the *scalar* output (13). Finally, let

$$\boldsymbol{u} = \boldsymbol{F}(y(t) - \tilde{y}(t)) = \boldsymbol{b}(y - \tilde{y}) \tag{16}$$

be the output feedback control input vector in (12). Therefore the equations of the drive and response systems become:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -a \\ 0 \\ 0 \end{bmatrix} f(x_1) \tag{17}$$

$$y = f(x_1) + k_1x_1 + k_2x_2 + k_3x_3 \tag{18}$$

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \end{bmatrix} = \begin{bmatrix} 0 & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} + \begin{bmatrix} -a \\ 0 \\ 0 \end{bmatrix} f(\tilde{x}_1) + \\ + \begin{bmatrix} -a \\ 0 \\ 0 \end{bmatrix} (y - \tilde{y}) \end{aligned} \tag{19}$$

$$\tilde{y} = f(\tilde{x}_1) + k_1\tilde{x}_1 + k_2\tilde{x}_2 + k_3\tilde{x}_3 \,. \tag{20}$$

The circuit parameters reported in [5] are: $\boldsymbol{a}$ = 9.78, b = 14.97, a = -1/6, and b = 1/16. These values guarantee that the pair ($\boldsymbol{A}$, $\boldsymbol{b}$) be controllable. As a consequence, the eigenvalues of the error system (9) can be placed anywhere in the left half plane. This implies that the two Chua's circuits are globally synchronized. Namely, synchronization is achieved regardless of the initial conditions of the response system.
Now, three different cases are discussed.

***Case 1*):** $k_1 \neq 0, k_2 \neq 0, k_3 \neq 0$.

By placing the eigenvalues in

{–4.638, –2.093±j1.390},

it results

$\boldsymbol{k}$ = [–0.8000 –1.3000 –0.6000].

The time behavior of the synchronization error e_1 is reported in Fig.2.

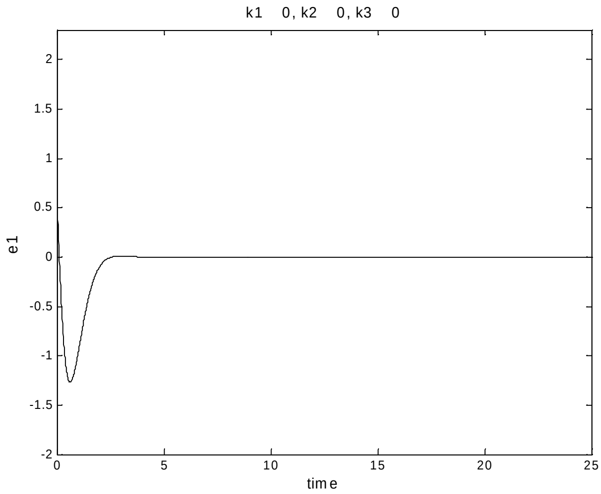


Fig. 2. Time behavior of the error e_1 in *Case 1*).

***Case 2*):** $k_1 \neq 0, k_3 \neq 0, k_2 = 0$.

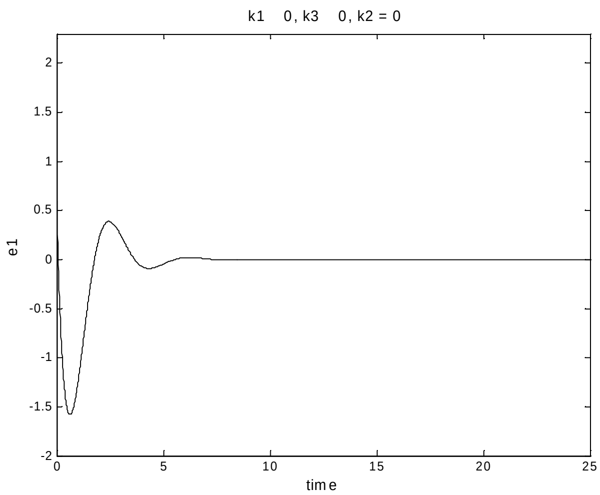
By placing the eigenvalues in

{–4.346, –0.772±j1.665},

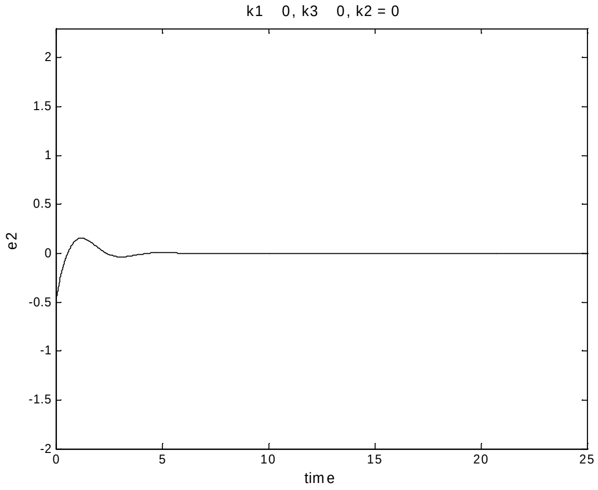
it results

$\boldsymbol{k}$ = [-0.5000 0 -0.4000].

The time behaviors of the errors e_1 and e_2 are reported in Fig.3.



(a)



(b)

Fig. 3. (a): time behavior of error e_1 in *Case 2*); (b): time behavior of error e_2 in *Case 2*).

***Case c*):** $k_1 \neq 0, k_2 = 0, k_3 = 0$.

By placing the eigenvalues in

$$\{-29.675, -3.332 \pm j3.833\},$$

it results

$$\boldsymbol{k} = [-0.3000 \quad 0 \quad 0].$$

The time behaviors of the synchronization errors are reported in Fig.4.

Remark

A significant feature of the proposed output feedback technique is the *flexibility of designing the transmitted signal in different ways*. In particular, note that in *Case 3)* synchronization is achieved by using only one state variable (i.e., x_1).

4 Hyperchaos Synchronization

Now, we show that the proposed approach can be applied to some well-known hyperchaotic circuits.

4.1 Matsumoto-Chua-Kobayashi circuit

This hyperchaotic circuit contains only one nonlinear element, a three-segment piecewise-linear resistor [7]. Its dynamics can be written as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -20 \\ 1 & 0 & 1 & 0 \\ 0 & 1.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 2 \\ -20 \\ 0 \\ 0 \end{bmatrix} g(x_2 - x_1)$$

where $g(\cdot)$ is the piecewise-linear function given by:

$$g(x_2 - x_1) = 3(x_2 - x_1) + \\ -1.6\left(|x_2 - x_1 - 1| - |x_2 - x_1 + 1|\right)$$

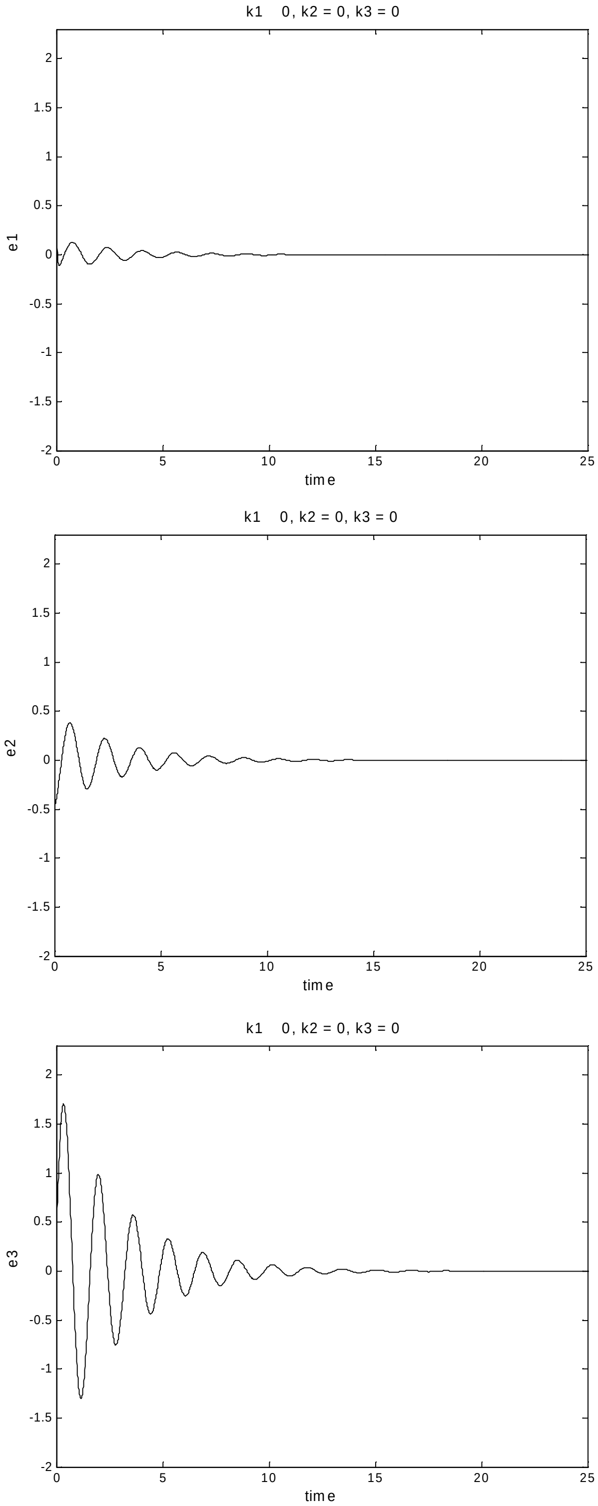


Fig. 4. Time behaviors of e_1, e_2 and e_3 in *Case 3*).

Since the pair ($\boldsymbol{A}$, $\boldsymbol{b}$) is controllable, the eigenvalues of the synchronization error can be moved anywhere. By placing them in

$$\{-0.3351\pm j6.6322, -0.1649\pm j0.8081\},$$

it results

$$\boldsymbol{k}=[1\ 0\ 0\ -0.5].$$

Notice that synchronization is achieved by using only two state variables in the scalar transmitted signal $y(t)$.

4.2 4D hyperchaotic oscillator

The 4D hyperchaotic oscillator proposed in [8] is now considered. Its dynamics can be written in dimensionless form as in [8]:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0.7 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -30 \end{bmatrix} (x_4 - 1) H(x_4 - 1)$$

where $H(u) = 0$ if $u<0$; $H(u) = 1$ if $u \geq 0$.

Also in this case ($\boldsymbol{A}$, $\boldsymbol{b}$) is controllable, that is, the eigenvalues of the synchronization error can be placed anywhere. By placing them in

$$\{-26.2436, -2.8894, -0.0835\pm j1.1396\},$$

it results

$$\boldsymbol{k}=[0\ 0\ 1\ -1].$$

Again, global synchronization is achieved by using only two state variables in the scalar transmitted signal.

Remark

The proposed method can be successfully applied to a wide class of chaotic and hyperchaotic systems. In particular, by computing the rank of the controllability matrices, it can be shown that Chua's circuit [9], Chua's oscillator [9], their modified versions [10]–[11], Rössler system [12], the oscillator with gyrators in [13] and the circuit with hysteretic nonlinearity in [14] can be globally asymptotically synchronized via output feedback method.

5 Conclusions

This paper has shown that output feedback technique represents an effective and practical tool for achieving chaos synchronization. In particular, the paper has proved that a large class of chaotic and hyperchaotic systems can be systematically synchronized via a scalar signal. In order to show how the technique works, the method has been applied to Chua's circuit with $x|x|$ nonlinearity.

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