AN INTRODUCTION TO THE ALGEBRAIC ANALYSIS OF LINEAR MULTIDIMENSIONAL CONTROL SYSTEMS

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Abstract: - The purpose of this plenary lecture is first to provide an introduction to "*algebraic analysis*". This fashionable though quite difficult domain of pure mathematics today has been pioneered by V.P. Palamodov, M. Kashiwara and B. Malgrange around 1970, after the work of D.C. Spencer on the formal theory of systems of partial differential equations. We shall then focus on its application to control theory in order to study linear multidimensional control systems by means of new methods from module theory and homological algebra. We shall revisit a few basic concepts and prove, in particular, that controllability, contrary to a well established engineering tradition or intuition, is an intrinsic "built in" property of a control system, not depending on the choice of inputs and outputs among the control variables or even on the presentation of the control systems. Our exposition will be rather elementary as we shall insist on the main ideas and methods while illustrating them through explicit examples. Meanwhile, we want to stress out the fact that these new techniques bring totally striking results even on classical control systems of Kalman type !.

Key-Words: - Algebraic Analysis, Partial Differential Equations, Module Theory, Torsion Module, Projective Equivalence, Homological Algebra, Extension Functor, Control Theory, Controllability, Observability.

1 INTRODUCTION

Ordinary differential (OD) control theory studies input/output relations defined by systems of ordinary differential (OD) equations. In this case, with standard notations, if a control system is defined by input/state/output relations:

$$\dot{x} = Ax + Bu \qquad , \qquad y = Cx + Du$$

with $dim(x) = n$, this system is "*controllable*" if $rk(B, AB, ..., A^{n-1}B) = n$ and "*observable*" if $rk(\tilde{C}, \tilde{A}\tilde{C}, ..., \tilde{A}^{n-1}\tilde{C}) = n$ where the tilde sign indicates the transpose of a matrix [3]. Accordingly, the so-called "*dual system*":

$$\dot{x}_a = -\tilde{A}x_a - \tilde{C}u_a \qquad , \qquad y_a = \tilde{B}x_a + \tilde{D}u_a$$

is controllable (observable) if and only if the given system is observable (controllable). However, and despite many attempts, such a dual definition still seems purely artificial as one cannot avoid introducing the state. The same method could be applied to delay systems with constant coefficients.

Now, let us consider the system of two OD equations for three unknowns where a is a constant parameter:

$$\dot{y}^1 - ay^2 - \dot{y}^3 = 0 \qquad , \qquad y^1 - \dot{y}^2 + \dot{y}^3 = 0,$$

Whether we choose $y^1 = x^1, y^2 = x^2, y^3 = u$ or $y^1 = x^1, y^2 = u, y^3 = x^2$ while choosing in both cases $\bar{x}^1 = x^1 - u, \bar{x}^2 = x^2 - u$, we get two quite different systems in Kalman form , though both are controllable if and only if $a \neq 0$ and $a \neq 1$.

Two problems are raised at once.

First of all, if the derivatives of the inputs do appear in the control system, for example in the SISO system $\dot{x} - \dot{u} = 0$, not a word is left from the original functional definition of controllability which is only valid for systems in "*Kalman form*" and the same comment can be made for the corresponding duality.

Secondly, we understand from the above example that *controllability must be a structural property of a control system*, neither depending on the choice of the inputs and outputs among the system variables, nor even on the presentation of the system (change of the variables eventually leading to change the order of the system).

More generally, "*partial differential (PD) control theory*" will study input/output relations defined by systems of partial differential (PD) equations. At first sight, we have no longer a way to generalize the Kalman form and not a word of the preceding approach is left as, in most cases, the number of arbitrary parametric derivatives playing the rôle of state could be infinite. We also understand that a good definition of controllability and duality should also be valid for control systems with variable coefficients. A similar comment can be made for the definition of the transfer matrix.

Keeping aside these problems for the moment, let us now turn for a few pages to the formal theory of systems of OD or PD equations.

In 1920, M. Janet provided an effective algorithm for looking at the formal (power series) solutions of systems of ordinary differential (OD) or partial differential (PD) equations [2]. The interesting point is that this algorithm also allows to determine the *compatibility conditions* $\mathcal{D}_1\eta = 0$ for solving (*formally* again but this word will not be repeated) inhomogeneous systems of the form $\mathcal{D}\xi = \eta$ when $\mathcal{D}$ is an OD or PD operator and ξ, η certain functions. Similarly, one can also determine the compatibility conditions $\mathcal{D}_2\zeta = 0$ for solving $\mathcal{D}_1\eta = \zeta$, and so on. With no loss of generality, this construction of a "*differential sequence*" can be done in such a canonical way that we successively obtain $\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_n$ from $\mathcal{D}$ and $\mathcal{D}_n$ is surjective when n is the number of independent variables.

With no reference to the above work, D.C. Spencer developed, from 1965 to 1975, the *formal theory* of systems of PD equations by relating the preceding results to *homological algebra* and *jet theory* [19]. However, this tool has been *largely ignored* by mathematicians and, "*a fortiori*", by engineers or even physicists. Therefore, the module theoretic counterpart, today known as "*algebraic analysis*", which has been pioneered around 1970 by V.P. Palamodov for the constant coefficient case [10], then by M. Kashiwara [4] and B. Malgrange [7] for the variable coefficient case, as it heavily depends on the previous difficult work and looks like even more abstract, has been *totally ignored* within the range of any application before 1990, when U. Oberst revealed its importance for control theory [9].

The purpose of this lecture will essentially be to repare this gap by explaining, in a self-contained way on a few explicit examples, what is the powerfulness of this new approach for understanding both the structural and input/output properties of linear PD control systems, also called multidimensional or simply n-dimensional. Meanwhile, the reader will evaluate the price to pay for such a better understanding. Needless to say that many results obtained could not even be imagined without this new approach, dating back to 1986 when we gave for the first time the formal definition of controllability of a control system [11] but now largely acknowledged by the control community [20,21].

As we always use to say, the difficulty in studying differential modules is not of an algebraic nature but rather of a differential geometric nature. This is the reason for which the study of algebraic analysis is at once touching delicate points of jet theory, the main one being *formal integrability.* We now explain this concept on a tricky motivating example.

Motivating Example 1: With two independent variables (x^1, x^2), one unknown y and standard notations, we consider the following third order system of PD equations with second member (u, v):

$$\left\{ \begin{array}{lcll} Py & \equiv & d_{222}y + x^2 y & = u \\ Qy & \equiv & d_2 y + d_1 y & = v \end{array} \right.$$

We check the identity $QP - PQ \equiv 1$ and obtain easily:

$$y = Qu - Pv = d_2 u + d_1 u - d_{222} v - x^2 v$$

Substituting in the previous PD equations, we should obtain the generating 6^{th}-order compatibility conditions for (u, v) in the form:

$$\left\{ \begin{array}{lll} A \equiv & PQu - P^2 v - u & = 0 \\ B \equiv & Q^2 u - QPv - v & = 0 \end{array} \right.$$

These two compatibility conditions are differentially dependent as we check at once $QA - PB \equiv 0$. Finally, setting $u = 0, v = 0$, we notice that the preceding homogeneous system can be written in the form $\mathcal{D}y = 0$ and admits the only solution $y = 0$.

Motivating Example 2: Again with two independent variables (x^1, x^2) and one unknown y, let us consider the following second order system with constant coefficients:

$$\left\{ \begin{array}{lll} Py \equiv & d_{22}y & = u \\ Qy \equiv & d_{12}y - y & = v \end{array} \right.$$

We obtain at once:

$$y = d_{11}u - d_{12}v - v$$

and could hope to obtain the 4^{th}-order generating compatibility conditions by substitution, that is to say:

$$\left\{ \begin{array}{lll} A \equiv & d_{1122}u - d_{1222}v - d_{22}v - u & = 0 \\ B \equiv & d_{1112}u - d_{11}u - d_{1122}v & = 0 \end{array} \right.$$

However, *in this particular case*, we notice that there is an unexpected *unique second order* generating compatibility condition of the form:

$$C \equiv d_{12}u - u - d_{22}v = 0$$

as we have indeed $A \equiv d_{12}C + C$ and $B \equiv d_{11}C$, a result leading to $C \equiv d_{22}B - d_{12}A + A$. Accordingly, the systems $A = 0, B = 0$ on one side and $C = 0$ on the other side are completely different though they have the same solutions in u, v.

2 ALGEBRAIC ANALYSIS

It becomes clear that there is a need for classifying the properties of systems of PD equations in a way that does not depend on their presentations and *this is the purpose of algebraic analysis.*

We recall a few basic facts from jet theory [13,14].

Let X be a manifold of dimension n with local coordinates $x = (x^1, ..., x^n)$ and E be a vector bundle over X with local coordinates (x^i, y^k), where $i = 1, ..., n$ for the independent variables, $k = 1, ..., m$ for the dependent variables, and projection $(x, y) \rightarrow x$. A (local) section $\xi : X \rightarrow E : x \rightarrow (x, \xi(x))$ is defined locally by $y^k = \xi^k(x)$. Under any change of local coordinates $(x, y) \rightarrow (\bar{x} = \varphi(x), \bar{y} = A(x)y)$ the section changes according to $\bar{y}^l = \bar{\xi}^l(\bar{x})$ in such a way that $\bar{\xi}^l(\varphi(x)) \equiv A^l_k(x)\xi^k(x)$ and we may differentiate successively each member in order to obtain, though in a more and more tedious way, the transition rules for the derivatives $\xi^k(x), \partial_i\xi^k(x), \partial_{ij}\xi^k(x), ...$ up to order q. As usual, we shall denote by $J_q(E)$ and call q-*jet bundle* the vector bundle over X with the same transition rules and local *jet coordinates* (x, y_q) with $y_q = (y^k, y^k_i, y^k_{ij}, ...)$ or, more generally y^k_μ with $1 \leq \mid \mu \mid \leq q$ where $\mu = (\mu_1, ..., \mu_n)$ is a multi-index of length $\mid \mu \mid = \mu_1 + ... + \mu_n$ and $\mu + 1_i = (\mu_1, ..., \mu_{i-1}, \mu_i + 1, \mu_{i+1}, ..., \mu_n)$. The reader must not forget that the above definiions are standard ones in physics or mechanics because of the use of tensors in electromagnetism or elasticity.

Definition 1: A system of PD equations on E is a vector sub-bundle $R_q \subset J_q(E)$ locally defined by a constant rank system of linear equations $A^{\tau\mu}_k(x)y^k_\mu = 0$.

Substituting the derivatives of a section in place of the corresponding jet coordinates, then differentiating once with respect to x^i and substituting the jet coordinates, we get the *first prolongation* $R_{q+1} \subset J_{q+1}(E)$, defined by the previous equations and by the new equations $A^{\tau\mu}_k(x)y^k_{\mu+1_i} + \partial_i A^{\tau\mu}_k(x)y^k_\mu = 0$, and, more generally, the r-prolongations $R_{q+r} \subset J_{q+r}(E)$ which need not be vector bundles ($xy_x - y = 0 \Longrightarrow xy_{xx} = 0$).

Definition: R_q is said to be *formally integrable* if the R_{q+r} are vector bundles and all the generating PD equations of order $q + r$ are obtained by prolonging R_q exactly r-times only, $\forall r \geq 0$.

We now specify the correspondence:

$$\begin{aligned} \text{SYSTEM} \quad &\Leftrightarrow \text{OPERATOR} \\ &\Leftrightarrow \text{MODULE} \end{aligned}$$

in order to show later on that certain concepts, which are clear in one framework, may become quite obscure in the others and conversely (check this for the formal integrability and torsion concepts for example !).

Having a system of order q, say $R_q \subset J_q(E)$, we can introduce the canonical projection $\Phi : J_q(E) \longrightarrow J_q(E)/R_q = F$ and define a linear differential operator $\mathcal{D} : E \longrightarrow F : \xi(x) \longrightarrow \eta^\tau(x) = A^{\tau\mu}_k(x)\partial_\mu\xi^k(x)$. When $\mathcal{D}$ is given, the compatibility conditions for solving $\mathcal{D}\xi = \eta$ can be described in operator form by $\mathcal{D}_1\eta = 0$ and so on. In general (see the preceding examples), if a system is not formally integrable, it is possible to obtain a formally integrable system, having the same solutions, by "*saturating*" conveniently the given PD equations through the adjunction of new PD equations obtained by various prolongations and *such a procedure must absolutely be done before looking for the compatibility conditions.*

In order to study differential modules, for simplicity we shall forget about changes of coordinates and consider trivial bundles. Let K is a *differential field* with n commuting derivations $\partial_1, ..., \partial_n$ (say $\mathbb{Q}, \mathbb{Q}(x^1, ..., x^n)$ or $\mathbb{Q}(a)$ in the previous examples). If $d_1, ..., d_n$ are *formal derivatives* (pure symbols in computer algebra packages !) which are only supposed to satisfy $d_i a = a d_i + \partial_i a$ in the operator sense for any $a \in K$, we may consider the (non-commutative) ring $D = K[d_1, ..., d_n]$ of differential operators with coefficients in K. If now $y = (y^1, ..., y^m)$ is a set of differential indeterminates, we let D act formally on y by setting $d_\mu y^k = y^k_\mu$ and set $Dy = Dy^1 + ... + Dy^m$. Denoting simply by $D\mathcal{D}y$ the subdifferential module generated by all the given OD or PD equations and all their formal derivatives, we may finally introduce the D-module $M = Dy/D\mathcal{D}y$. Here we recall that M is a *module* over a ring A or an A-module if $\forall a \in A, \forall m, n \in M \Rightarrow am, m + n \in M$.

Example: In the Motivating Examples, we get $M = 0$.

Before entering the heart of the paper, we need a few technical definitions and results from homological algebra [5,8,14,18].

First of all, we recall that a sequence of modules and maps is exact if the kernel of any map is equal to the image of the map preceding it.

If A is a noetherian integral domain, we denote by $K = Q(A)$ the quotient field of A and we have the short exact sequence:

$$0 \longrightarrow A \longrightarrow K \longrightarrow K/A \longrightarrow 0$$

If now M is a left A-module, we may tensor this sequence by M on the right with $A \otimes M = M$ but we do not get in general an exact sequence. The defect of exactness *on the left* is nothing else but the *torsion submodule* $t(M) = \{m \in M \mid \exists 0 \neq a \in A, am = 0\} \subseteq M$ and we have the long exact sequence:

$$0 \longrightarrow t(M) \longrightarrow M \longrightarrow K \otimes_A M \longrightarrow K/A \otimes_A M \longrightarrow 0$$

as we may describe the central map as follows:

$$m \longrightarrow 1 \otimes m = \frac{a}{a} \otimes m = \frac{1}{a} \otimes am \quad , \quad \forall a \neq 0$$

Such a result based on the so-called *localization* technique allows to understand why controllability has to do with the so-called "simplification" of the *transfer matrix* but this is out of our scope [14,16]. In particular, a module M is said to be a *torsion module* if $t(M) = M$ and a *torsion-free module* if $t(M) = 0$.

We now introduce the *extension functor* in an elementary manner, using the standard notation $hom_A(M, A) = M^*$. First of all, by a *free resolution* of an A-module M, we understand a long exact sequence:

$$... \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0$$

where $F_0, F_1, ...$are free modules, that is to say modules isomorphic to powers of A and $M = coker(d_1) = F_0/im(d_1)$. We may *take out* M and obtain the *deleted sequence*:

$$... \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

which is of course no longer exact. If N is any other A-module, we may apply the functor $hom_A(\bullet, N)$ and obtain the sequence:

$$... \xleftarrow{d_2^*} hom_A(F_1, N) \xleftarrow{d_1^*} hom_A(F_0, N) \longleftarrow 0$$

in order to state:

Definition: $ext^0_A(M, N) = ker(d_1^*) = hom_A(M, N)$, $ext^i_A(M, N) = ker(d^*_{i+1})/im(d^*_i), \forall i \geq 1$

One can prove that the extension modules do not depend on the resolution of M chosen and have the following two main properties, the first of which only is classical [14,17,18].

PROPOSITION: If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a short exact sequence of A-modules, then we have the following *connecting long exact sequence*:

$$\begin{aligned} 0 \rightarrow hom_A(M'', N) \rightarrow hom_A(M, N) \\ \rightarrow hom_A(M', N) \rightarrow ext^1_A(M'', N) \rightarrow ... \end{aligned}$$

of extension modules.

PROPOSITION: $ext^i_A(M, A)$ is a torsion module, $\forall i \geq 1$.

Proof: Let F be a maximal free submodule of M. From the short exact sequence:

$$0 \longrightarrow F \longrightarrow M \longrightarrow M/F \longrightarrow 0$$

where M/F is a torsion module, we obtain the long exact sequence:

$$\begin{aligned} ... \rightarrow ext^{i-1}_A(F, A) \rightarrow ext^i_A(M/F, A) \\ \rightarrow ext^i_A(M, A) \rightarrow ext^i_A(F, A) \rightarrow ... \end{aligned}$$

From the definitions, we obtain $ext^i_A(F, A) = 0, \forall i \geq 1$ and thus $ext^i_A(M, A) \simeq ext^i_A(M/F, A), \forall i \geq 2$. Now it is known that the tensor by the field K of any exact sequence is again an exact sequence. Accordingly, we have from the definition:

$$\begin{aligned} K \otimes_A ext^i_A(M/F, A) \simeq ext^i_A(M/F, K) \\ \simeq ext^i_K(K \otimes_A M/F, K) = 0, \forall i \geq 1 \end{aligned}$$

and we finally obtain from the above sequence $K \otimes_A ext^i_A(M, A) = 0 \Rightarrow ext^i_A(M, A)$ torsion, $\forall i \geq 1$.
Q.E.D.

As we have seen in the Motivating Examples, the same module may have many very different presentations. In particular, we have [5,14]:

Schanuel Lemma: If $F'_1 \xrightarrow{d'_1} F'_0 \rightarrow M \rightarrow 0$ and $F''_1 \xrightarrow{d''_1} F''_0 \rightarrow M \rightarrow 0$ are two *presentations* of M, there exists a presentation $F_1 \xrightarrow{d_1} F_0 \rightarrow M \rightarrow 0$ of M projecting onto the preceding ones.

Definition: An A-module P is *projective* if there exists a free module F and another (thus projective) module Q such that $P \oplus Q \simeq F$. Any free module is projective.

Definition: A short exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

splits if $M \simeq M' \oplus M''$ or, equivalently, if one can find a *lift* $u : M \rightarrow M'$ such that $u \circ f = id'_M$ or a *lift* $v : M'' \rightarrow M$ such that $g \circ v = id_{M''}$.

Proposition: The above short exact sequence splits whenever M'' is projective.

Proposition: When P is a projective module and N is any module, we have $ext^i_A(P, N) = 0, \forall i \geq 1$.

3 PROBLEM FORMULATION

Though it seems that we are very far from any possible application, let us now present three problems which, both with the previous examples, look like unrelated with what we already said and between themselves.

Problem 1: Let a rigid bar of length L be able to slide horizontally and attach at the end of abcissa x (resp. $x+L$) a pendulum of length l_1 (resp. l_2) with mass m_1 (resp. m_2), making an angle θ_1 (resp. θ_2) with the downwards vertical axis. Projecting the dynamical equations on the perpendicular to each pendulum in order to eliminate the respective tension, we get:

$$m_1(\ddot{x}cos\theta_1 + l_1\ddot{\theta_1}) + m_1 g sin\theta_1 = 0$$

where g is the gravity. When θ_1 and θ_2 are small, we get the following two OD equations that only depend on l_1 and l_2 but no longer on m_1 and m_2:

$$\left\{ \begin{array}{ll} \ddot{x} + l_1\ddot{\theta}_1 + g\theta_1 & = 0 \\ \ddot{x} + l_2\ddot{\theta}_2 + g\theta_2 & = 0 \end{array} \right.$$

Now it is easy to check *experimentally* that, when $l_1 \neq l_2$, it is possible to bring any small amplitude motion $\theta_1 = \theta_1(t), \theta_2 = \theta_2(t)$ of the two pendula back to equilibrium $\theta_1 = 0, \theta_2 = 0$, just by choosing a convenient $x = x(t)$ and *the system is said to be controllable.* On the contrary, if $l_1 = l_2$ and unless $\theta_1(t) = \theta_2(t)$, then it is impossible to bring the pendula back to equilibrium and *the system is said to be uncontrollable.* A similar question can be asked when $l_1 = l_1(t), l_2 = l_2(t)$ are given, the variation of length being produced by two small engines hidden in the bar [14].
Hence, a much more general question concerns the controllability of control systems defined by systems of OD or PD equations as well, like in gasdynamic or magnetohydrodynamic.
In our case, setting $x_1 = x + l_1\theta_1, x_2 = x + l_2\theta_2$, we get:

$$\left\{ \begin{array}{ll} \ddot{x}_1 + (g/l_1)x_1 - (g/l_1)x & = 0 \\ \ddot{x}_2 + (g/l_2)x_2 - (g/l_2)x & = 0 \end{array} \right.$$

and may set $\dot{x}_1 = x_3, \dot{x}_2 = x_4$ in order to bring the preceding system to Kalman form with 4 first order OD equations. The controllability condition is then easily seen to be $l_1 \neq l_2$ but such a result not only seems to depend on the choice of input and output but cannot be extended to PD equations.

Problem 2: Any engineer knows about the first set of Maxwell equations:

$$\vec{\nabla}.\vec{B} = 0, \qquad \vec{\nabla} \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

and the fact that any solution can be written in the form:

$$\vec{B} = \vec{\nabla} \wedge \vec{A}, \qquad \vec{E} = -\vec{\nabla}.V - \frac{\partial \vec{A}}{\partial t}$$

for an arbitrary vector $\vec{A}$ and an arbitrary function V.
According to special relativity, these equations can be condensed on space-time by introducing a 1-form A for the *potential* and a 2-form F for the *field* in order to discover that the above Mawell equations can be written in the form $dF = 0$ and admit the "generic" solution $dA = F$ where d is the exterior derivative. Hence, we have "*parametrized*" the field equations by means of a "*potential*", that is the field equations generate the compatibility conditions of the inhomogeneous system allowing to express the field (right member) by means of the potential (left member).

Similarly, in 2-dimensional elasticity theory, if we want to solve the stress equations with no density of force, namely:

$$\partial_1\sigma^{11} + \partial_2\sigma^{21} = 0 \quad , \quad \partial_1\sigma^{12} + \partial_2\sigma^{22} = 0$$

we may use the first PD equation to get:

$$\exists\varphi \ , \ \sigma^{11} = \partial_2\varphi \quad , \quad \sigma^{21} = -\partial_1\varphi$$

and the second PD equation to get:

$$\exists\psi \ , \ \sigma^{12} = -\partial_2\psi \quad , \quad \sigma^{22} = \partial_1\psi$$

Now, $\sigma^{12} = \sigma^{21} \Rightarrow \exists\phi \ , \ \varphi = \partial_2\phi \quad , \quad \psi = \partial_1\phi$
and we finally get the generic parametrization by the Airy function:

$$\sigma^{11} = \partial_{22}\phi \quad , \quad \sigma^{12} = \sigma^{21} = -\partial_{12}\phi \quad , \quad \sigma^{22} = \partial_{11}\phi$$

The reader will have noticed that such a specific computation cannot be extended in general, even to 3-dimensional elasticity theory.

In 1970 J. Wheeler asked a similar question for Einstein equations in vacuum and we present the linearized version of this problem.
Indeed, if $\omega = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2$ with $x^4 = ct$, where c is the speed of light, is the Minkowski metric of space-time, we may consider a perturbation Ω of ω and the linearized Einstein equations in vacuum become equivalent to the following second order system with 10 equations for 10 unknowns:

$$\begin{array}{l} \omega^{rs}(d_{ij}\Omega_{rs} + d_{rs}\Omega_{ij} - d_{ri}\Omega_{sj} - d_{sj}\Omega_{ri}) \\ \qquad -\omega_{ij}(\omega^{rs}\omega^{uv}d_{rs}\Omega_{uv} - \omega^{ru}\omega^{sv}d_{rs}\Omega_{uv}) = 0 \end{array}$$

Surprisingly, till we gave the (negative) answer in 1995 [15], such a problem had never been solved before.
More generally, if one considers a system of the form $\mathcal{D}_1\eta = 0$, the question is to know whether one can parametrize or not the solution space by $\mathcal{D}\xi = \eta$ with arbitrary potential-like functions ξ, in such a way that $\mathcal{D}_1\eta = 0$ just generates the compatibility conditions of the parametrization. The problem of multiple parametrizations may also be considered, as an inverse to the construction of differential sequences. For example, in vector calculus, the *div* operator is parametrized by the *curl* operator which is itself parametrized by the *grad* operator (See [4,10,14] for more details).

Problem 3: When M is an A-module, there is a canonical morphism $\epsilon = \epsilon_M : M \longrightarrow M^{**}$ given by $\epsilon(m)(f) = f(m), \forall m \in M, \forall f \in M^*$ and M is said to be *torsionless* if ϵ is injective and *reflexive* if ϵ is bijective. Any finitely projective module is reflexive but a reflexive module may not be projective. We have $t(M) \subseteq ker(\epsilon)$ because, if $m \in M$ is a torsion element for $a \neq 0$, then $af(m) = f(am) = f(0) = 0 \Rightarrow f(m) = 0, \forall f \in M^*$ as before and ϵ fails to be injective. Hence, it just remains to study whether this inclusion is strict or not.

The striking result of this lecture is to prove that THESE THREE PROBLEMS ARE IDENTICAL !.

4 PROBLEM SOLUTION

The *main but highly not evident* trick will be to introduce the *adjoint operator* $\tilde{\mathcal{D}} = ad(\mathcal{D})$ by the formula of integration by part:

$$< \lambda, \mathcal{D}\xi > = < \tilde{\mathcal{D}}\lambda, \xi > + div(\quad)$$

where λ is a test row vector and $<>$ denotes the usual contraction. The adjoint can also be defined formally, as in computer algebra packages, by setting $ad(a) = a, \forall a \in K, ad(d_i) = -d_i, ad(PQ) = ad(Q)ad(P), \forall P, Q \in D$. Denoting by N the differential module defined from $ad(\mathcal{D})$ exactly like M was defined from $\mathcal{D}$, we have [4,10,14,15,20]:

Theorem: The following statements are equivalent:
- A control system is controllable.
- The corresponding operator is simply (doubly) parametrizable.
- The corresponding module is torsion-free (reflexive).

Proof: Let us start with a free presentation of M:

$$F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0$$

By definition, we have $M = coker(d_1) \Longrightarrow N = coker(d_1^*)$ and we may exhibit the following free resolution of N:

$$0 \longleftarrow N \longleftarrow F_1^* \xleftarrow{d_1^*} F_0^* \xleftarrow{d_0^*} F_{-1}^* \xleftarrow{d_{-1}^*} F_{-2}^*$$

where $M^* = ker(d_1^*) = im(d_0^*) \simeq coker(d_{-1}^*)$. The deleted sequence is:

$$0 \longleftarrow F_1^* \xleftarrow{d_1^*} F_0^* \xleftarrow{d_0^*} F_{-1}^* \xleftarrow{d_{-1}^*} F_{-2}^*$$

Applying $hom_A(\bullet, A)$ and using the canonical isomorphism $F^{**} \simeq F$ for any free module F, we get the sequence:

$$\begin{array}{ccccccc} 0 \longrightarrow F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{d_0} & F_{-1} & \xrightarrow{d_{-1}} & F_{-2} \\ & & \downarrow & & \uparrow & & \\ & & M & \xrightarrow{\epsilon} & M^{**} & & \\ & & \downarrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array}$$

Denoting as usual a coboundary space by B, a cocycle space by Z and the corresponding cohomology by $H = Z/B$, we get the commutative and exact diagram:

$$\begin{array}{ccccccccc} 0 \longrightarrow & B_0 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow 0 \\ & \downarrow & & \parallel & & \downarrow \epsilon & \\ 0 \longrightarrow & Z_0 & \longrightarrow & F_0 & \longrightarrow & M^{**} & \end{array}$$

An easy chase provides at once $H_0 = Z_0/B_0 = ext_A^1(N, A) \simeq ker(\epsilon)$. It follows that $ker(\epsilon)$ is a torsion module and, as we already know that $t(M) \subseteq ker(\epsilon) \subseteq M$, we finally obtain $t(M) = ker(\epsilon)$. Also, as $B_{-1} = im(\epsilon)$ and $Z_{-1} \simeq M^{**}$, we obtain $H_{-1} = Z_{-1}/B_{-1} = ext_A^2(N, A) \simeq coker(\epsilon)$. Accordingly, a torsion-free (reflexive) module is described by an operator that admits a single (double) step parametrization.

Q.E.D.

This proof also provides an effective test for applications by using D and ad instead of A and $*$ in the differential framework. In particular, a control system is controllable if it does not admit any "*autonomous element*", that is to say any finite linear combination of the control variables and their derivatives that satisfies, *for itself*, at least one OD or PD equation. More precisely, starting with the control system described by an operator $\mathcal{D}_1$, one MUST construct $\tilde{\mathcal{D}}_1$ and then $\mathcal{D}$ such that $\tilde{\mathcal{D}}$ generates all the compatibility conditions of $\tilde{\mathcal{D}}_1$. Finally, M is torsion-free if and only if $\mathcal{D}_1$ generates all the compatibility conditions of $\mathcal{D}$. Though striking it could be, *this is the true generalization of the standard Kalman test.*

Example: If $\mathcal{D}_1 : (\sigma^{11}, \sigma^{12} = \sigma^{21}, \sigma^{22}) \rightarrow (\partial_1\sigma^{11} + \partial_2\sigma^{21}, \partial_1\sigma^{12} + \partial_2\sigma^{22})$ is the stress operator, then $\tilde{\mathcal{D}}_1 : (\xi^1, \xi^2) \rightarrow (\partial_1\xi^1 = \epsilon_{11}, \frac{1}{2}(\partial_1\xi^2 + \partial_2\xi^1) = \epsilon_{12} = \epsilon_{21}, \partial_2\xi^2 = \epsilon_{22})$ is half of the Killing operator. The only compatibility condition for the strain tensor ϵ is $\tilde{\mathcal{D}}\epsilon = 0 \Leftrightarrow \partial_{11}\epsilon_{22} + \partial_{22}\epsilon_{11} - 2\partial_{12}\epsilon_{12} = 0$ and $\mathcal{D}$ describes the Airy parametrization.

Of course, keeping the same module M but changing its presentation or even using an isomorphic module M' (2 OD equations of order 2 or 4 OD equations of order 1 as in the case of the double pendulum), then N may change to N'. The following result, *totally unaccessible to intuition*, justifies "*a posteriori*" the use of the extension functor by proving that the above results are unchanged and are thus "*intrinsic*" [15,17].

Theorem: N and N' are *projectively equivalent*, that is to say one can find projective modules P and P' such that $N \oplus P \simeq N' \oplus P'$.

Proof: According to Schanuel lemma, we can always suppose, with no loss of generality, that the resolution of M projects onto the resolution of M'. The kernel sequence is a splitting sequence made up with projective modules because the kernel of the projection of F_i onto F'_i is a projective module P_i for $i = 0, 1$. Such a property still holds when applying duality. Hence, if C is the kernel of the epimorphism from P_1 to P_0 induced by d_1, then C is a projective module, C^* is also a projective module and we obtain $N \simeq N' \oplus C^*$.

Q.E.D.

Accordingly, using the properties of the extension functor, we get:

Corollary: $ext^i_A(N, A) \simeq ext^i_A(N', A) \qquad \forall i \geq 1$.

We finally apply these results in order to solve the three preceding problems.

Solution 1: As the operator $\mathcal{D}$ of the control system is surjective, it follows that the map d_1 of the presentation is injective. When $K = \mathbb{R}$ and $n = 1$, then D can be identified with a polynomial ring in one indeterminate and is therefore a principal ideal domain (any ideal can be generated by a single polynomial). In this case, it is well known [1,18] that any torsion-free module is indeed free and thus projective. The short exact sequence of the presentation splits, with a similar comment for its dual sequence. Accordingly, *M is torsion-free if and only if $N = 0$* and it just remains to prove that $\tilde{\mathcal{D}}$ is injective. We have to solve the system:

$$\left\{ \begin{array}{lcll} x & \longrightarrow & \ddot{\lambda}_1 + \ddot{\lambda}_2 & = 0 \\ \theta_1 & \longrightarrow & l_1 \ddot{\lambda}_1 + g\lambda_1 & = 0 \\ \theta_2 & \longrightarrow & l_2 \ddot{\lambda}_2 + g\lambda_2 & = 0 \end{array} \right.$$

Multiplying the second OD equation by l_2, the third by l_1 and adding them while taking into account the first OD equation, we get:

$$l_2 \lambda_1 + l_1 \lambda_2 = 0$$

Differentiating this OD equation twice while using the second and third OD equations, we get:

$$(l_2/l_1)\lambda_1 + (l_1/l_2)\lambda_2 + 0$$

The determinant of this linear system for λ_1 and λ_2 is just $l_1 - l_2$, hence the system is controllable if and only if $l_1 \neq l_2$.
Conversely, if $l_1 = l_2 = l$, the corresponding module has torsion elements. In particular, setting $\theta = \theta_1 - \theta_2$ and substracting the second dynamic equation from the first, we get $l\ddot{\theta} + g\theta = 0$. Hence θ is a torsion element which is solution of an *autonomous* OD equation, that is an OD equation *for itself* which cannot therefore be "controlled" by any means.

Solution 2: After a short computation left to the reader as an exercise, one checks easily that *the Einstein operator is self-adjoint* because the 6 terms are just exchanged between themselves. Then, it is well known that the compatibility condition is made by the standard divergence operator and its adjoint is the Killing operator (Lie derivative of the Minkowski metric) which admits the linearized Riemann curvature (20 PD equations) as compatibility conditions and not the Einstein equations (10 PD equations only). Hence, the Einstein operator cannot be parametrizable and it follows that *Einstein equations cannot be any longer considered as field equations* (For a computer algebra solution, see [21]).

Solution 3: It has already been provided by the preceding theorems.

Remark: Writing a Kalman type system in the form $-\dot{x} + Ax + Bu = 0$ and multiplying on the left by a test row vector λ, the kernel of the adjoint operator is defined by the system:

$$\dot{\lambda} + \lambda A = 0, \qquad \lambda B = 0$$

Differentiating the second equations, we get:

$$\dot{\lambda} B = 0 \Longrightarrow \lambda AB = 0 \Longrightarrow \lambda A^2 B = 0 \Longrightarrow ...$$

and we discover that *the Kalman criterion just amounts to the injectivity of the adjoint operator*. Hence, in any case, *controllability only depends on formal integrability*. Comparing with the Motivating Examples, we notice that, when a constant coefficient operator is injective, the fact that we can find differentially independent compatibility conditions is equivalent to the Quillen-Suslin theorem saying roughly that a projective module over a polynomial ring is indeed free (See [5,18] for details). More generally, by duality we obtain at once $t(M) \simeq ext^1_A(N, A) \Leftrightarrow t(N) \simeq ext^1_A(M, A)$ and this result is coherent with the introduction of this lecture provided we say that a control system is "*observable*" if $ext^1_A(M, A) = 0$.

5 CONCLUSION

We hope to have convinced the reader that the results presented are striking enough to open a wide future for applications of computer algebra. The systematic use of the adjoint operator has allowed to relate together results as far from each other as the Quillen-Suslin theorem in module theory and the controllability criterion in control theory. A similar criterion for projective modules does exist and relies on the possibility to have finite length differential sequences [14,15]. We believe that the corresponding symbolic packages will be available in a short time. It will thus become possible to classify (differential) modules, having in mind that such a classification always describes hidden but deep concepts in the range of applications.

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