On the Transient, Busy and Idle Period Analysis of Statistical Multiplexers with N Input Links and Train Arrivals

Faouzi Kamoun
College of Information Technology
Dubai University College
P.O. Box: 14143, Dubai
United Arab Emirates

Abstract: - In this paper, we present an exact transient (discrete-time) queuing analysis of a statistical multiplexer with a finite number of input links and whose arrival process is correlated and consists of a train of a fixed number of fixed-length packets. The functional equation describing this queuing model is manipulated and transformed into a mathematical tractable form. This allows us to derive the transient probability generating function (pgf) of the buffer occupancy. From this transient pgf, time-dependent performance measures such as transient probability of empty buffer, transient mean of buffer occupancy and instantaneous packet overflow probabilities can be derived. The transient analysis provides useful insights into the derivation of the busy period distribution function, which will be illustrated for the simpler GI/D/1 case. We also present closed-form expressions for the idle period distribution of the queuing model under consideration. The paper presents significant new results on the transient and busy-period analysis of statistical multiplexers with N input links and train arrivals.

Key-Words: - ATM multiplexers, Transient analysis, Train arrivals, busy-period, idle-period, Discrete-time queues.

1 Introduction

In this paper, we consider a statistical multiplexer with N input links, having the same transmission rate, and one output link. The arrival process to this multiplexer is correlated and consists of a fixed-length packet-train arrival process. The main thrust behind our interest in investigating the impact of the above train arrival process on the performance of switching elements stems from the fact that such train arrival models are often encountered in the performance evaluation of large-scale ATM switching networks. For example, in some ATM environments, large external data frames (e.g voice or IP frames) are segmented at the edge of an ATM network into fixed-length ATM cells (mini-cells). Discrete-time queuing models with correlated train arrivals are also encountered in various other applications whereby customers are messages (eg. Frames or jumbo packets) composed of multiple fixed-length packets, see eg. [1-2].

In this paper, we model an ATM multiplexer as a discrete-time queue, whose arrival process consists of mini-cell arrivals (thereafter referred to train arrivals). A functional equation describing this system has been derived in [1].We manipulated and transformed the functional equation describing this queuing model into a mathematical tractable form. This allows us to extract the transient pgf of the queue length, from which transient performance measures such as probabilities of an empty buffer, transient mean of buffer occupancy and instantaneous packet overflow probabilities can be derived. The proposed transform approach is an extension of an earlier approach [3] in the analysis of ATM multiplexers with correlated arrivals. Further, using the GI/D/1 queuing model as a reference, we show how our transient results allow us to characterize the distribution of the busy period of the queuing model under consideration. We also characterize the idle period distribution of our model in terms of its system's parameters and illustrate our solution techniques through some numerical examples.

2 Queuing Model and Functional Equation

In this paper, we consider a discrete-time queuing system (figure 1) with infinite buffer capacity, N input links, one output link and a single (FCFS) deterministic server. The time axis is divided into equal length slots and packet transmission is synchronized to occur at the slot boundaries. Here a slot is the time period required to transmit exactly one packet from the buffer, and a message enters the buffer as a train at a fixed rate of one packet per slot. We further assume that each message is composed of a fixed number of *m* packets. In addition, traffic on different input links is assumed to be independent and with the same statistical characteristics.

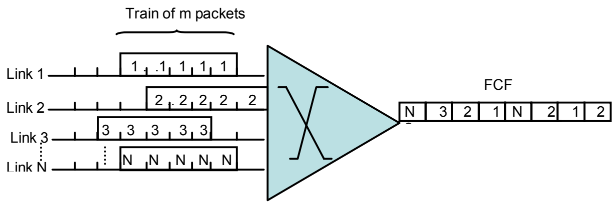

Fig.1. Statistical multiplexer with N input links and *m* packets/train

On any input link, the probability that the first (leading) packet of a message enters the buffer in any given slot is q if the first packet of the previous message on this link did not enter the buffer during the previous $(m-1)$ slots and it is zero, otherwise

Further, let $\{c_j ; j \geq 1\}$ be a series of independent and identical Bernoulli random variables with pgf

$$C(z) = 1-q + q.z$$

The queuing model under consideration can be formulated as a discrete-time m-dimensional Markov chain. The state of the system is defined by the state vector $(l_k, a_{1,k}, a_{2,k}, .. a_{m-1,k})$ where l_k is the queue length at the end of slot k and $a_{n,k}$ $(0<n<m)$ is the number of input links having sent the n^{th} packet of a message to the buffer in slot k.

Next let:

$$Q_k(z, x_1, x_2, x_3, ... x_{m-1}) = E\left(z^{l_k} . x_1^{a_{1,k}} . x_2^{a_{2,k}} . x_{m-1}^{a_{m-1,k}}\right)$$

denote the joint pgf of the system state vector.

This type of statistical multiplexers with correlated train arrivals was modeled in [1] and a functional equation describing the pgf of the system state vector was derived and is given by the following expression [1]:

$$Q_{k+1}(z, x_1, x_2, x_3, ... x_{m-1}) = [C(x_1 z)]^N \left\{ \frac{Q_k\left(z, \frac{x_2 z}{C(x_1 z)}, \frac{x_3 z}{C(x_1 z)}, ... \frac{x_{m-1} z}{C(x_1 z)}, \frac{z}{C(x_1 z)}\right) - p_k(0)}{z} + p_k(0) \right\} \quad (1)$$

Where $p_k(0)$ = Prob $(l_k=0)$ is the probability of an empty buffer at the end of the k^{th} slot

In [1], a technique for deriving an explicit expression for the steady-state mean buffer occupancy is provided and an approximate method to get a tight upper bound for the tail distribution of the buffer occupancy is presented. In the sequel, an exact transient analysis of this queuing model will be presented, along with new results related to its busy and idle period distributions.

3 Idle Period Analysis

We first start with the idle period analysis of the queuing model under consideration, since it is easier to tackle. Recall [4] that an idle period starts at the departure instant of the last packet from the buffer (which leaves the system empty) and ends at the end of the first subsequent slot during which at least one arrival occurs. Let the random variable I denote the length of an arbitrary idle period, expressed in number of slots, and let $I(z)$ be the corresponding pgf.

For the idle period to last for k consecutive slots, there must be no arrivals during the first $(k$-1) slots and at least one arrival must occur in the k^{th} slot. Further, recall from the queuing model's description in the previous section that when the multiplexer is empty, all the links must be in an 'idle' ('off') state and must remain so for the first $(k$-1) slots. At the last slot (k), at least one link must switch to an 'active' ('on') state. Because of the independence assumption among all the links, it follows that:

$$\Pr[I = k\ slots] = \left[(1-q)^N\right]^{k-1} . \left[1-(1-q)^N\right]$$

with probability generating function:

$$I(z) = \frac{z.\left[1-(1-q)^N\right]}{1-z(1-q)^N}$$

The above shows that the idle periods of the queuing model under consideration are geometrically distributed with parameter $(1-q)^N$, mean:

$$\bar{I} = \frac{1}{1-(1-q)^N}$$

and variance:

$$\sigma_I^2 = \frac{(1-q)^N}{(1-(1-q)^N)^2}$$

4 Transient Probabilities ($p_k(0)$'s) of an Empty Buffer

In this section, we derive expressions for the transient probabilities ($p_k(0)$'s) of an empty system for the queuing model under consideration. First, we illustrate the solution technique by considering the simpler GI/D/1 queuing model and then highlight the strong resemblance in the general expressions of $p_k(0)$ between the two discrete models. Note that the analysis in this section will provide valuable insights into the busy period analysis, discussed in section 5.

4.1 The GI/D/1 Queue Case

Consider a GI/D/1 queuing system, and denote by

$V(z)$ the pgf of the number of packet arrivals in steady-state. Also let $P_k(z)$ denote the pgf of the buffer occupancy at the end of the k^{th} slot. Without any loss of generality, assume that the system is initially empty ($P_0(z)) = 1$). The imbedded Markov Chain analysis of the GI/D/1 queue yields the following well known equation relating the pgf of the queue length between two consecutive slots:

$$P_k(z) = V(z)\left[\frac{P_{k-1}(z) - p_{k-1}(0)}{z} + p_{k-1}(0)\right] \qquad (2)$$

In the sequel, we show how to extract the transient probabilities of an empty buffer, $p_k(0)$'s from (2). First, we zero initial conditions, we can re-write (2) as follows:

$$P_k(z) = \left[\frac{V(z)}{z}\right]^k + (z-1)\sum_{j=1}^{k}\left[\frac{V(z)}{z}\right]^j p_{k-j}(0) \qquad (3)$$

where the only unknowns are the transient probabilities $p_k(0)$'s. To evaluate these, we proceed as follows: First let us define the following transforms ($|w| \le 1$):

$$P(z,w) = \sum_{k=0}^{\infty} P_k(z) w^k \;\; ; \;\; P(w) = \sum_{k=0}^{\infty} p_k(0) w^k \qquad (4)$$

Now substituting $P_k(z)$ from (3) into $P(z,w)$, as defined in (4):

$$P(z,w) = \frac{z + w(z-1)P(w)V(z)}{z - wV(z)}$$

Using Rouchés theorem, and taking into the account the analytical property of P(z,w) inside the poly-disk ($(|z| \le 1; |w| < 1)$, we get:

$$P(w) = \frac{1}{1 - z^*} \qquad (5)$$

where z^* is the unique root inside the unit circle of the equation

$$z - wV(z) = 0 \qquad (6)$$

Applying Lagrange's theorem [Appendix] to equation (5) allows us to derive the following expression for the transient probabilities of an empty buffer for the GI/D/1 queue:

$$p_k(0) = \frac{1}{k!}\frac{d^{k-1}}{dz^{k-1}}\left[\frac{V(z)^k}{(1-z)^2}\right]\Bigg|_{z=0} \qquad (7)$$

Using the Leibniz's rule for the k^{th} derivative of a product, allows us to re-write (7) as follows:

$$p_k(0) = \frac{1}{k}\sum_{i=0}^{k-1}\frac{(k-i)}{i!}\frac{d^i\left[V(z)^k\right]}{dz^i}\Bigg|_{z=0} \quad (\forall k \ge 1) \qquad (8)$$

For instance, applying (8) to the M/D/1 case, with $V(z) = e^{-\rho(1-z)}$ gives:

$$p_k(0) = (1-\rho) + \frac{e^{-\rho k}.\rho.(\rho k)^k H_g(2, k+2, \rho k)}{(k+1)!}$$

where H_g is the Barnes's extended Hyper-geometric function, which for integers n and d is defined by:

$$H_g(n,d,z) = \sum_{i=0}^{\infty}\frac{1}{i!}\frac{(n+i-1)!}{(n-1)!}.\frac{(d-1)!}{(d+i-1)!}z^i$$

4.2 The Case of Our Queuing Model

The aim of this section is to re-write the marginal pgf of the buffer content of the queuing model under consideration in a form similar to (2) and then use similar arguments to extract the transient probabilities of an empty buffer, $p_k(0)$'s.

4.2.1 Theorem :

Under zero initial conditions, where the buffer is initially empty with all links being 'idle', the functional equation (1) describing the queuing model under consideration can be written as follows:

$$Q_k(z, x_1, x_2, \ldots x_{m-1}) = \frac{[J(k)]^N}{z^k} + (z-1)\sum_{j=1}^{k}\frac{[J(j)]^N}{z^j} p_{k-j}(0) \qquad (9)$$

where the summation is taken to be empty for k=0 and the sequence $J(k)$ is defined by the m^{th}-order linear homogeneous "difference" equation:

$$J(k) = (1-q)J(k-1) + qz^m J(k-m) \quad (k \ge m) \qquad (10)$$

With the following m initial conditions:

$$J(0) = 1$$
$$J(k) = (1-q)J(k-1) + qx_k z^k \qquad (k < m)$$

The proof of the above proposition is readily obtained by induction and can be found in [5].

Next, let $P_k(z) = Q_k(z, x_1, x_2, \ldots x_{m-1})\big|_{x_1 = x_2 = \ldots x_{m-1} = 1}$ denotes the marginal pgf of the buffer occupancy at the end of the k^{th} slot, assuming zero initial conditions. From (9):

$$P_k(z) = \frac{\left[\tilde{J}(k)\right]^N}{z^k} + (z-1)\sum_{j=1}^{k}\frac{\left[\tilde{J}(j)\right]^N}{z^j} p_{k-j}(0) \quad (11)$$

where $\tilde{J}(k) = J(k)\big|_{x_1 = x_2 = \ldots x_{m-1} = 1}$.

It is interesting to note that the transient joint pgf of the queuing model under consideration, as expressed in (11) is now explicitly defined in terms the sequences $\tilde{J}(k)$ as well as the transient probabilities of an empty buffer $p_k(0)$.

4.2.2 Proposition:

The function $\tilde{J}(k) = J(k)\big|_{x_1 = x_2 = \ldots x_{m-1} = 1}$, appearing in (11) is given by the following formula:

$$\tilde{J}(k) = \sum_{i=1}^{m} C_i \lambda_i^{-k} \qquad (12)$$

where:

$$C_i = \frac{(1-q)(z-1)\lambda_i}{(1-\lambda_i z)[(m-1)(1-q)\lambda_i - m]} \quad (13)$$

and λ_i's ($i=1,2, ..m$) are the m distinct roots of the characteristic equation:

$$q(z\lambda)^m + (1-q)\lambda - 1 = 0 \quad (14)$$

The proof of the above proposition is readily obtained via standard transform techniques [5].

From (14), it is obvious that one of the roots has the property that $\lambda|_{z=1} = 1$. This particular root is thereafter denoted by λ_m

Next substituting $P_k(z)$ from (11) into $P(z,w)$, as defined in (4):

$$P(z,w) = \sum_{k=0}^{\infty} \frac{\left[\tilde{J}(k)\right]^N}{z^k} w^k + (z-1)\sum_{k=1}^{\infty}\sum_{j=1}^{k} \frac{\left[\tilde{J}(j)\right]^N}{z^j} p_{k-j}(0) w^k$$

Substituting for $\tilde{J}(k)$ from (12) into the above, substituting for the Multinomial expansion and simplifying the resulting expression, yields [5]:

$$P(z,w) = \sum_{n_1+n_2+...+n_m=N} \frac{N!}{n_1!.n_2!...n_m!} \cdot \frac{z.\prod_{i=1}^{m}(C_i)^{n_i}}{z - w\prod_{i=1}^{m}\lambda_i^{-n_i}}$$

$$+ (z-1)P(w).w \sum_{n_1+n_2+...+n_m=N} \frac{N!}{n_1!.n_2!...n_m!} \cdot \frac{\prod_{i=1}^{m}\left(\frac{C_i}{\lambda_i}\right)^{n_i}}{z - w\prod_{i=1}^{m}\lambda_i^{-n_i}} \quad (15)$$

Next, we determine the unknown boundary $P(w)$, by invoking the analytical property of $P(z,w)$ inside the polydisk $(|z| \le 1; |w| < 1)$. First, by defining $\beta_i = 1/\lambda_i$, we can re-write (15):

$$P(z,w) = \sum_{n_1+n_2+...+n_m=N} \frac{N!}{n_1!.n_2!...n_m!} \cdot \frac{z.\prod_{i=1}^{m}(C_i)^{n_i}}{z - w\prod_{i=1}^{m}\beta_i^{n_i}}$$

$$+ (z-1)P(w).w \sum_{n_1+n_2+...+n_m=N} \frac{N!}{n_1!.n_2!...n_m!} \cdot \frac{\prod_{i=1}^{m}(C_i\beta_i)^{n_i}}{z - w\prod_{i=1}^{m}\beta_i^{n_i}} \quad (16)$$

where from (14), the β_i's are the m distinct roots of the characteristic equation:

$$\beta^m - (1-q)\beta^{m-1} - qz^m = 0 \quad (17)$$

Furthermore, from (17), one of these roots has the property that $\beta|_{z=1} = 1$ This particular root is thereafter denoted by β_m.

Next, from Rouchés theorem, it can be shown that for a small $\varepsilon > 0$, the equation

$$z - w\prod_{i=1}^{m}\beta_i^{n_i} \quad (18)$$

has a unique root inside $|z| = 1 + \varepsilon$. Moreover, from the definitions of the β_i's, it is easy to show that since $\forall i < m : \beta_i|_{z=0} = 0$, then the unique root of (18) inside the unit disk is $z^* = 0$, which also appears in the numerator of (16) since $\forall i < m : C_i|_{z=0} = 0$. Therefore these roots do not give us the equation we need to solve for $P(w)$. For the remaining case $(i = m)$, we note that since $\beta_m|_{z=0} = 1 - q$ and $C_m|_{z=0} = 1$ then the corresponding term in (16) is given by:

$$P_0(z,w) = \frac{z.C_m^N + (z-1)P(w)wC_m^N\beta_m^N}{z - w\beta_m^N} \quad (19)$$

Now, let $H(z) = \beta_m^N$ and denote by z^* the unique root of the equation $z = w.H(z) = w.\beta_m^N$ inside $|z| \le 1$. Since $P(z,w)$ is bounded on $(|z| \le 1; |w| < 1)$, the numerator of (19) must also be zero at z*, which implies:

$$P(w) = \frac{1}{1 - z^*} \quad (20)$$

where z* is the unique root inside the unit circle of:

$$z - w.\beta_m^N = 0 \quad (21)$$

From here we note the similarity between the expression of $P(w)$ as given in (20-21) and the corresponding expression (5-6) for the GI/D/1 queue. Using the same approach, presented in section 4.1, we can express the transient probabilities of an empty buffer of the queuing model under consideration as follows:

$$p_k(0) = \frac{1}{k}\sum_{i=0}^{k-1}\frac{(k-i)}{i!}\frac{d^i\left[H(z)^k\right]}{dz^i}\Bigg|_{z=0} \quad (\forall k \ge 1) \quad (22)$$

For instance, for $m=2$, it is easy to show from (17) that:

$$\beta_1 = \left[\frac{1-q}{2} - \frac{\sqrt{(1-q)^2 + 4qz^2}}{2}\right]$$

$$\beta_2 = \left[\frac{1-q}{2} + \frac{\sqrt{(1-q)^2 + 4qz^2}}{2}\right]$$

and therefore:

$$H(z) = \beta_2^N = \left[\frac{1-q}{2} + \frac{\sqrt{(1-q)^2 + 4qz^2}}{2}\right]^N \quad (23)$$

In this case (m=2), a closed-form expression for the transient probabilities $p_k(0)$'s can be obtained from (22) by first expanding $H(z)^k$ using the binomial theorem and then using a proof by induction to obtain the i^{th} derivative of the corresponding expansion at z=0 [5]. Substituting the result back into (22), gives the following expression for the transient probabilities of an empty buffer:

$$p_k(0)=\left[\frac{1-q}{2}\right]^{N.k}\left\{1+\frac{1}{k\sqrt{\pi}}\sum_{i=0}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\sum_{j=1}^{N.k}\frac{\binom{N.k}{j}(k-2i)(-16q)^i\Gamma(i+\frac{1}{2})\Gamma\left(i-\frac{j}{2}\right)}{(2i)!(1-q)^{2i}\Gamma(-\frac{j}{2})}\right\} \quad (24)$$

where $\lfloor\ \rfloor$ denotes the *floor* function and $\Gamma(x)$ is the Gamma function defined by $\Gamma(x)=\int_0^{\infty} z^{x-1}e^{-z}dz$ which, for positive x, satisfies $\Gamma(\frac{1}{2})=\sqrt{\pi}$ and $\Gamma(x)=(x-1)\Gamma(x-1)$, among others [6].

In this case (m=2), equations (12) and (24), allow us to fully characterize the transient pgf $P_k(z)$ of the queue length. From this pgf, time-dependent performance measures such as transient mean $\overline{N}_k=\left.\frac{dP_k(z)}{dz}\right|_{z=1}$ and transient overflow probabilities of buffer occupancy can be derived.

5. Busy Period Analysis

The main aim of this section is derive an expression for the pgf of the busy period of the multiplexer under consideration in terms of its system parameters. Generally speaking, the analysis of the busy periods is far more complicated than that of the idle periods. Further, the lengths and the positions of the idle and busy periods on the time axis are not affected by the queuing discipline, as long as it is work conserving [4]. We first illustrate the solution technique by considering the simpler case of the GI/D1 queue and show how the same approach is applicable to the our original queuing model. This approach has been used in [7] to analyze the busy period of an ATM multiplexer whose arrival process consists of the superposition of the traffic generated by independent binary Markov sources.

5.1 The GI/D/1 Queue Case

Consider a GI/D/1 queuing system, and, again, denote by $V(z)$ be the pgf of packet arrivals in steady-state. Also let $P_k(z)$ denote the pgf of the buffer occupancy at the end of the k[th] slot. Without any loss of generality, assume that the system is initially empty ($P_0(z)$) = 1). The tree diagram, shown in figure 2 below will be used to explain the general approach, where for purpose of illustrations we have assumed a maximum of two packet arrivals/slot. Note that in figure 2, numbers above edges represent number of arrivals/slot and node labels represent buffer length at the end of the corresponding slot.

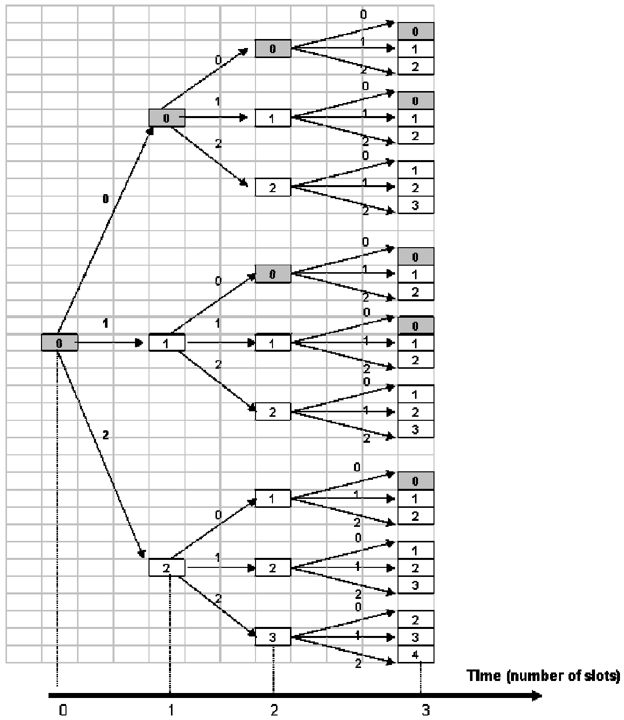

Fig.2 An illustrating example

From figure 2, we can see that the event of having an empty buffer at the end of the k^{th} slot (grayed nodes) can be expressed as the sum of k mutually exclusive events. For instance, in reference to figure 2, at the end of slot number 1, the probability of an empty buffer, $p_1(0)$, is equal to the probability that the buffer was initially empty and there were no arrivals. In other words:

For k=1, $p_1(0)=p_0(0).V(0)$

Similarly, from figure 2, the probability of having an empty buffer at the end of the second slot is equal to the probability that the buffer was empty at the end of the first slot and there were no arrivals plus the probability that the buffer was initially empty and there were one and then zero arrivals (these two events are mutually exclusive). In mathematical terms, this translates to:

$$p_2(0)=p_1(0).V(0)+p_0(0).\left.\frac{dV(z)}{dz}\right|_{z=0}.V(0)$$
$$=p_1(0).V(0)+.\frac{1}{2!}\left.\frac{d\left(V(z)^2\right)}{dz}\right|_{z=0}p_0(0)$$

Similarly, for k=3, it is easy to verify from figure 2 that:

$$p_3(0)=p_2(0).V(0)+p_1(0).\left.\frac{dV(z)}{dz}\right|_{z=0}.V(0)$$
$$+p_0(0).\left[\left.\frac{d\left(V(z)^2\right)}{dz}\right|_{z=0}V(0)+\frac{1}{2!}\left.\frac{d^2\left(V(z)^2\right)}{dz^2}\right|_{z=0}.V(0)^2\right]$$

and in general, by induction, we can write:

$$p_k(0) = \sum_{j=1}^{k} \Omega(j).p_{k-j}(0) \qquad (25)$$

where:

$$\Omega(j) = \begin{cases} V(0) & (j=1) \\ \sum_{i=1}^{j-1} \frac{V(0)^i}{i!} \frac{d^i (V(z)^{j-i})}{dz^i}\bigg|_{z=0} & (j \geq 2) \end{cases} \qquad (26)$$

In (25), we have expressed the probability of an empty buffer at the end of the k[th] slot as the sum of *k* mutually exclusive events. Further, in (25), $p_{k-j}(0)$ is interpreted as the probability that the system was empty for the last time at the end of the $(k\text{-}j)^{th}$ slot and therefore the function $\Omega(j)$ is the probability that the system is busy for *(j-1)* slots. Further, with simple algebra, we can prove by induction that (26) can be further simplified to yield:

$$\Omega(j) = pr[system\, is\, busy\, for\, (j-1)\, slots] = \frac{1}{j!} \frac{d^{j-1}}{dz^{j-1}} \left[V(z)^j\right]\bigg|_{z=0}$$

and therefore:

$$pr[system\, is\, busy\, for\;\; j\, slots] = \frac{1}{(j+1)!} \frac{d^j}{dz^j} \left[V(z)^{j+1}\right]\bigg|_{z=0}$$

Note that the above expression allows a busy period to consist of zero slots, and since in general we define the busy period of a queuing system as the time between two consecutive idle periods, then the busy period must consist of at least one slot (i.e initiated by at least one arrival). Under this convention, let the random variable *b* denote the length of an arbitrary busy period in number of slots and let B(z) be the corresponding pgf. Then:

$$pr[b = j\, slots] = \frac{1}{[1-V(0)]} \frac{1}{(j+1)!} \frac{d^j}{dz^j} \left[V(z)^{j+1}\right]\bigg|_{z=0} \qquad (27)$$

and the corresponding pgf is therefore:

$$B(z) = \sum_{j=1}^{\infty} pr[b=j] z^j$$

$$= \frac{1}{[1-V(0)]} \cdot \frac{1}{z} \left\{ \sum_{k=1}^{\infty} \frac{z^k}{k!} \frac{d^{k-1}}{dz^{k-1}} \left[V(z)^k\right]\bigg|_{z=0} - zV(0) \right\}$$

Further, from Lagrange's theorem (Appendix), we can write:

$$\sum_{k=1}^{\infty} \frac{z^k}{k!} \frac{d^{k-1}}{dz^{k-1}} \left[V(z)^k\right]\bigg|_{z=0} = \sigma^*$$

where σ^* is the unique solution of the equation:

$$\sigma = z.V(\sigma) \qquad (28)$$

inside the unit circle. Hence the pgf of the busy period for the GI/D/1 queue is:

$$B(z) = \frac{\sigma^* - zV(0)}{z[1-V(0)]} \qquad (29)$$

where σ^* is as defined in (28). The mean length of the busy period is readily obtained from (28-29), giving:

$$\bar{b} = \frac{dB(z)}{dz}\bigg|_{z=1} = \frac{V'(1)}{(1-V'(1))(1-V(0))} \qquad (30)$$

5.2 The Multiplexer with Train Arrivals Case

In section (4.2), it was found that under zero initial conditions, the general expression of transient probability of empty buffer, $p_k(0)$, for our model, is the same as that of the GI/D/1 queue, with $V(z) = H(z) = \beta_m{}^N$. Hence equations (25-26) also hold for the correlated case (this has also been verified through symbolic computation using the Maple computational system [8]). Hence, for our case, the distribution of the busy period is characterized by the probabilities:

$$pr[b = j\, slots] = \frac{1}{\left[1-(1-q)^N\right]} \frac{1}{(j+1)!} \frac{d^j}{dz^j} \left[H(z)^{j+1}\right]\bigg|_{z=0} \qquad (31)$$

and the corresponding pgf:

$$B(z) = \frac{\sigma^* - z(1-q)^N}{z\left[1-(1-q)^N\right]} \qquad (32)$$

where σ^* is the unique solution of the equation $\sigma = zH(\sigma)$ inside the unit circle. Further, from (30), the mean busy period of the multiplexer is:

$$\bar{b} = \frac{\rho}{(1-\rho)(1-(1-q)^N)}$$

where $\rho = H'(1) = \frac{Nqm}{1+(m-1)q}$ is the load of the system at steady-state (readily obtained from (17)).

Finally, we note that our definition of the busy and idle periods implies that a slot will belong to an idle period if and only if the multiplexer is empty at the beginning of this slot; otherwise it belongs to a busy period. Hence, from the previous expressions for $\bar{b}$ and $\bar{I}$, the fraction of slots belonging to an idle period is given by:

$$\frac{\bar{I}}{\bar{I} + \bar{b}} = 1 - \rho$$

which, as expected, equals the steady-state probability of an empty buffer.

For the case (*m*=2), closed-form expressions for the busy-period's probability distribution can be derived from (31), giving:

$$pr[b = j\ slots] = \begin{cases} \dfrac{1}{\left[1-(1-q)^N\right]}\dfrac{1}{(j+1)!}\cdot \left[\dfrac{\Gamma(j+1)(-16q)^{j/2}(1-q)^{N(j+1)-j}}{2^{N(j+1)}.\sqrt{\pi}}\displaystyle\sum_{i=0}^{N(j+1)}\binom{N(j+1)}{i}\dfrac{\Gamma\left(\frac{j-i}{2}\right)}{\Gamma\left(\frac{-i}{2}\right)}\right] & (j\ even) \\ 0 \qquad (j\ odd) & \end{cases}$$

In this case, the fact that the busy periods consists of an even number of slots is expected as each message consists of a fixed number of m=2 packets.

6.0 Numerical Results

In this section, we illustrate our analysis approach through some numerical examples. In figure 3, we plot the transient probabilities of an empty buffer as a function of time, with the number of input links as a parameter. We kept the steady-state load constant at $\rho = 80\%$ and assumed m=2. As may be seen, for the same steady-state load, different probabilities are obtained for different values of N. Also note that the transient probabilities of an empty buffer approach the steady-state value of $1-\rho = 0.2$ as time increases.

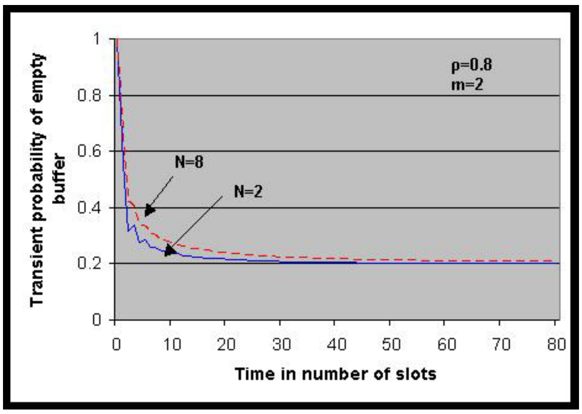

Fig. 3 Transient probabilities of an empty buffer

In figure 4, we plot the corresponding transient mean of the queue length. In particular, we note that the exponential rise behavior in the transient mean-time curve, depicted in figure 4, is typical in many other queuing systems [3-9].

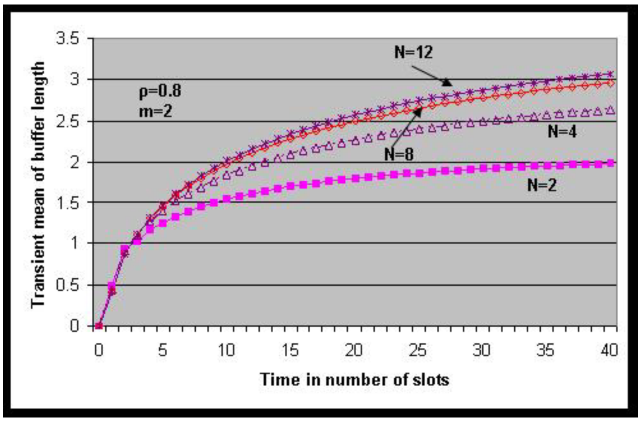

Fig. 4 Transient mean of queue length

A very useful measure to estimate the instantaneous packet overflow probabilities due to a finite buffer size (n) is the transient probability that occupancy in an infinite buffer system exceeds the proposed buffer size, $pr\left[i_k > n\right]$. These probabilities can be computed from the transient pgf $P_k(z)$ as given in (11), by also noting that $pr\left[i_k > n\right]$ corresponds to the coefficient of z^n in the polynomial $\dfrac{1-P_k(z)}{1-z}$.

For m=2, these probabilities are displayed in figure 5, below. As expected, the transient probabilities of overflow increase as time evolves, and this reflects the fact that when the system starts from zero initial conditions, the queue waiting room builds up progressively.

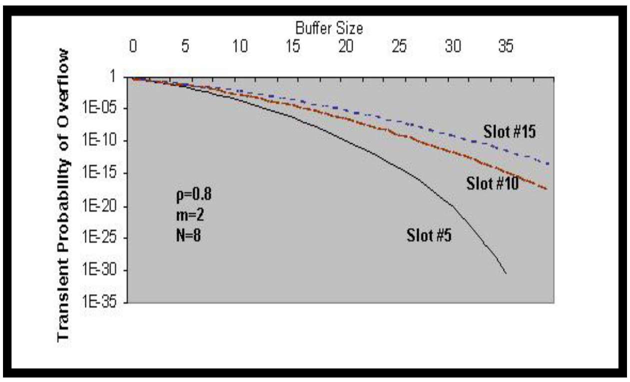

Fig.5 Transient probabilities of packet overflow

Finally, figure 6 depicts the corresponding probability distribution function of the busy period, which exhibits zero 'discontinuities' at odd numbered slots.

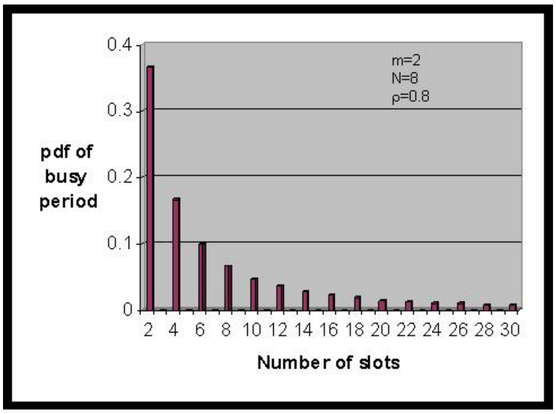

Fig.6 Probability distribution function of busy Period

7. Conclusions

In this paper, we have carried an exact transient analysis of a statistical multiplexer with a finite number of input links and whose arrival process consists of a train of a fixed number of fixed-length packets. By means of a generating functions approach, coupled with functional transformation techniques, we were able to extract the expression for the transient pgf of the queue length. From this pgf, several time-dependent performance measures were derived. Using the GI/D/1 queue as a reference, we showed how the transient analysis allows us to derive expressions for the probability distribution of the busy period and its corresponding pgf. Results for the idle period distribution were also provided.
The transform approach used in the present analysis provides a general framework under which similar types of queuing models with correlated arrivals can be analyzed. Finally, we note that our transient analysis approach can also be generalized to cover non-zero initial conditions. More importantly, our transient analysis provides a key solution technique to extract the corresponding steady-state pgf [5].

Appendix : Lagrange's Theorem [10]

If $\Psi(z)$ *and* $g(z)$ are functions of z, analytical on and inside and on a closed contour *C* surrounding a point *a*, and if *w* is such that $|w.g(z)| < |z-a|$ is satisfied at all points z on the perimeter C, then the equation:

$$z = a + w.g(z)$$

regarded as an equation in z, has exactly one root in the interior of *C*. Further any function $\Psi(z)$ of *z* analytical on and inside *C* can be expanded as power series in *w* by the formula:

$$\Psi(z) = \Psi(a) + \sum_{k=1}^{\infty} \frac{w^k}{k!} \left[\frac{d^{k-1}\left(\Psi'(z) g(z)^k\right)}{dz^{k-1}} \right]_{z=a}$$

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