THE MULTIDIMENSIONAL DELSARTE TRANSMUTATION OPERATORS, THEIR DIFFERENTIAL-GEOMETRIC STRUCTURE AND APPLICATIONS. PART 1

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This paper is dedicated to the memory of the 85-th birthday and the 10-th death anniversaries of the mathematics and physics giant of the previous century academician ***Nikolay Nikolayevich Bogoliubov***

Abstract. A differential-geometric structure of Delsarte transmuation operators in multi-dimension is described, application to the inverse spectral transform problem is discussed.

1. Introduction

Consider the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^m; Hom(\mathbb{C}^k; \mathbb{C}^N)), k, m, N \in \mathbb{Z}_+$, with the natural bilinear (not scalar here) form on $\mathcal{H}^* \times \mathcal{H}$

$$(< \varphi, \psi >) := \int_{\mathbb{R}^m} Sp(\varphi(x)^\intercal \psi(x) dx \tag{1.1}$$

for any pair $(\varphi, \psi) \in \mathcal{H}^* \times \mathcal{H}$, where, evidently $\mathcal{H}^* \simeq \mathcal{H}$, sign "$\intercal$"is the matrix transposition and "Sp" denotes the usual matrix trace. Take now $\mathcal{H}_0$ and $\tilde{\mathcal{H}}_0$ as two closed subspaces of $\mathcal{H}$ and two linear operators L and $\tilde{L}$ from $\mathcal{H}$ into $\mathcal{H}$.

Definition 1.1. (J. Delsarte and J. Lions [2]) A linear invertible operator $\hat{\Omega}$ defined on the whole $\mathcal{H}$ and acting from $\mathcal{H}_0$ onto $\tilde{\mathcal{H}}_0$ is called a Delsarte transmutation operator for the pair of operators L and $\tilde{L}$, if the following two conditions hold:

- the operator $\hat{\Omega}$ and its invertible $\hat{\Omega}^{-1}$ are continuos in $\mathcal{H}$;
- the operator identity

$$L\hat{\Omega} = \hat{\Omega}\tilde{L} \tag{1.2}$$

is satisfied.

Such transmutation operators were for the first time introduced in [1, 2] for the case of one-dimensional second order differential operators. In particular, for the Sturm-Liouville and Dirac operators the complete structure of the corresponding Delsarte transmutation operators was described in [3, 4], where also the extensive applications to spectral theory were given.

As it has just become clear recently, some special cases of the Delsarte transmutation operators were constructed a lot before by Darboux and Crum (see [5]). A special generalization of the Delsarte-operators for the two-dimensional Dirac operators was done for

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the first time in [6], where its applications to inverse spectral theory and solving some nonlinear two-dimensional evolution equations were also presented.

Recently some progress in this direction has been made in [7, 8] due to analyzing a special operator structure of Darboux type transformations which appeared in [9].

In this paper we give, in some sense, a complete description of multi-dimensional Delsarte transmutation operators based on natural generalization of the differential-geometric approach devised in [8], and discuss how one can apply these operators to studying spectral properties of linear multi-dimensional differential operators.

2. A generalized Lagrangian identity and its differential-geometric structure.

Let a multi-dimensional linear differential operator $L:\mathcal{H}\to\mathcal{H}$ of order $n(L)\in\mathbb{Z}_+$ be of the form

$$L:=\sum_{|\alpha|=0}^{n(L)}a_\alpha(x)\partial^{|\alpha|}/\partial x^\alpha, \tag{2.1}$$

where, as usual, $\alpha\in\mathbb{Z}_+^m$ is a multi-index, $x\in\mathbb{R}^m$, and for brevity one assumes that coefficients $a_\alpha\in\mathcal{S}(\mathbb{R}^m;\mathbb{C}^N)$. Consider the following easily derivable generalized Lagrangian identity for the operator (2.1) :

$$< L^*\varphi,\psi > - < \varphi, L\psi >= Sp(\sum_{i=1}^m(-1)^{i+1}\frac{\partial}{\partial x_i}Z_i[\varphi,\psi]), \tag{2.2}$$

where $(\varphi,\psi)\in\mathcal{H}^*\times\mathcal{H}$, mappings $Z_i:\mathcal{H}^*\times\mathcal{H}\to Hom\mathbb{C}^k$, $i=\overline{1,m}$, are bilinear due to the construction and $L^*:\mathcal{H}^*\to\mathcal{H}^*$ is the corresponding formally conjugated to (2.1) differential expression, that is

$$L^*:=\sum_{|\alpha|=0}^{n(L)}(-1)^{|\alpha|}\frac{\partial^{|\alpha|}}{\partial x^\alpha}\cdot a_\alpha(x). \tag{2.3}$$

Having multiplied the identity (2.2) by the usual oriented Lebesgue measure $dx=\wedge_{\overrightarrow{j=\overline{1,m}}}dx_j$, we get from that

$$< L^*\varphi,\psi > dx- < \varphi, L\psi > dx = Sp d(Z^{(m-1)}[\varphi,\psi])$$

for all $(\varphi,\psi)\in\mathcal{H}^*\times\mathcal{H}$, where

$$Z^{(m-1)}[\varphi,\psi]:=\sum_{i=1}^m dx_1\wedge dx_2\wedge...\wedge dx_{i-1}\wedge Z_i[\varphi,\psi]dx_{i+1}\wedge...\wedge dx_m \tag{2.4}$$

is an $(m-1)-$differential matrix form on $\mathbb{R}^m$.

Consider now all such pairs $(\varphi,\psi)\in\mathcal{H}^*\times\mathcal{H}$ that the matrix differential form (2.4) is exact, that is there exists such an $(m-2)-$differential matrix form $\Omega^{(m-2)}[\varphi,\psi]$ on $\mathbb{R}^m$ satisfying the condition

$$Z^{(m-1)}[\varphi,\psi]=d\Omega^{(m-2)}[\varphi,\psi]. \tag{2.5}$$

Assume also that for any fixed element $\varphi\in\mathcal{H}^*$ the set $\mathcal{H}_\varphi\subset\mathcal{H}$ of functions $\psi\in\mathcal{H}$ satisfying the condition (2.5) is dense in $\mathcal{H}$, that is $\overline{\mathcal{H}_\varphi}=\mathcal{H}$. Since the relationship (2.5) is bilinear in $(\varphi,\psi)\in\mathcal{H}^*\times\mathcal{H}$, one gets easily that (2.5) holds for any pair $(\varphi,\psi)\in\mathcal{H}^*\times\mathcal{H}$. Thus, taking into account that $d^2=0$, it follows from (2.3) by integration over $\mathbb{R}^m$ that for any pair $(\varphi,\psi)\in\mathcal{H}^*\times\mathcal{H}$ the identity $(< L^*\varphi,\psi >)-(< \varphi,L\psi >)$ holds, that is the operator (2.1) possesses its adjoint L^* in $\mathcal{H}^*$.

Let now $S(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})$ denote an $(m-1)-$dimensional piece-wise smooth hypersurface in $\mathbb{R}^m$ such that its boundary $\partial S(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) = \sigma_x^{(m-2)} - \sigma_{x_0}^{(m-2)}$, where $\sigma_x^{(m-2)}$ and $\sigma_{x_0}^{(m-2)}$ are some $(m-1)-$dimensional homological cycles in $\mathbb{R}^m$, parametrized formally by means of two points $x, x_0 \in \mathbb{R}^m$. Then from (2.5) based on the general Stokes theorem one gets easily that

$$\int_{S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi,\psi] = \int_{\partial S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \Omega^{(m-2)}[\varphi,\psi] =$$

$$\int_{\sigma_x^{(m-2)}} \Omega^{(m-2)}[\varphi,\psi] - \int_{\sigma_{x_0}^{(m-2)}} \Omega^{(m-2)}[\varphi,\psi] := \Omega_x[\varphi,\psi] - \Omega_{x_0}[\varphi,\psi] \tag{2.6}$$

for all $(\varphi,\psi) \in \mathcal{H}^* \times \mathcal{H}$, where matrix functionals $\Omega_x[\varphi,\psi]$ and $\Omega_{x_0}[\varphi,\psi]$ are assumed further to be nondegenerate and satisfying the regularity condition $\lim_{x\to x_0} \Omega_x[\varphi,\psi] = \Omega_{x_0}[\varphi,\psi]$, define now actions of the following two linear Delsarte permutations operators $\hat{\Omega} : \mathcal{H} \to \mathcal{H}$ and $\hat{\Omega}^* : \mathcal{H}^* \to \mathcal{H}^*$ still upon a fixed pair of functions $(\varphi,\psi) \in \mathcal{H}^* \times \mathcal{H}$:

$$\tilde{\psi} = \hat{\Omega}(\psi) := \psi(\Omega_x[\varphi,\psi])^{-1}\Omega_{x_0}[\varphi,\psi],$$

$$\tilde{\varphi} = \hat{\Omega}(\varphi) := \varphi(\Omega_x^{\intercal}[\varphi,\psi])^{-1}\Omega_{x_0}^{\intercal}[\varphi,\psi]. \tag{2.7}$$

Making use of the expressions (2.7), based on arbitrariness of the chosen pair of functions $(\varphi,\psi) \in \mathcal{H}^* \times \mathcal{H}$, we can easily retrieve the corresponding operator expressions for $\hat{\Omega}$ and $\hat{\Omega}_*$ forcing the constant matrix $\Omega_{x_0}[\varphi,\psi]$ to variate:

$$\hat{\Omega}(\psi) = \psi(\Omega_x[\varphi,\psi])^{-1}(\Omega_x[\varphi,\psi] - \int_{S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi,\psi] =$$

$$\psi - \psi(\Omega_x[\varphi,\psi])^{-1}\Omega_{x_0}[\varphi,\psi](\Omega_{x_0}[\varphi,\psi])^{-1} \int_{S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi,\psi] =$$

$$\psi - \tilde{\psi}(\Omega_{x_0}[\varphi,\psi])^{-1} \int_{S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi,\psi] =$$

$$(\mathbf{1} - \tilde{\psi}(\Omega_{x_0}[\varphi,\psi])^{-1} \int_{S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi,\cdot])\,\psi := \hat{\Omega}\cdot\psi;$$

$$\hat{\Omega}_*(\varphi) = \varphi(\Omega_x^{\intercal}[\varphi,\psi])^{-1}(\Omega_x^{\intercal}[\varphi,\psi] - \int_{S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1),\intercal}[\varphi,\psi] =$$

$$\varphi - \varphi(\Omega_x^{\intercal}[\varphi,\psi])^{-1}\Omega_{x_0}^{\intercal}[\varphi,\psi](\Omega_{x_0}^{\intercal}[\varphi,\psi])^{-1} \int_{S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1),\intercal}[\varphi,\psi] =$$

$$(\mathbf{1} - \tilde{\varphi}(\Omega_{x_0}^{\intercal}[\varphi,\psi])^{-1} \int_{S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1),\intercal}[\cdot,\psi])\,\varphi := \hat{\Omega}_*\cdot\varphi, \tag{2.8}$$

where, by definition,

$$\hat{\Omega} := \mathbf{1} - \tilde{\psi}(\Omega_{x_0}[\varphi,\psi])^{-1} \int_{S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi,\cdot],$$

$$\hat{\Omega}_* := \mathbf{1} - \tilde{\varphi}(\Omega_{x_0}^{\intercal}[\varphi,\psi])^{-1} \int_{S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1),\intercal}[\cdot,\psi] \tag{2.9}$$

are of Volterra type multidimensional integral operators. It is to be noted here that now elements $(\varphi,\psi) \in \mathcal{H}^* \times \mathcal{H}$ and $(\tilde{\varphi},\tilde{\psi}) \in \mathcal{H}^* \times \mathcal{H}$ inside the operator expressions (2.9) are arbitrary but fixed. Therefore, the operators (2.9) realize extension of their actions (2.7) on fixed pair of functions $(\varphi,\psi)\in\mathcal{H}^*\times\mathcal{H}$ upon the whole functional space $\mathcal{H}^*\times\mathcal{H}$.

Due to the symmetry of expressions (2.7) and (2.9) with respect to two pairs of functions $(\varphi,\psi)\in\mathcal{H}^*\times\mathcal{H}$ and $(\tilde{\varphi},\tilde{\psi})\in\mathcal{H}^*\times\mathcal{H}$,it is very easy to state the following lemma.

Lemma 2.1. *Operators (2.9) are invertible of Volterra type expressions on $\mathcal{H}^*\times\mathcal{H}$ whose inverse are given as follows:*

$$\hat{\Omega}^{-1}:=\mathbf{1}-\psi(\tilde{\Omega}_{x_0}[\tilde{\varphi},\tilde{\psi}])^{-1}\int_{S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}\tilde{Z}^{(m-1)}[\tilde{\varphi},\cdot]),$$

$$\hat{\Omega}_*^{-1}:=\mathbf{1}-\varphi(\Omega_{x_0}^{\intercal}[\tilde{\varphi},\tilde{\psi}])^{-1}\int_{S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}\tilde{Z}^{(m-1),\intercal}[\cdot,\tilde{\psi}], \tag{2.10}$$

where pairs of functions $(\varphi,\psi)\in\mathcal{H}^*\times\mathcal{H}$ and $(\tilde{\varphi},\tilde{\psi})\in\mathcal{H}^*\times\mathcal{H}$ are taken as arbitrary but fixed.

For the expressions (2.10) to be compatible with mappings (2.7) the following actions must hold:

$$\psi=\hat{\Omega}^{-1}\cdot\psi=\tilde{\psi}(\tilde{\Omega}_x[\tilde{\varphi},\tilde{\psi}])^{-1}\tilde{\Omega}_{x_0}[\tilde{\varphi},\tilde{\psi}]),$$

$$\varphi=\hat{\Omega}_*^{-1}\cdot\tilde{\varphi}=\varphi(\tilde{\Omega}_x^{\intercal}[\tilde{\varphi},\tilde{\psi}])^{-1}\tilde{\Omega}_{x_0}^{\intercal}[\tilde{\varphi},\tilde{\psi}]), \tag{2.11}$$

where for any two pairs of functions $(\varphi,\psi)\in\mathcal{H}^*\times\mathcal{H}$ and $(\tilde{\varphi},\tilde{\psi})\in\mathcal{H}^*\times\mathcal{H}$ the following relationships are satisfied:

$$<\tilde{L}^*\tilde{\varphi},\tilde{\psi}>dx-<\tilde{\varphi},\tilde{L}\tilde{\psi}>dx=S\,pd(\tilde{Z}^{(m-1)}[\tilde{\varphi},\tilde{\psi}]),$$

$$\tilde{L}:=\hat{\Omega}L\hat{\Omega}^{-1},\ \ \tilde{L}^*:=\hat{\Omega}_*L^*\hat{\Omega}_*^{-1},\ \ \tilde{Z}^{(m-1)}[\tilde{\varphi},\tilde{\psi}]=d\tilde{\Omega}^{(m-2)}[\tilde{\varphi},\tilde{\psi}]. \tag{2.12}$$

Moreover, the expressions $\tilde{L}:\mathcal{H}\to\mathcal{H}$ and $\tilde{L}^*:\mathcal{H}^*\to\mathcal{H}^*$ must be differential too. Since this condition determines properly Delsarte transmutation operators (2.10), we need to state the following theorem.

Theorem 2.2. *The operator expressions $\tilde{L}:=\hat{\Omega}L\hat{\Omega}$ and $\tilde{L}^*:=\hat{\Omega}_*L^*\hat{\Omega}_*^{-1}$ are purely differential on $\mathcal{H}^*\times\mathcal{H}$ for any suitably chosen hyper-surfaces $S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})$.*

Proof. For proving the theorem it is necessary to show that the formal pseudo-differential expressions corresponding to operators $\tilde{L}$ and $\tilde{L}^*$ contain no integral elements. Making use of an idea devised in [8, 6], one can formulate the following lemma. □

Lemma 2.3. *A pseudo-differential operator $L:\mathcal{H}\to\mathcal{H}$ is purely differential iff the following equality*

$$(<,(L\frac{\partial^{|\alpha|}}{\partial x^\alpha})_+f>)=(<,L_+\frac{\partial^{|\alpha|}}{\partial x^\alpha}f>) \tag{2.13}$$

holds for any $|\alpha|\in\mathbb{Z}_+$ and all $(h,f)\in\mathcal{H}^*\times\mathcal{H}$, that is the condition (2.13) is equivalent to the equality $L_+=L$, where, as usual, the sign "$(...)_+$" means the purely differential part of the corresponding expression inside the bracket.

Proof. Based now on this Lemma and exact expressions of operators (2.9) , similarly to calculations done in [8], one shows right away that operators $\tilde{L}$ and $\tilde{L}^*$, depending only on a pair of homological cycles $\sigma_x^{(m-2)}$ and $\sigma_{x_0}^{(m-2)}$ marked by points $x,x_0\in R^m$, are purely differential, thereby finishing the proof.► □

3. The general structure of Delsarte transmutation operators

Consider the expression (2.9) in case when the dimension $k = \dim \mathbb{C}^k$ tends to infinity. Then we shall consider the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^m; (L_2^{(\mu)}(\mathbb{R}^p;\mathbb{C}))^N)$, $p \in \mathbb{Z}_+$, and the corresponding bilinear form on $\mathcal{H}^* \times \mathcal{H}$ given as

$$(< \varphi, \psi >) := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^p} d\mu(\lambda)\varphi^\intercal(x;\lambda)\psi(x;\lambda), \tag{3.1}$$

where μ is some Lebesgue measure on the space of values of a vector parameter $\lambda \in \mathbb{R}^p$. Subject to the bilinear form (3.1) one can derive an analog of the Lagrangian identity (2.2):

$$< L^*\varphi, \psi > - < \varphi, L\psi >= Tr(\sum_{i=1}^m (-1)^{i+1}\frac{\partial}{\partial x_i} Z_i[\varphi,\psi](\lambda|\xi)), \tag{3.2}$$

where for any pair $(\varphi,\psi) \in \mathcal{H}^* \times \mathcal{H}$ kernels $Z_i[\varphi,\psi](\lambda|\xi)$, $i = \overline{1,m}$, being bilinear functionals on $\mathcal{H}^* \times \mathcal{H}$, are assumed to be of the trace class scalar operators from $L_2^{(\mu)}(\mathbb{R}^p;\mathbb{C})$ into $L_2^{(\mu)}(\mathbb{R}^p;\mathbb{C})$ with the standard Tr-operation given as follows: $Tr(A(\lambda|\xi) := \int_{\mathbb{R}^p} A(\lambda|\lambda)d\mu(\lambda)$; the variable $x \in \mathbb{R}^n$ is assumed here as a parameter. Correspondingly, there exists a differential $(m-1)$–form $Z^{(m-1)}[\varphi,\psi](\lambda|\xi)$ like (2.4), one finds that under suitable conditions similar to those formulated before, there exists a differential $(m-2)$–form $\Omega^{(m-2)}[\varphi,\psi](\lambda|\xi)$ for all $\lambda, \xi \in \mathbb{R}^p$, so that

$$Z^{(m-1)}[\varphi,\psi](\lambda|\xi) = d\Omega^{(m-2)}[\varphi,\psi](\lambda|\xi). \tag{3.3}$$

Making use of (3.3), we can derive now the fundamental expression generating Delsarte transmutation operators:

$$\int_{S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi,\psi](\lambda|\xi) = \int_{\partial S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \Omega^{(m-2)}[\varphi,\psi](\lambda|\xi) =$$

$$\int_{\sigma_x^{(m-2)}} \Omega^{(m-2)}[\varphi,\psi](\lambda|\xi) - \int_{\sigma_{x_0}^{(m-2)}} \Omega^{(m-2)}[\varphi,\psi](\lambda|\xi) := \Omega_x[\varphi,\psi](\lambda|\xi) - \Omega_{x_0}[\varphi,\psi](\lambda|\xi) \tag{3.4}$$

for any pair $(\lambda,\xi) \in \mathbb{R}^p \times \mathbb{R}^p$. The bilinear functionals $\Omega_x[\varphi,\psi](\lambda|\xi)$ and $\Omega_{x_0}[\varphi,\psi](\lambda|\xi)$, parametrized by points x and $x_0 \in \mathbb{R}^m$, should be also considered as kernels of some scalar integral operators $\Omega_x[\varphi,\psi], \Omega_{x_0}[\varphi,\psi] : L_2^{(\mu)}(\mathbb{R}^p;\mathbb{C}) \to L_2^{(\mu)}(\mathbb{R}^p;\mathbb{C})$. For instance, by definition,

$$\Omega_x[\varphi,\psi]f(\lambda) := \int_{\mathbb{R}^p} d\mu(\xi)\Omega_x[\varphi,\psi](\lambda|\xi)f(\xi) \tag{3.5}$$

for any $f \in L_2^{(\mu)}(\mathbb{R}^p;\mathbb{C})$. Thereby, one can define the corresponding generalized Delsarte transformation operators actions as follows:

$$\tilde{\psi}(x;\lambda) = (\hat{\Omega}\psi)(x;\lambda) := \int_{\mathbb{R}^p\times\mathbb{R}^p} d\mu(\xi)d\mu(\eta)\Omega_{x_0}[\varphi,\psi](\lambda|\eta)\Omega_x^{-1}[\varphi,\psi](\eta|\xi)\psi(x;\xi),$$

$$\tilde{\varphi}(x;\lambda) = (\hat{\Omega}_*\varphi)(x;\lambda) := \int_{\mathbb{R}^p\times\mathbb{R}^p} d\mu(\xi)d\mu(\eta)\Omega_{x_0}^\intercal[\varphi,\psi](\lambda|\eta)\Omega_x^{-1,\intercal}[\varphi,\psi](\eta|\xi)\psi(x;\xi), \tag{3.6}$$

where $(x;\lambda) \in \mathbb{R}^m \times \mathbb{R}^p$ and integral scalar operators $\Omega_x[\varphi,\psi], \Omega_{x_0}[\varphi,\psi] : L_2^{(\mu)}(\mathbb{R}^p;\mathbb{C}) \to L_2^{(\mu)}(\mathbb{R}^p;\mathbb{C})$ are assumed to be invertible. Based on (3.6) and on the method of varying constant invertible integral operator $\Omega_{x_0}[\varphi,\psi] : L_2^{(\mu)}(\mathbb{R}^p;\mathbb{C}) \to L_2^{(\mu)}(\mathbb{R}^p;\mathbb{C})$, one can easily finds that expressions

$$\begin{aligned}\hat{\Omega}(x;\lambda) &= \mathbf{1} - \int_{\mathbb{R}^p\times\mathbb{R}^p} d\mu(\xi)d\mu(\eta)\tilde{\varphi}(x;\eta)\Omega_{x_0}^{-1}[\varphi,\psi](\xi|\eta)\times \\ &\int_{S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi,\cdot](\lambda|\xi)\Omega_{x_0}^{-1}[\varphi,\psi](\xi|\eta), \\ \hat{\Omega}_*(x;\lambda) &= \mathbf{1} - \int_{\mathbb{R}^p\times\mathbb{R}^p} d\mu(\xi)d\mu(\eta)\tilde{\psi}(x;\eta)\Omega_{x_0}^{-1,\intercal}[\varphi,\psi](\xi|\eta)\times \\ &\int_{S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1),\intercal}[\cdot,\psi](\lambda|\xi)\Omega_{x_0}^{-1,\intercal}[\varphi,\psi](\xi|\eta)\end{aligned} \tag{3.7}$$

for all parameters $(x;\lambda)\in\mathbb{R}^m\times\mathbb{R}^p$ and fixed pairs $(\varphi,\psi)\in\mathcal{H}^*\times\mathcal{H}$ are purely invertible integral operators of Volterra type on $\mathcal{H}^*\times\mathcal{H}$. Applying the same arguments as in section1, one can also show that correspondingly transformed operators $\tilde{L} := \hat{\Omega}L\hat{\Omega}^{-1}$, $\tilde{L}^* := \hat{\Omega}_*L^*\hat{\Omega}_*^{-1}$ appear to be purely differential too. Thereby, one can formulate the following theorem.

Theorem 3.1. *The generalized operator expressions (3.7) are invertible integral Delsarte transmutation operators of Volterra type onto $\mathcal{H}^*\times\mathcal{H}$, transforming correspondingly given operators L and L^* into the pure differential ones $\tilde{L} := \hat{\Omega}L\hat{\Omega}^{-1}$ and $\tilde{L}^* := \hat{\Omega}_*L^*\hat{\Omega}_*^{-1}$. Moreover, the suitable subspaces $\mathcal{H}_0\subset\mathcal{H}$ and $\tilde{\mathcal{H}}_0\subset\tilde{\mathcal{H}}$, so that $\hat{\Omega}\mathcal{H}_0 \rightleftarrows \tilde{\mathcal{H}}_0$, depend strongly on the topological structure of basic homological cycles $\sigma_x^{(m-2)}$and $\sigma_{x_0}^{(m-2)}$ parametrized by points $x, x_0\in\mathbb{R}^m$, generating a hypersurface $S(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})$ via spanning them.*

4. Discussion.

Consider a differential operator $L:\mathcal{H}\to\mathcal{H}$ in the form (2.1) and assume that its spectrum $\sigma(L)$ consists of the discrete $\sigma_d(L)$ and continuous $\sigma_c(L)$ parts. By means of general the general form of the Delsarte transmutation operators (3.7) one can construct a new differential operator $\tilde{L} := \hat{\Omega}L\hat{\Omega}^{-1}$ in $\mathcal{H}$, so that its continuos spectrum $\sigma_c(\tilde{L}) = \sigma_c(L)$ but $\sigma_d(L)\neq\sigma_d(\tilde{L})$. Thereby these Delsarte transformed operators can be used for constructing a wide class of differential operators with a fixed spectrum.

As it was shown before in [6] for the two-dimensional Dirac operator, the kernel of the corresponding Delsarte transmutation operator satisfies necessarily some special linear of Fredholm type integral equations called the Gelfand-Levitan-Marchenko ones, which are very important for solving the corresponding inverse spectral problem, having a lot of applications in modern mathematical physics.

One believes that such equations can be constructed for our multidimensional case too, thereby, making it possible to pose the corresponding inverse spectral problem for describing a wide class of multidimensional operators with given spectral characteristics. In, particular, similar to [6, 10], one can use such results for studying so called completely integrable nonlinear evolution equations, especially for constructing their exact solutions like solitons and many others. This activity is in progress now and the corresponding results will be published later.

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