# Enumerative and Structural Aspects of Incomplete Generating Languages

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*Abstract:* In this paper we launch a systematic study of a subclass of $\infty$-regular languages called incomplete generating languages. These languages represent a non-deterministic output of a computer with finite operational memory, where the decisions are made by observing previous $k$ states.

*Key-Words:* Incomplete generating language, Finite automaton, Regular language.

## 1 Introduction

Richard Büchi [1] gave a decision procedure for the sequential calculus, by showing that each well formed formula of the system is equivalent to a formula which says something about the infinite input history of a finite automaton. Similar concepts were also discovered by others while studying a problem in asynchronous switching theory. This motivates a notion of an $\omega$-language constructed by a finite acceptor. McNaughton [5] proved an analogue of Kleene's theorem for $\omega$-languages constructed by finite acceptors (so called $\omega$-regular languages). Further studies concerning finite and infinite state $\omega$-languages are found in Nivat [11],[12], Cohen and Gold [2], Niwinski [13], Wolper [17], Staiger [16], among others. In [14] Novotný provided a new characterization of three special classes of sets constructed by infinite acceptors and a new characterization of regular languages.

Pawlak's machine introduced in [15] and studied by Kwasowiec [3] and [4], is a special type of a finite acceptor. The $\infty$-language constructed by Pawlak's machine is called a generable set. Kwasowiec gives a characterization of generable sets and proves that generable sets are closed under intersection but not under Boolean operations in general. In [6] Mezník introduced the notion of a G-machine which is a non-deterministic version of Pawlak's machine. He showed that the family of $\infty$-languages generated by G-ma- chines is strictly greater then the family of generable sets. He solved the equivalence problem for G-machines and gave necessary and sufficient conditions for an arbitrary $\infty$-language to be a language generated by some G-machine. Yet another generalization of Pawlak's machine is a G$k$-machine (in this terminology G-machine is a G1-machine) introduced under the name of [$k$]-machine in [8]. It was showed in [7] that G1-languages form a lattice, a property which is not shared by G$k$-languages for $k > 1$. In [9] Mezník proved that G$k$-languages over a fixed alphabet form an upper semilattice and he obtained formulas for the maximal lengths of chains and antichains. He also solved the inclusion and equivalence problems of G$k$-machines.

An IG-machine is a generalization of a G$k$-machine (or a [k]-machine) and was first introduced in [10]. IG-machine is a machine operating in a discrete time scale; at any time instant it remembers $k$ previous states and undeterministically passes into another one. Thus, an IG-machine $M$ generates finite and infinite sequences of its states. The set of all sequences (finite or infinite) of states generated by $M$ is referred to as IG-*language*. It was shown [10] that IG-languages form a subclass of the class of $\infty$-*regular languages* [12]. An IG-machine may serve as a model for various technical and electronic devices, such as a computer with a fixed memory size.

We introduce the notions of IG$k$ and IG-*closures* of an $\infty$-language $L$. We will see that these will be very useful tools in our investigations. An IG$k$-closure of an $\infty$-language $L$ is the least IG$k$-language containing $L$ provided it exists. Similarly an IG-closure of an $\infty$-language is the least IG-language containing $L$ provided it exists. It is easy to see that the IG$k$-closure of the language $L$ exist if and only if the length of every word in $L$ is $\geq k$. We provide a new characterization of IG$k$-languages. Our result fully generalizes the result of Kwasowiec [3] who obtained characterization of generable sets. We study IG-languages with respect to Boolean operations. We show that IG and IG$k$-languages are closed under intersection but, in general, are not closed under union or difference. This result generalizes results obtained by Kwasowiec [3] for generable sets. Note that languages generated by G$k$-machines are not closed under intersection. We give an algorithm to construct the IG-machine which generates the intersection language. We study the poset structure of IG-languages and prove that the family of IG$k$ languages forms a lattice. As we have remarked earlier according to Meznḱ [9] G$k$-languages form an upper semilattice but not a lattice. For IG-languages we show we have only lower semilattice structure. We also solve inclusion and equivalence problems for IG-languages. We compute the maximal length of the chains in $IGk(\Sigma)$ and we find a lower bound for the width of $IGk(\Sigma)$. Note that Mezník in [9] computed exactly the width of the set of G$k$-languages. We will show why his arguments do not apply to IG$k$ languages.

## 2 Preliminaries and Definitions

By $N$ we denote the set of *all non negative integers* and by $\omega$ the *least infinite ordinal.* The set theoretical operations of *union, intersection* and *difference* are denoted by $\cup, \cap$ and $-$ respectively, and they are referred to as *Boolean operations.* $\subseteq$ ($\subset$) denotes the *(proper) set inclusion.* The *empty set* is denoted by $\emptyset$, the *power set* of the set $A$ is denoted by $2^A$ and $\mathrm{Card}(A)$ denotes the cardinality of the set $A$. By definition an *alphabet* is a finite set; elements of an alphabet $\Sigma$ are called *letters*. Denote by $\Sigma^*$ ,resp., $\Sigma^\omega$ the set of all finite ,resp., infinite sequences of elements of $\Sigma$. By $\lambda$ we denote the empty sequence and be definition we let $\lambda \in \Sigma^*$. $\lambda$ is called the *empty word.* Denote by $\Sigma^\infty$ the set $\Sigma^* \cup \Sigma^\omega$. The elements of $\Sigma^*, \Sigma^\omega$ and $\Sigma^\infty$, respectively, are called *words, $\omega$-words and $\infty$-words. For $a \in \Sigma^\infty - \{\lambda\}$ denote by $a_n$ the $(n+1)$th element of $a$. The length of an $\infty$-word $a \in \Sigma^\infty$, in symbols $|a|$, is the length of the sequence $a$. We write $a_0a_1a_2\ldots$ instead of $a$ and we formally identify both expressions, i.e. $a = a_0a_1a_2\ldots$. Catenation* of a word $a$ and an $\infty$-word $b$ is denoted by $ab$. We define $a\lambda = \lambda a = a$. Given $a, b \in \Sigma^\infty$, $a$ is a *factor* of $b$ if and only if there are $x \in \Sigma^*$ and $y \in \Sigma^\infty$, such that $b = xay$. Moreover, if $x = \lambda$, resp., $y = \lambda$ then $a$ is *left factor,* resp., *right factor* of $b$. Denote by $F(a), LF(a), RF(a)$ the set of all factors, left factors, right factors of $a$, respectively. Subsets of $\Sigma^*$, $\Sigma^\omega$ and $\Sigma^\infty$ are called *languages, $\omega$-languages* and *$\infty$-languages over* $\Sigma$. Let $L$ be an $\infty$-language. Denote $m(L) = \inf\{|a|; a \in L\}$. Let $L_1, L_2, L$ be languages. Define *catenation* (or *product*) of $L_1$ and $L_2$ by $L_1L_2 = \{xy;\ x \in L_1, y \in L_2\}$, and $L^0 = \{\lambda\}$, $L^{n+1} = L^nL$. Denote the catenation closure of $L$ by $L^*$. By definition $L^*$ is the union of all $L^i$, $i \in N$. Denote by $L^+ = L^* - \{\lambda\}$.

**Definition 2.1.** An *incomplete generating machine over* $\Sigma$, or briefly an IG-*machine* or more precisely IG$k$-*machine*, is a quadruple $M = \langle \Sigma, H, k, S\rangle$, where $\Sigma$ is an alphabet, $H \notin \Sigma, k \geq 1, S \subseteq \Sigma^k \times (\Sigma \cup \{H\})$. The elements of $\Sigma$ are referred to as *states*, $H$ is called the *halt state, $k$* is called the *depth of memory,$S$* is called the *successor operator.* Define $\mathrm{Dom}\, S = \{a \in \Sigma^k; (\exists b \in \Sigma \cup \{H\})(ab \in S)\}$, $\mathrm{Pos}\, S = \{a \in \Sigma^{k-1} \times (\Sigma \cup \{H\}); (\exists b \in \Sigma)(ba \in S)\}$, and $\mathrm{Top}\, S = \mathrm{Pos}\, S - \mathrm{Dom}\, S$.

$M$ operates in discrete time scale $N$ and at the $i$-th time instant the successor operator $S$ is applied. By this way $\infty$-words are obtained forming the $\infty$-language of $M$ as given in the following definitions.

**Definition 2.2.** An *output word (of $M$)* is defined recursively as follows:

(1) $a_0 \ldots a_{k-1} \in \mathrm{Dom}\, S$ is an output word;

(2) $a_0 \ldots a_n (n \geq k)$ is an output word if $a_0 \ldots a_{n-1}$ is an output word and $a_{n-k} \ldots a_n \in S$ with $a_n \in \Sigma$.

(3) All output words are obtained by (1) and (2). Denote by $\mathrm{OUT}(M)$ the set of all output words.

**Definition 2.3.** A word $a$, of length $n$ is generated by $M$ if and only if $a$ is an output word and $(a_{n-k} \ldots a_{n-1}H \in S \quad \text{or} \quad a_{n-k} \ldots a_{n-1} \notin \mathrm{Dom}\, S)$.

An $\omega$-word $a$ is generated by $M$ if and only if $a_0 \dots a_n$ is an output word of $M$ for any $n \geq k$.

The $\infty$-*language generated* by $M$ is the set of all words and $\omega$-words generated by $M$ and is denoted by $L^\infty(M)$. $\infty$-languages generated by IG-machines and IG$k$-machines are referred to as IG-*languages* and IG$k$-*language*. Denote by $IGk(\Sigma)$ and $IG(\Sigma)$ the sets of all $\infty$-languages generated by IGk-machines over $\Sigma$, and IG-machines over $\Sigma$.

The easiest way to introduce the notion of a *finite automaton* is perhaps to view it as a labeled digraph, where each edge is labeled by one or several letters of the alphabet $\Sigma$. Furthermore two subsets of nods are specified called the set of *initial* and *final* nods. We say that a word $a$ over $\Sigma$ is *accepted* by a finite automaton $\mathcal{A}$ if there is a path from an intial node to a final node labeled by $a$. The language $L(\mathcal{A})$ *accepted by* $\mathcal{A}$ is defined to be the set of all words accepted by $\mathcal{A}$. An infinite path is called an *omega-path*. The $\omega$-path is called *successful* if it begins from an initial node and passes infinitely many times through final nodes. For a finite automaton $\mathcal{A}$ we define an $\omega$-language $L_\omega(\mathcal{A})$ accepted by $A$ as $L_\omega(\mathcal{A}) = \{a \in \Sigma^\omega;\ a$ is a label for some successful $\omega$-path $\}$. Languages accepted by finite automata are called *regular;* $\omega$-languages accepted by finite automata are referred to as $\omega$-*regular* languages. A language obtained by a union of a regular and an $\omega$-regular language is termed $\infty$*regular*. To an IG-machine $M = \langle \Sigma, H, k, S\rangle$ we can easily construct two finite automata $\mathcal{A}_1$ and $\mathcal{A}_2$ such that the $\infty$-language generated by $M$ is a union of $L(\mathcal{A}_1)$ and $L_\omega(\mathcal{A}_2)$. For example we construct $\mathcal{A}_2$. let $V$ denote the set of nods and $E$ denote the set of arcs. Set $V = \mathrm{Dom}\, S \cup \mathrm{Top}\, S$, and $E = \{ (a_0 \dots a_{k-1}, a_1 \dots a_k)\ ;\ a_0 \dots a_k \in S, \}$. An arc $(a_0 \dots a_{k-1}, a_1 \dots a_k)$ is labelled by $a_0$. Nods in $\mathrm{Dom}\, S$ are initial nods and every nod is a final nod. Hence every IG-language is also $\infty$-regular. The converse is not true. For instance the $\infty$-regular language $\{ a^{2n}\ ;\ n \geq 1 \}$ is not generated by any IG-machine. The proof of the following lemma is left to the reader.

**Lemma 2.4.** *Let $M = \langle \Sigma, H, k, S\rangle$ be an IG-machine. The following statements are true.*

*(i)* $\lambda \notin L^\infty(M)$.

*(ii)* $m(L^\infty(M)) \geq k$.

*(iii)* $(\forall a \in OUT(M))(\exists x \in L^\infty(M))(a \in LF(x))$.

*(iv)* $(\forall a \in L^\infty))(\forall x \in \Sigma^\infty)(|a| > k$ *and* $a \in RF(x) \Rightarrow a \in L^\infty(M))$.

*(v)* $(\forall u, a, b, c, d \in \Sigma^\infty)(|u| = k)$ $(aub \in L^\infty(M)$ *and* $cud \in L^\infty(M) \Rightarrow aud \in L^\infty(M))$.

*(vi) Let $a \in \Sigma^\omega$. If for every $i \in N$ there are $x^{(i)} \in \Sigma^*$, $y^{(i)} \in \Sigma^\infty$, such that $x^{(i)}a_i \dots a_{i+k}y^{(i)} \in L^\infty(M)$ then $a \in L^\infty(M)$.*

*(vii) Let $a \in \Sigma^\omega$. If for every $i \in N$ there is $y^{(i)} \in \Sigma^\infty$, such that $a_0 \dots a_{i+k}y^{(i)} \in L^\infty(M)$ then $a \in L^\infty(M)$.*

*Proof.* (i) and (ii) are trivial consequences of Definition 1.3. (iii) was proved in [10]. To prove (iv) let $ya = x \in L^\infty(M)$ for some $y \in \Sigma^*$. First, assume $\omega > |a| = n > k$. Hence $a$ is an output word and $a_{n-k} \dots a_{n-1}H \in S$ or $a_{n-k} \dots a_{n-1} \notin \mathrm{Dom}\, S$. Thus, $a \in L^\infty(M)$. Second, let $a \in \Sigma^\omega$. Since $ya \in L^\infty(M)$ we get $a_i \dots a_{i+k} \in S$ for all $i \in N$, which implies $a \in L^\infty(M)$. (v). First we assume that $a = d = \lambda$. If, moreover, $b = \lambda$ then $aud = aub \in L^\infty(M)$. If $b \neq \lambda$ then $u \in \mathrm{Dom}\, S$. Since $cu \in L^\infty(M)$ then $uH \in S$. Thus, $aud = u \in L^\infty(M)$. Second, we assume that $a \neq \lambda$ or $d \neq \lambda$. By Definition 1.3 $au \in \mathrm{OUT}(M)$ provided $a \neq \lambda$, and $ud \in \mathrm{OUT}(M)$ provided $d \neq \lambda$, and so $aud \in \mathrm{OUT}(M)$. If $d$ is an $\omega$-word then obviously $aud \in L^\infty(M)$. Assume $d$ is a word. Denote by $g = g_0 \dots g_{n-1}$ the word $aud$. Since $|aud| > k$ then by 1.3 $g_{n-k} \dots g_{n-1}H \in S$ or $g_{n-k} \dots g_{n-1} \notin \mathrm{Dom}\, S$. Thus, again $aud \in L^\infty(M)$. (vi). Let $x^{(i)}a_i \dots a_{i+k}y^{(i)} \in L^\infty(M)$ for all $i \in N$. Then it holds $(a_i \dots a_{i+k} \in S)$ for all $i \in N$ and so $a \in L^\infty(M)$. (vii) is an immediate consequence of (vi) letting $x_0 = \lambda$ and $x^i = a_0 \dots a_{i-1}$.

# 3 IG and IG$k$-Closures

This section introduces notions of IG$k$ and IG-closures. We find necessary and sufficient conditions for the existence of the IG$k$-closure of a given $\infty$-language. We also give an new characterization of IG$k$-languages.

**Definition 3.1.** Let $L$ be an $\infty$-language over $\Sigma$ and $k \geq 1$. Define an IG$k$-machine $M(L,k) = \langle \Sigma, H, k, S(L,k)\rangle$, where $S(L,k) = \{a \in \Sigma^{k+1}; (\exists x \in L)(a \in F(x))\} \cup \{aH; a \in \Sigma^k \cap L\} \cup \{aH; (\exists x \in L)(a \in \Sigma^k \cap RF(x))$ and $(\exists y \in L)(\exists a_k \in \Sigma)(aa_k \in F(y))\}$.

The following lemma is immediate.

**Lemma 3.2.** *Let $L$ be an $\infty$-language. Let $a \in L^\infty(M(L,k))$, $|a| = n < \omega$. Then there is $x \in L$, such that $a_{n-k} \dots a_{n-1} \in RF(x)$.*

**Theorem 3.3.** *Let $L$ be an $\infty$-language, $1 \leq k \leq m(L)$. Then $L^\infty(M(L,k))$ is the least IG$k$-language containing $L$ (with respect to the set inclusion).*

*Proof.* It is straightforward to verify that $L \subseteq L^\infty(M(L,k))$. Let $L' \in IGk$ and assume $L \subseteq L'$. Denote by $M' = \langle \Sigma, H, k, S' \rangle$ the IG-machine that generates $L'$. We shall prove $L^\infty(M(L,k)) \subseteq L'$. First, let $a \in L^\infty(M(L,k)), |a| = k$. It holds $aH \in S(L,k)$. Since $L \subseteq L'$ then $a \in L'$ or $((\exists x \in L')(a \in RF(x))$ and $(\exists y \in L')(\exists a_k \in \Sigma)(aa_k \in F(y)))$. Assume the latter possibility holds true. Then $aa_k \in S'$ and there is $x' \in \Sigma^*$, such that $x = x'a \in L'$. Obviously $aa_k \in \mathrm{OUT}(M')$ and by Lemma 1.5(iii) there is $z' \in \Sigma^\infty$, $aa_kz' \in L'$. By Lemma 1.5(v) $a \in L'$. Second, let $a \in L^\infty(M(L,k))$, $\omega > |a| = n > k$. By Definition 3.1 $(\forall i = 0, \dots, n-k-1)(a_i \dots a_{i+k} \in S(L,k))$ and by Lemma 3.2 there exists $x \in L$ such that $a_{n-k} \dots a_{n-1} \in RF(x)$. Since $L \subseteq L'$ then we deduce $(\forall i = 0, \dots, n-k-1)(\exists x^{(i)} \in L')(a_i \dots a_{i+k} \in F(x^{(i)}))$ and so $(\forall i = 0, \dots, n-k-1)(a_i \dots a_{i+k} \in S')$ and

$$a_{n-k} \dots a_{n-1}H \in S' \quad \text{or} \quad a_{n-k} \dots a_{n-1} \notin \mathrm{Dom}\, S'. \tag{5}$$

It follows that $a \in L'$. Third, assume $a \in L^\infty(M(K, k))$, $|a| = \omega$. It holds $a_i \dots a_{i+k} \in S(L,k)$ for all $i \in N$. From Definition 3.1 it follows for all $i \in N$ there is $x^{(i)} \in L$, such that $a_i \dots a_{i+k} \in F(x^{(i)})$. Since $L \subseteq L'$, then $a_i \dots a_{i+k} \in S'$ and so $a \in L'$. We conclude that $L^\infty(M(L,k)) \subseteq L'$.

**Definition 3.4.** Let $L$ be an $\infty$-language, $k \geq 1$. If there is the least IG$k$-language containing $L$, then we call it the IG$k$-*closure* of L and denote it by $\overline{L}^k$.

**Corollary 3.5.** *Let $L$ be an $\infty$-language. The IG$k$-closure $\overline{L}^k$ of $L$ exists if and only if $m(L) \geq k$. Moreover, $\overline{L}^k = L^\infty(M(L,k))$.*

*Proof.* Consider the first statement. Assume IG$k$-closure $\overline{L}^k$ of $L$ exists. By Definition 3.4 $L \subseteq \overline{L}^k \in IGk$ and by Theorem 1.5(ii) $m(L) \geq m(\overline{L}^k) \geq k$. The reverse implication and the second statement follow readily from Theorem 3.3.

The following theorem gives a new characterization of IG-languages and generalizes the result of Kwasowiec [3], Theorem 1.

**Theorem 3.6.** *Let $L$ be an $\infty$-language. Then the following statements (i), (ii) are equivalent:*

*(i)* $L \in IGk$.

*(ii)* *(1)* $m(L) \geq k$.

*(2)* $(\forall x \in L)(\forall y \in \Sigma^\infty)(|y| > k$ *and* $y \in RF(x) \Rightarrow y \in L)$.

*(3)* $(\forall u, a, b, c, d \in \Sigma^\infty)(|u| = k)(aub \in L$ *and* $cud \in L \Rightarrow aud \in L)$.

*(4) Let $a$ be an $\omega$-word. If for every $i \in N$ there is $y^{(i)} \in \Sigma^\infty$, such that $a_0 \dots a_{i+k}y^{(i)} \in L$, then $a \in L$.*

*Proof.* (i)$\Rightarrow$(ii) holds true due to Lemma 1.5. (ii)$\Rightarrow$(i). By Theorem 3.5 $\overline{L}^k$ exists and by Definition 3.4 $L \subseteq \overline{L}^k$. We shall prove the reverse inclusion. First, suppose $a \in \overline{L}^k$, $|a| = k$. By Theorem 3.5 $aH \in S(L,k)$ and from Definition 3.1 it follows that either $a \in L$ or there are $x, y \in \Sigma^*$ and $z \in \Sigma^\infty$, $z \neq \lambda$, such that $xa \in L$ and $yaz \in L$. Assume the latter possibility holds true. Then by (2) $az \in L$ and using (3) follows $a \in L$. Second, suppose $a \in \overline{L}^k$, $\omega > |a| = n > k$. Hence $a_i \dots a_{i+k} \in S(L,k))$ for all $i = 0, \dots, n-k-1$. By Definition 3.1 there are $x^{(i)} \in \Sigma^*$, $y^{(i)} \in \Sigma^\infty$, such that $x^{(i)}a_i \dots a_{i+k}y^{(i)} \in L$ for every $i = 0, \dots, n-k-1$. Applying (3) $(n-k-1)$-times we get $x^{(0)}a_0 \dots a_{n-1}y^{(n-k-1)} \in L$. From (2) follows $a_0 \dots a_{n-1}y^{(n-k-1)} \in L$. Using Lemma 3.2 observe that there is $x' \in \Sigma^*$, such that $x'a_{n-k} \dots a_{n-1} \in L$. Using again (3) we arrive at $a = a_0 \dots a_{n-1} \in L$. Third, suppose $a \in \overline{L}^k$, $|a| = \omega$. Then $(\forall i \in N)(a_i \dots a_{i+k} \in S(L,k))$. By Definition 3.1 there exist $x^{(i)} \in \Sigma^*, y^{(i)} \in \Sigma^\infty$, such that $x^{(i)}a_i \dots a_{i+k}y^{(i)} \in L$. By (2) $a_i \dots a_{i+k}y^{(i)} \in L$ and so applying (3) i-times we get $a_0 \dots a_{i+k}y^{(i)} \in L$ for all $i \in N$. Consequently from (4) follows $a = a_0a_1a_2 \dots \in L$. We conclude that $\overline{L}^k \subseteq L$.

Define the map $c_r$ from the set of all $\infty$-languages into itself by $c_r(L) = \{a \in L; |a| \geq r\}$, for an $\infty$-language $L$. From Theorem 3.6 immediately follows the following theorem.

**Lemma 3.7.** *Let $L \in IGk$, $r \geq k$. Then $c_r(L) \in IGr$.*

We will now explicitely construct the IG-machine that for a given $M$ generates $c_r(L^\infty(M)$. First we introduce the operation of *overlaping catenation* which is a generalization of the usual catenation operation. Let $L, L'$ be $\infty$-languages, $r \in N$. Define $L \textcircled{r} L' = \{xuy; |u| = r, xu \in L, uy \in L'\}$. By $\textcircled{r}_{k \leq i \leq n} L_i$ we abbreviate $L_k \textcircled{r} L_{k+1} \textcircled{r} \ldots \textcircled{r} L_n$; we write $\textcircled{r}_{k \leq i \leq n} L$ instead of $\textcircled{r}_{k \leq i \leq n} L_i$ in case $L_k = L_{k+1} = \cdots = L_n = L$. Observe that the operation $\textcircled{r}$ is associative.

Let $M = \langle \Sigma, H, k, S \rangle$, $r \in N$. Define an IG-machine $M/r = \langle \sigma, H, k/r, S/r \rangle$ as follows:

$$k/r = \begin{cases} k, & \text{for } r \leq k \\ r, & \text{for } r > k\,, \end{cases}$$

$$S/r = \begin{cases} S, & \text{for } r \leq k \\ \{aH; a \in \textcircled{k}_{k \leq i \leq r-1} S \cap \Sigma^r,\ a_{r-k} \ldots a_{r-1} \\ \notin \mathrm{Dom} S\} \cup \textcircled{k}_{k \leq i \leq r} S, & \text{for } r > k\,. \end{cases}$$

**Lemma 3.8.** *Let $M = \langle \Sigma, H, k, S \rangle$ be an IG-machine, $r \in N$, $L = L^\infty(M)$. Then $c_r(L^\infty(M)) = L^\infty(M/r)$.*

*Proof:* For $r \leq k$ the statement is straightforward. Assume the contrary. By Lemma 3.7 $c_r(L) = L^\infty(M$ $(c_r(L), r))$. We shall prove that $M(c_r(L), r) = M/r$. Obviously $a \in S(c_r(L), r) \cap \Sigma^{r+1}$ if and only if there is $x \in c_r(L) \subseteq L$, such that $a \in F(x)$ if and only if $a \in \textcircled{k}_{k \leq i \leq r} S$ iff $a \in S/r$. Let $aH \in S(c_r(L), r)$. By Definition 3.1 there is $y \in c_r(L) \subseteq L$, such that $a \in RF(y)$. Hence $a_i \ldots a_{i+k} \in S$ for all $i = 0, \ldots, r-k-1$ and $(a_{r-k} \ldots a_{r-1} H \in S$ or $a_{r-k} \ldots a_{r-1} \notin \mathrm{Dom} S)$. By Definition $a \in \textcircled{k}_{k \leq i \leq r-1} S$ and $aH \in S/r$. To prove the reverse inclusion assume $aH \in S/r$. Hence either $aH \in \textcircled{k}_{k \leq i \leq r} S$ or $(a \in \textcircled{k}_{k \leq i \leq r-1} S$ and $a_{r-k} \ldots a_{r-1} \notin \mathrm{Dom} S)$. In both cases $a \in L$ which implies $aH \in S(c_r(L), r)$.

Using Lemma 3.7 it is easy to see that if $1 \leq r \leq s \leq m(L)$, then $\overline{L}^s \subseteq \overline{L}^r$ and so

$$\overline{L}^1 \supseteq \overline{L}^2 \supseteq \cdots \supseteq \overline{L}^{m(L)} \quad \text{for} \quad 1 \leq m(L) < m,$$

$$\text{and} \overline{L}^1 \supseteq \overline{L}^2 \supseteq \overline{L}^3 \supseteq \ldots \quad \text{for} \quad m(L) = \omega.$$

**Definition 3.9.** Let $L$ be an $\infty$-language. If there is the least IG-language containing $L$ then we call it the IG-*closure* of L and denote it by $\overline{L}$.

Note that if $1 \leq m(L) < \omega$, then the IG-closure $\overline{L}$ of $L$ exists and $\overline{L} = \overline{L}^{m(L)}$.

**Theorem 3.10.** *Let $m(L) = \omega$. Then the following three conditions are equivalent:*

*(1) The IG-closure of $L$ exists.*

*(2) There exists $r \geq 1$ such that for all $s > r, \overline{L}^r = \overline{L}^s$. Moreover, $\overline{L} = \overline{L}^r$.*

*(3) $\bigcap_{k=1}^{\omega} \overline{L}^k \in IG$. Moreover, $\bigcap_{k=1}^{\omega} \overline{L}^k = \overline{L}$.*

*Proof.* (1) $\Rightarrow$ (2). Let $\overline{L}$ exists. Then $\overline{L} \in IGr$ for some $r \geq 1$ and so $\overline{L} = \overline{L}^r \supseteq \overline{L}^{r+1} \supseteq \overline{L}^{r+2} \supseteq \ldots$. Hence $\overline{L}^r = \overline{L}^{r+1} = \overline{L}^{r+2} = \ldots$. (2) $\Rightarrow$ (3) is immediate. (3) $\Rightarrow$ (1). Denote $\bigcap_{k=1}^{\omega} \overline{L}^k$by$L_0$. Obviously $L \subseteq L_0$. Consider an arbitrary IG-language $L'$, $L \subseteq L'$. Then there is $r \geq 1$, such that $L' \in IGr$. By definition $L_0 \subseteq \overline{L}^r \subseteq L'$ and so $L_0$ is the IG-closure of $L$.

**Example 3.11.** Consider an $\omega$-regular language $L = \{ba^\omega\} \cup \{a^n b^\omega; n > 0\}$. By Definition 3.1

$$S(L, k) = \{ba^k, a^{k+1}, a^k b, a^{k-1} b^2, \ldots, ab^k, b^{k+1}\}.$$

Hence $ba^k b^\omega \in \overline{L}^k$. Assume $ba^k b^\omega \in \overline{L}^{k+1}$. Then $ba^k b \in S(L, k+1)$. Thus, $ba^k b$ is a factor or some $\omega$-word of $L$. This yields a contradiction and so we conclude that $\overline{L}^k \neq \overline{L}^{k+1}$ for all $k \geq 1$. By Theorem 3.10 IG-closure of $L$ does not exist.

# 4 Inclusion and Equivalence

In this section we solve the inclusion and equivalence problems for IG-languages.

**Theorem 4.1.** *Let $M = \langle \Sigma, H, k, S \rangle$, $M' = \langle, \Sigma, H, k, S' \rangle$. Then the following statements (i),(ii) are equivalent:*

*(i) $L^\infty(M) \subseteq L^\infty(M')$*

*(ii) $S \subseteq S'$ and $Dom\, S' \cap Top\, S \subseteq \{a; aH \in S'\}$.*

*Proof.* (i)$\Rightarrow$(ii). First, assume $a \in S \cap \Sigma^{k+1}$. Hence $a \in \mathrm{OUT}(M)$. By Lemma 1.5(iii) there exists $x \in L^\infty(M)$, such that $a \in LF(x)$. Using (i) we get $x \in L^\infty(M')$ and by Definition 1.3 $a \in S'$. Second, if $aH \in S$, then it holds $a \in L^\infty(M) \subseteq L^\infty(M')$, which implies $aH \in S'$ and so we conclude that $S \subseteq S'$. Now assume that $a \in \mathrm{Dom}\, S' \cap \mathrm{Top}\, S$. By Definition 1.3 there is $b \in \Sigma$, $ba \in S$ and $a \notin \mathrm{Dom}\, S$ and so $ba \in L^\infty(M)$. By (i) $ba \in L^\infty(M')$. Due to $a \in \mathrm{Dom}\, S'$ we obtain $aH \in S'$. (ii)$\Rightarrow$(i). First, let $a \in L^{\infty}(M)$, $|a| = k$. Then $aH \in S \subseteq S'$ which implies $a \in L^{\infty}(M')$. Second, let $a \in L^{\infty}(M)$, $\omega > |a| = n > k$. Hence $a_i \dots a_{i+k} \in S \subseteq S'$, for all $i = 0, \dots, n-k-1$ and ($a_{n-k} \dots a_{n-1}H \in S \subseteq S'$ or $a_{n-k} \dots a_{n-1} \notin \mathrm{Dom}\, S$). If $a_{n-k} \dots a_{n-1}H \in S \subseteq S'$ or $a_{n-k} \dots a_{n-1} \notin \mathrm{Dom}\, S'$ then $a \in L^{\infty}(M')$. Assume the contrary, i.e. $a_{n-k} \dots a_{n-1}H \notin S$ and $a_{n-k} \dots a_{n-1} \in \mathrm{Dom}\, S'$. Then $a_{n-k} \dots a_{n-1} \in \mathrm{Top}\, S$ and $a_{n-k} \dots a_{n-1}H \in S'$ and so $a \in L^{\infty}(M')$. Third, let $a \in L^{\infty}(M)$, $|a| = \omega$. It holds $(\forall i \geq 0)(a_i \dots a_{i+k} \in S \subseteq S')$ and so $a \in L^{\infty}(M')$.

**Theorem 4.2.** *Let $M = \langle \Sigma, H, k, S \rangle$, $M' = \langle \Sigma, H, k', S' \rangle$, $r = \max(k, k')$. Then the following statements (i),(ii) are equivalent:*

*(i)* $L^{\infty}(M) \subseteq L^{\infty}(M')$.

*(ii)* *(1)* $S/_r \subseteq S'/_r$ *and*

*(2)* $\mathit{Dom}(S'/_r) \cap \mathit{Top}(S/_r) \subseteq \{a; aH \in S'/_r\}$ *and*

*(3)* $m(L^{\infty}(M)) \geq r$.

*Proof.* (i)⇒(ii) By Lemma 3.7. $L^{\infty}(M/_r) = c_r(L^{\infty}(M)) \subseteq c_r(L^{\infty}(M')) = L^{\infty}(M'/_r)$. By Theorem 4.1 the conditions (1),(2) are satisfied. Let $a \in L^{\infty}(M)$. Then $|a| \geq k$. Since $L^{\infty}(M) \subseteq L^{\infty}(M')$ then also $|a| \geq k'$. Consequently $|a| \geq r$. (ii)⇒(i) From (1),(2), using theorem 4.1, it follows $L^{\infty}(M/_r) \subseteq L^{\infty}(M'/_r)$. By Lemma 3.7. $c_r(L^{\infty}(M)) \subseteq c_r(L^{\infty}(M'))$ and by (3) $L^{\infty}(M) = c_r(L^{\infty}(M)) \subseteq c_r(L^{\infty}(M')) \subseteq L^{\infty}(M')$. ∎

The following statement is a straightforward consequence of Theorem 4.2.

**Theorem 4.3.** *Let $M = \langle \Sigma, H, k, S \rangle$, $M' = \langle \Sigma, H, k', S' \rangle$, $k \leq k'$. Then the following statements (i),(ii) are equivalent:*

*(i)* $L^{\infty}(M) = L^{\infty}(M')$.

*(ii)* $S/_{k'} = S'$ *and* $m(L^{\infty}(M)) \geq k'$.

In particular if $k' = k$ Theorem 4.3 has the following simple form.

**Theorem 4.4.** *Let $M = \langle \Sigma, H, k, S \rangle$, $M' = \langle \Sigma, H, k, S' \rangle$. Then the following statements (i) (ii) are equivalent:*

*(i)* $L^{\infty}(M) = L^{\infty}(M')$.

*(ii)* $S = S'$.

# 5 Boolean Operations

n this section we study the closure properties of IG-languages under Boolean operations. We show that the classes of IG$k$/languages and IG-languages are closed under finite intersection but, in general, they are not closed under infinite intersection, finite union and difference.

**Theorem 5.1.** *The sets $IGk(\Sigma)$ and $IG(\Sigma)$ are closed under finite intersection.*

*Proof.* Let $L_1, L_2 \in IGk(\Sigma)$. One can easily verify that the conditions (1),(2) (3) and (4) in Theorem 3.6 are satisfied for $L_1 \cap L_2$. for example we prove that condition (4) is satisfied. Let $a$ be an $\omega$-word and for every $i \in N$ let there be $y^{(i)} \in \Sigma^{\infty}$, such that $a_0 \dots a_{i+k} y^{(i)} \in L_1 \cap L_2$. Then $a \in L_1$ and $a \in L_2$ and so $a \in L_1 \cap L_2$. Thus $L_1 \cap L_2 \in IGk(\Sigma)$. Let $L_i \in IGk_i(\Sigma)$, $i = 1, 2$ and assume $k_1 \geq k_2$. Then $L_1 \cap L_2 - c_{k_2}(L_1) \cap L_2$. The theorem now follows from its firs part.

**Theorem 5.2.** *The set $IG(\Sigma)$ is not closed under infinite intersection with the only exception when $Card(\Sigma) = 1$.*

*Proof.* Let $a, b \in \Sigma$ and $a \neq b$. Consider the language $L$ from Example 3.11. We have $\overline{L}^k \in IG(\Sigma)$ for all $k \geq 1$ and we know that IG-closure of $L$ does not exist, hence by Lemma 3.10 $\bigcap_{k=1}^{\omega} \overline{L}^k \notin IG$. Let $\Sigma = \{a\}$. It is easy to see $IG(\{a\}) = \{\emptyset, \{a^{\omega}\}\} \cup \{\{a^k\}; k \geq 1\} \cup \{\{a^n; n \geq k\} \cup \{a^{\omega}\}; k \geq 1\}$. Consider an infinite sequence $L_i \in IG(\{a\})$, $i \in N$. If there is $n \in N$, such that $\mathrm{Card}(L_n) = 1$ or $L = \emptyset$, then $\bigcap_{i \in N} L_i$ equals $L_n$ or $\emptyset$ which both are in $IG(\{a\})$. Assume the contrary. Then there are $k_i$, such that $L_i = \{a^n; n \geq k_i\} \cup \{a^{\omega}\}$ for all $i \in N$. Thus,

$$\bigcap_{i \in N} L_i = \{a^n;\ n \geq \sup\{k_i; i \in N\}\} \cup \{a^{\omega}\}$$

which again is in $IG(\{a\})$.

**Theorem 5.3.** *The classes $IGk(\Sigma)$ and $IG(\Sigma)$ are not closed under union and difference.*

*Proof.* If $a \in \Sigma$, then $\{a^k\}$, $\{a^{\omega}\}$, $L = \{a^n; n \geq k\} \cup \{a^{\omega}\} \in IGk(\Sigma)$. One easily observes that $\{a^k\} \cup \{a^{\omega}\} \notin IG$,$L - \{a^{\omega}\} = \{a^n; n \geq k\} \notin IG$ and $\{a, b\}^{\omega} - \{a^{\omega}\} \notin IG$.

Let $L_i \in IG$, for $1 \leq i \leq n$. We now explicitly construct the IG-machine that generates $\bigcap_{i=1}^{n} L_i$.

**Lemma 5.4.** *Let $M_i = \langle \Sigma, H, k, S_i\rangle$, $i = 1, 2$. Then $Top(S_1 \cup S_2) \subseteq Top\, S_1 \cup Top\, S_2$.*

*Proof.* Let $aH \in \mathrm{Top}(S_1 \cup S_2)$. Then there is $b \in \Sigma$, such that $baH \in S_1 \cup S_2$. Hence $baH \in S_1$ or $baH \in S_2$ and so $aH \in \mathrm{Top}\, S_1$ or $aH \in \mathrm{Top}\, S_2$. Let $a \in \mathrm{Top}(S_1 \cup S_2) \cap \Sigma^k$. Hence $a \notin Dom(S_1 \cup S_2)$ and there is $b \in \Sigma$, $ba \in S_1 \cup S_2$. So $a \notin \mathrm{Dom}\, S_i$ for $i = 1, 2$ and $ba \in S_1$ or $ba \in S_2$. Thus, $a \in \mathrm{Top}\, S_1$ or $a \in \mathrm{Top}\, S_2$.

**Theorem 5.5.** *Let $M_i = \langle \Sigma, H, k, S_i\rangle$, $L_i = L^\infty(M_i)$ for $1 \le i \le n$. Then $M = \langle \Sigma, H, k, S\rangle$, $L^\infty(M) = \bigcap_{i=1}^{n} L_i$, where $S = \bigcup_{S' \in \Theta} S'$ where $S' \in \Theta$ if and only if the following two conditions are satisfied:*

$$(1) \qquad S' \subseteq \bigcap_{i=1}^{n} S_i \qquad \textit{and}$$

$$(2) \qquad Top\, S' \subseteq \bigcap_{i=1}^{n} (Top\, S_i \cup \{a; aH \in S_i\}).$$

*Proof.* By Theorem 5.1 there is an IG=machine $M = \langle \Sigma, H, k, S\rangle$, such that $L^\infty(M) = \bigcap_{i=1}^{n} L_i$.

Claim 1. $S \in \Theta$. Since $L^\infty(M) \subseteq L^\infty(M_i)$, then by Theorem 4.1 $S \subseteq S_i$ for all $1 \le i \le n$. Hence condition (1) holds for $S$. Let $aH \in \mathrm{Top}\, S$. Then there is $b \in \Sigma$, $baH \in S$. By (1) $baH \in S_i$ for all $1 \le i \le n$, and so $aH \in \mathrm{Top}\, S_i$ for all $1 \le i \le n$. If $a \in \mathrm{Top}\, S \cap \Sigma^k$, then there is $b \in \Sigma$, such that $ba \in S$ and so $ba \in \bigcap_{i=1}^{n} L_i$. Assume $a \notin \mathrm{Top}\, S_i$. Since $ba \in L_i$, then $a \in Dom\, S_i$ and so $aH \in S_i$. Consequently (2) holds for $S$.

Claim 2. If $S' \in \Theta$ then $L^\infty(M') \subseteq \bigcap_{i=1}^{n} L_i$ where $M' = \langle \Sigma, H, k, S'\rangle$. By (2) $\mathrm{Dom}\, S_i \cap \mathrm{Top}\, S' \subseteq \mathrm{Dom}\, S_i \cap (\mathrm{Top}\, S_i \cup \{a; aH \in S_i\}) = \{a; aH \in S_i\}$. From Theorem 4.1 follows $L^\infty(M') \subseteq L^\infty(M_i) = L_i$.

Claim 3. If $S', S'' \in \Theta$, then also $S' \cup S'' \in \Theta$. The claim follows easily from Lemma 5.4.

From claims 2 and 3 follow that $\Theta$ has the greatest element, namely $\bigcup_{S' \in \Theta} S'$.By claim 2 $L^\infty(M') \subseteq L^\infty(M)$ for any $S' \in \Theta$ and so by Theorem 4.1 $S' \subseteq S$. By claim 1 $S \in \Theta$ and so $S$ is the greatest element of $\Theta$.

We now provide an algorithm for constructing $S$. Denote

$$\Delta = \bigcap_{i=1}^{n} (\mathrm{Top}\, S_i \cup \{a; aH \in S_i\}).$$

Define recursively the sequence

$$S^{(0)} = \bigcap_{i=1}^{n} S_i,$$

$$S^{(m+1)} = S^{(m)} - \{ba;\ b \in \Sigma, a \in \mathrm{Top}\, S^{(m)} - \Delta\}.$$

Whenever $\mathrm{Top}\, S^{(m)} \subseteq \Delta$ we claim that $S^{(m)} = S$.

To prove this we first observe that $S^{(0)}$ is finite, $S^{(m+1)} \subseteq S^{(m)}$ and $\mathrm{Top}\, S^{(m)} \subseteq \Delta$ if and only if $S^{(m)} = S^{(m+1)}$. Thus, there is $m_0$ such that $S^{(m_0)} = S^{(m_0+1)} = S^{(m_0+2)} = \dots$. We shall now prove that $S \subseteq S^{(m)}$ for every $m \in N$. By Theorem 5.5 $S \subseteq S^{(0)}$. Proceeding inductively we assume $S \subseteq S^{(m)}$. By contraposition assume $S \not\subseteq S^{(m+1)}$. Hence there is $a \in S - S^{(m+1)}$. By assumption $a \in S^{(m)} - S^{(m+1)}$. First, assume $a$ is of the form $bH$, $b = b_0 \dots b_{k-1}$. For all $bH \in S^{(m)}$ follows $b \in L_i$ which implies $bH \in S_i$ for all $1 \le i \le n$. Hence $b_1 \dots b_{k-1}H \in \mathrm{Top}\, S_i$ for all $1 \le i \le n$ and so $b_1 \dots b_{k-1}H \in \Delta$. Thus, $a = bH \in S^{(m+1)}$ which yields a contradiction. Second, assume $a \in \Sigma^{k+1}$. By the definition of $S^{(m+1)}$ we obtain $a_1 \dots a_k \notin \Delta$ and $a_1 \dots a_k \in \mathrm{Top}\, S^{(m)}$ and so $a_1 \dots a_k \notin \mathrm{Dom}\, S^{(m)}$. Since $S \subseteq S^{(m)}$, then also $a_1 \dots a_k \notin \mathrm{Dom}\, S$. Hence $a_1 \dots a_k \in \mathrm{Top}\, S$. By the definition of $S$, $\mathrm{Top}\, S \subseteq \Delta$ and so $a_1 \dots a_k \in \Delta$. This yields a contradiction. We have proved $S \subseteq S^{(m)}$ for all $m \in N$. If $S^{(m)} = S^{(m+1)}$, then $S^{(m)} \subseteq \Delta$ and so $S^{(m)} \in \Theta$. $S$ is the greatest element of $\Theta$ and consequently $S = S^{(m)}$.

**Theorem 5.6.** *Let $M_i = \langle \Sigma, H, k_i, S_i\rangle$, $L_i = L^\infty(M_i)$ for $1 \le i \le n$. Put $k = \max\{k_i; 1 \le i \le n\}$. There is $M = \langle \Sigma, H, k, S\rangle$, such that $L^\infty(M) = \bigcap_{i=1}^{n} L_i$ and $S = \bigcup_{S' \in \Theta} S'$ where $S' \in \Theta$ if and only if the following two conditions are satisfied:*

$$(1) \qquad S' \subseteq \bigcap_{i=1}^{n} S_i/k,$$

$$(2) \qquad Top\, S' \subseteq \bigcap_{i=1}^{n} (Top(S_i/k) \cup \{a; aH \in S_i/k\}).$$

*Proof.* Observe that $\bigcap_{i=1}^n L_i = \bigcap_{i=1}^n c_k(L_i)$. By Lemma 3.7 $c_k(L_i) = L^\infty(M_i/k) \in IGk$. The Theorem now follows from Theorem 5.5.

Theorem 5.6 yields an algorithm for constructing the IG-machine $M$ which generates $\bigcap_{i=1}^n L_i$ : To each IG-machine $M_i$ we construct the IG$k$-machine $M_i/k$. Then we apply the above algorithm.

# 6 Poset Structures

Let $(P, \leq)$ be partially ordered set and $S$ a subset of $P$. If the partial ordering $\leq$ of the set $P$ is known, we formally identify $P$ with $(P, \leq)$ to simplify the notation. This simplification will be commonly used throughout this section. We say that an element $a \in P$ is a *join,* resp., *meet* of $S$ if $a$ is the least upper, resp., the greatest lower bound of $SA$ partially ordered set in which every pair of elements has a join, resp., meet is called an *upper,* resp., *lower semilattice.* A partially ordered set in which every pair of elements has a join and meet is called a *lattice.*

**Lemma 6.1.** *Let $M_i = \langle \Sigma, H, k, S_i\rangle$, $L_i = L^\infty(M_i)$, $1 \leq i \leq n$. Consider the IG$k$-machine $M = \langle \Sigma, H, k, S\rangle$, where*

$$S = \bigcup_{i=1}^n S_i \cup \{aH;\ a \in (\bigcup_{i=1}^n Top\, S_i) \cap Dom(\bigcup_{i=1}^n S_i)\}.$$

*Then $L^\infty(M) = \overline{\bigcup_{i=1}^n L_i}^k$.*

*Proof.* Using Definition 3.1 one easily verifies that $S = S(\bigcup_{i=1}^n L_i, k)$. Since $m(\bigcup_{i=1}^n L_i) \geq k$ the lemma follows from Corollary 3.5.

**Theorem 6.2.** *The set $IGk(\Sigma)$ is a lattice with meet $L_1 \cap L_2$ and join $\overline{L_1 \cup L_2}^k$. The least element of $IGk(\Sigma)$ is $\emptyset$ and the greatest element is $\Sigma^\infty - \bigcup_{n=0}^{k-1} \Sigma^n$.*

*Proof.* Consider the first statement. By Theorem 5.1 the set $IGk(\Sigma)$ is closed under finite intersection. Obviously $L_1 \cap L_2$ is a meet of $\{L_1, L_2\}$. By Theorem 3.5 $\overline{L_1 \cup L_2}^k$ is a join of $\{L_1, L_2\}$. The second statement is immediate.

**Example 6.3.** We give an example of an T $\omega$-language over a two letter alphabet which is the union of two IG1-languages and for which IG-closure does not exist. Consider two IG$\omega$-machines $M_i = \langle \Sigma, H, 1, S_i\rangle$, $i = 1, 2$, where $S_1 = \{a^2, ab, b^2\}$, $S_2 = \{b^2, ba, a^2\}$. Define $L = L^\infty(M_1) \cup L^\infty(M_2) = \{a^n b^\omega, b^n a^\omega;\ n \in N\}$. Let $k \geq 1$. Then by Definition 3.1 $a^k b$, $a^{k-1}b^2, \ldots, ab^k, b^k a, b^{k-1}a^2, \ldots, ba^k \in S(L, k)$. Hence $(a^k b^k)^\omega \in \overline{L}^k$. Assume $\overline{L}^k = \overline{L}^{k+1}$ for some $k \geq 1$. Then $ab^k a \in S(L, k+1)$. By Definition 3.1 $S(L, k+1)$ consists of all factors of length $k+2$ of $\omega$-words in $L$. Obviously $ab^k a$ is not a factor of any $\omega$-word in $L$. This yields a contradiction. We conclude $\overline{L}^k \neq \overline{L}^{k+1}$ for all $k \geq 1$. By Lemma 3.10 IG-closure of $L$ does not exist.

**Theorem 6.4.** *If $Card(\Sigma) \geq 2$, then the set $IG(\Sigma)$ is a lower semilattice, with the meet $L_1 \cap L_2$, but not lattice. For $Card(\Sigma) = 1$ the set $IG(\Sigma)$ is a lattice. In both cases the least element of $IG(\Sigma)$ is $\emptyset$; the greatest element of $IG(\Sigma)$, is $\Sigma^\infty - \{\lambda\}$.*

*Proof.* Consider the first statement. It is immediate from Theorem 5.1 that $IG(\Sigma)$ is a lower semilattice. Let $a, b \in \Sigma$. Consider the two IG1-languages $L_1 = \{a^n b^\omega; n \in N\}$, $L_2 = \{b^n a^\omega; n \in N\}$ from Example 6.3 and assume that the join of $\{L_1, L_2\}$ exists. Hence there is the least IG-language containing $L_1 \cup L_2$ and so IG-closure of $L_1 \cup L_2$ exists. This yields a contradiction with Example 6.3. It is easy to see that $IGX(\{a\})$ is a lattice. The last statement is obvious.

# 7 Enumeration Results

Let $(P, \leq)$ be partially ordered set and $S$ a subset of $P$. If $S$ is a subset of $P$ with the property that any two elements in $P$ are comparable, resp., noncomparable, then set $S$ is called a *chain,* resp., *antichain.* The *length* of the chain $S$ is $\mathrm{Card}(S) - 1$. $P$ is said to be of *length* $n$, in symbols $l(P) = n$, if there is a chain of length $n$ and all the chains are of length $\leq n$. The *width* of $P$ is $n$, in symbols $w(P) = n$, if there is an antichain of $n$ elements and all antichains have $\leq n$ elements. From Theorem 4.4 immediately follows $\mathrm{Card}(IGk(\Sigma)) = 2^{n^k(n+1)}$.

**Theorem 7.1.** *If $Card(\Sigma) = n$, then $l(IGk(\Sigma)) = n^k(n+1)$.*

*Proof.* Consider an arbitrary chain $L_0 \subset L_1 \subset \cdots \subset L_m$ of elements of $IGk(\Sigma)$. Denote by $M_i = \langle \Sigma, H, k, S_i\rangle$ the IG$k$-machine over $\Sigma$ that generates $L_i$. By Theorem 4.1 $S_0 \subset S_1 \subset \cdots \subset S_m \subseteq \Sigma^k \times$

$(\Sigma \cup \{H\})$. Hence $m \leq n^k(n+1)$. To end the proof we shall construct a chain of IG$k$-languages over $\Sigma$ of length $n^k(n+1)$. Assume a linear ordering on $\Sigma^k = \{x_1, \ldots, x_{n^k}\}$ and $\Sigma^{k+1} = \{y_1, \ldots, y_{n^{k+1}}\}$. Define recursively $S_0 = \emptyset$, $S_i = S_{i-1} \cup \{x_i H\}$ for $1 \leq i \leq n^k$, $S_i = S_{i-1} \cup \{y_{i-n^k}\}$ for $n^k + 1 \leq i \leq n^k(n+1)$. Put $M_i = \langle \Sigma, H, k, S_i \rangle$. By Theorem 4.1 $L^\infty(M_i) \subset L^\infty(M_{i+1})$. Thus, $L^\infty(M_0) \subset L^\infty(M_1) \subset \cdots \subset L^\infty(M_{n^k(n+1)})$ is the chain we sought.

**Theorem 7.2.** *If Card*$(\Sigma) = n$, *then*

$$w(IGk(\Sigma)) \geq \binom{n^k(n+1)}{\frac{n^k(n+1)}{2}}.$$

*Proof.* We shall construct an antichain in $IGk(\Sigma)$ of $\binom{n^k(n+1)}{\frac{n^k(n+1)}{2}}$ elements. Consider the set $A^{(m)} = \{L^\infty(M);\ M = \langle \Sigma, H, k, S \rangle,\ \mathrm{Card}(S) = m\}$. By Theorem 4.1 elements of $A^{(m)}$ are noncomparable and $\mathrm{Card}(A^{(m)}) = \binom{n^k(n+1)}{m}$. To end the proof put $m = \frac{n^k(n+1)}{2}$.

The set $A^{\binom{\frac{n^k(n+1)}{2}}{}}$ is the greatest set (with respect to cardinality) of all $A^{(m)}$, $0 \leq m \leq n^k(n+1)$. Notice that, unlike the case of G$k$-languages, the set $A^{(m)}$ is not always an antichain of maximal length in $IGk(\Sigma)$ and this is why the argument used by Mezník in [9] does not apply in the above case. hence instead obtaining the width of $IGk(\Sigma)$ we only obtain the lower bound on width. Indeed, consider $\Sigma = \{a, b\}$, $n = k = m = 2$ and $S = \{abb, bba, bab\}$, $M = \langle \Sigma, H, k, S \rangle$. Then $L^\infty(M) = RF((abb)^\omega) = L$. We shall prove that $L$ is non-comparable with all elements of $A^{(2)}$. Arguing indirectly assume the contrary, i.e that there is $L' = L^\infty(M')$, $M' = \langle \Sigma, H, 2, S' \rangle$, $\mathrm{Card}(S)' = 2$ such that $L \subseteq L'$ or $L' \subseteq L$. Since $\mathrm{Card}(S) = 3$ then $S \not\subseteq S'$ and so by Theorem 4.1 $L \not\subseteq L'$. Thus, $L' \subseteq L$ and by definition $L' \neq \emptyset$. Hence $L'$ contains at least one of the words $(abb)^\omega$, $(bba)^\omega$, $(bab)^\omega$. In either case $abb, bba, bab \in S'$ which is a contradiction. Thus, $A^{(2)}$ is not a maximal antichain in $IGk(\Sigma)$.

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