Finite Valued Control Laws Design

G. CELENTANO, R. IERVOLINO
Dipartimento di Informatica e Sistemistica
Università degli Studi di Napoli, Federico II
Via Claudio, 21, 80125 Napoli
ITALY

Abstract: - The possibility of implementing finite valued control laws more and more easily by making use of microcontrollers, the consequent costs reduction of multi-level actuators and the obtainable high efficiencies motivate the development of new methodologies for the design of multi-level control laws for a class of systems that is wider than the class of systems that can be approximated by first order models.
In this paper a method for the synthesis of a discrete valued control law is provided that makes use of diagrams and tables. Such diagrams and tables are obtained by an analytical/numerical optimisation technique with respect to a minimal number of normalised parameters that characterise both the plant as well as the controller.
The effectiveness of the method is illustrated by the solution of a practical engineering problem.

Keywords: - Nonlinear control systems, hybrid systems, finite valued control

1 Introduction

There are many industrial plants (especially power systems) that, for construction simplicity and/or in order to attain better efficiencies, may be controlled through control signals assuming a finite number of different values, with a relatively slow switching. Moreover, the advent of modern microcontrollers and the relatively lower costs of multi-level actuators allow the easy implementation of control strategies based on the appropriate switching of the control values chosen from a finite discrete set.
The above considerations have motivated the interest of the authors in the development of new methodologies for the design of discrete valued control laws for a class of systems that is sufficiently wide. In particular, it is treated here the problem of control law synthesis for the sufficiently general class of hybrid systems consisting of continuous-time dynamical systems, approximated by high order linear time-invariant models, whose control inputs take value from a finite discrete set.
The specific literature about hybrid control laws synthesis is relatively recent and the recently published papers document how both theoretical and practical problems are still open (see [7] and the rich bibliography therein). In [5],[6],[8] the controllers with control signals without constraints on their amplitude, but constant in prescribed intervals of time, are discussed. Vice-versa, in [3] control laws are proposed with two or infinite number of levels, with an infinitely fast switching (see also [4],[9] for a detailed discussion).
In this paper a practical and systematic method is presented for the design of a control law with a finite number of levels, which guarantees the fulfillment of a given set of requirements in terms of both system performance and control effort. The proposed approach is based on the construction of diagrams and tables by a mixed analytical/numerical optimisation technique of the number of parameters that characterise the plant and the controller.
A numerical example illustrates the effectiveness of the technique.

2 Problem Statement

The block diagram of Figure 1 shows a typical control feedback system:

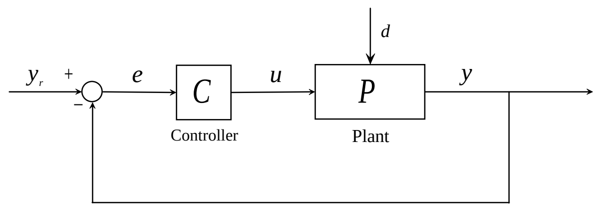

Fig. 1: The block diagram of a typical control feedback system

where: $y_r \in \Re$ denotes the constant reference signal, $u \in \Re$ and $y \in \Re$ indicate the plant scalar input and output respectively, $d \in \Re$ is the constant disturbance signal and $e = y_r - y$ is the regulation error.

The controller that is examined is a generalization of the classic two-valued (on/off) relay control with hysteresis, in the sense that the controller output may now assume one of three or more levels, according to both the current value of the error signal and the output value of the controller after the last switching. For the sake of illustrative simplicity and without loosing generality, the input-output characteristic of the considered N+1-valued controller with hysteresis (with the inclusion of the null level) is the one depicted in Figure 2 relative to the case N=2, where $U_i, \varepsilon_i,\ i = 1, 2$ denote the control (non null) levels and the (symmetric) error thresholds, respectively.

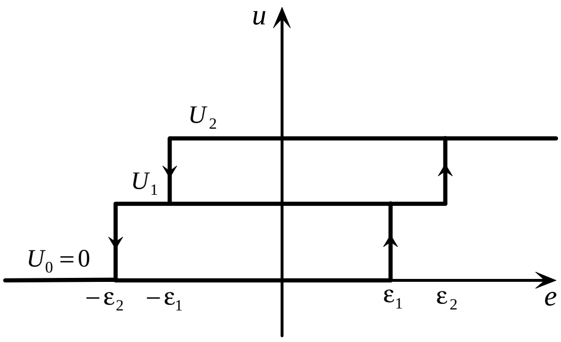

Fig. 2: The characteristic of the multi-valued controller for the case N = 2

It is easy to verify that an alternative and more expressive description of the controller characteristic is the transition graph of Figure 3. Such a graph is relative to an automaton that, among other things, offers a method of realization via software or digital circuits of the controller.

For what concerns the plant model, it is supposed to be linearised about one of its equilibrium points and that the corresponding linear system is asymptotically stable, represented by the following input/state/output model:

$$\begin{cases} \dot{x} = Ax + Bu + Fd \\ y = Cx + Hd \end{cases} \quad , x(t_0) = 0\,, \qquad (1)$$

where $A \in \Re^{n \times n}$, $B, F \in \Re^{n \times 1}$, $C \in \Re^{1 \times n}$ and $H \in \Re$, being n the order of the plant.

The aim of the paper is to solve the following problem.

Problem. Consider the feedback control system of Figure 1 with the finite discrete valued controller with hysteresis described by the example case of Figure 2 (or of Figure 3) and the linear plant (1). For a given constant reference value y_r, in the presence of a constant disturbance d, design the multi-valued controller parameters U_i and ε_i, $i = 1, 2, \ldots, N$, in order to satisfy the following requirements about:

Transient:

1. the 0-100% rise time, $T_s \in (T_{s\min}, T_{s\max})$, $T_{s\max} \geq T_{s\min} \geq 0$;
2. the overshoot/undershoot, $S \in (S_{\min}, S_{\max})$, $S_{\max} \geq S_{\min} \geq 0$;

Steady-state:

3. the maximum absolute steady state regulation error, $E \in (E_{\min}, E_{\max}), E_{\max} \geq E_{\min} \geq 0$;

Control effort:

4. the control signal period $T \in (T_{\min}, T_{\max})$, $T_{\max} \geq T_{\min} \geq 0$.

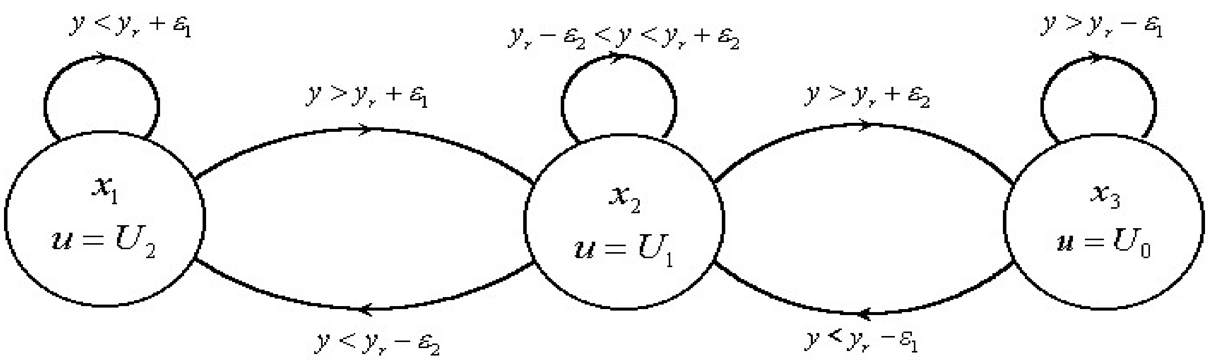

Fig. 3: The transition graph of the multi-valued controller for the case N = 2

It is evident that, due to the limited number of degrees of freedom, the solution of the Problem does not always exist; moreover, in the case that such a solution exist, it may be not unique and the determination of the link between the controller parameters and the prescribed requirements, for a given reference signal and a given disturbance, presents notable difficulties.
In the next Section a mixed analytical-numerical method is presented for the general case of high order plant models that is based on the normalisation of a reduced order representation of the plant.

3 Synthesis procedure

In the case of high order plants, explicit formulae for the design of the controller are not available; it is possible to proceed numerically and found those relationships in a table/graphical format. In order to reduce the amount of simulations, the next preliminary steps are performed:

1) The input/state/output model (1) is converted into an equivalent input/output model:

$$W(s) = C(sI - A)^{-1}B = \\ = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n},\ m < n \qquad (2)$$

with the disturbance signal (supposed known) taken to act at the plant output as shown in Figure 4, where $y_d = \left(H + C(-A)^{-1}F\right)d$.

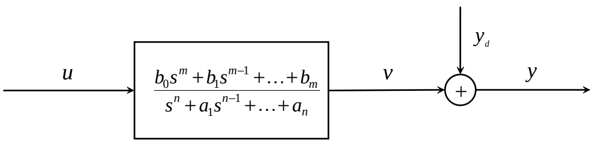

Fig. 4: Alternative representation of the plant

2) The n-th order transfer function (2) is reduced to a second order equivalent model of the type:

$$W(s) = \frac{b_0 s + b_1}{s^2 + a_1 s + a_2} \qquad (3)$$

by using an optimal model reduction technique (see for example [1]).

3) the second order model (3) is normalized as in Figure 5 (see Appendix):

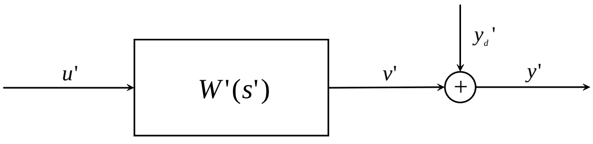

Fig. 5: Reduced order and normalized model

where:

$$t' = \sqrt{a_2}\, t \quad , \qquad (4)$$

$$u'(t') = u(t'/\sqrt{a_2}) \quad , \qquad (5)$$

$$v'(t') = \frac{v(t'/\sqrt{a_2})}{b_1/a_2} \quad , \qquad (6)$$

$$y'_d = \frac{y_d}{b_1/a_2} \quad , \qquad (7)$$

$$y'(t') = \frac{y(t'/\sqrt{a_2})}{b_1/a_2} \quad , \qquad (8)$$

and

$$W'(s') = \frac{1 + s'\alpha}{1 + 2\zeta s' + s'^2}, \qquad (9)$$

with:

$$\alpha = \frac{b_0\sqrt{a_2}}{b_1}, \qquad (10)$$

$$\zeta = \frac{a_1}{2\sqrt{a_2}}. \qquad (11)$$

If $\zeta > 1$ the plant can be also normalised as follows:

$$W'(s') = \frac{1 + s'\beta}{(1 + s')(1 + s'\tau_n)} \qquad (12)$$

with:

$$t' = \frac{t}{\tau_{\max}}, \qquad (13)$$

$$\beta = \frac{b_0}{b_1 \tau_{\max}}, \qquad (14)$$

$$\tau_n = \frac{\tau_{\min}}{\tau_{\max}}, \qquad (15)$$

being $\tau_{\max}$ and $\tau_{\min}$ the time constants of the reduced plant (3).

4) The controller parameters U_i and ε_i are normalized as follows:

$$\sigma_i = \frac{U_i}{U_n},\ i = 1,2,\ldots,N, \qquad (16)$$

$$\chi_i = \frac{\varepsilon_i}{|y_r - y_d|},\ i = 1,2,\ldots,N, \qquad (17)$$

where:

$$U_n = (y_r - y_d)/G, \qquad (18)$$

is the *nominal* control level (i.e. the level that, if available, would guarantee a zero steady-state regulation error), being G the plant static gain. Note that in order to assure the regulation the control level U_N must satisfy the following conditions:

$$\mathrm{sgn}(GU_N) = \mathrm{sgn}(y_r - y_d)\,, \qquad (19)$$

$$\left|GU_N\right| > \left|y_r - y_d\right| + \varepsilon_N\,, \qquad (20)$$

i.e. the maximum normalised level σ_N is always greater than 1.

Once the above Steps are performed, the following quality indices of the control system are considered:

$T_s^{'}$ = rise time,
S' = overshoot,
E' = max $\{E_+^{'}, |E_-^{'}|\}$ maximum steady-state error,
T' = steady-state control signal period.

Such quality indices are nonlinear functions of the controller parameters σ_i and χ_i and of the plant parameters α and ζ (or also, if ζ >1, of β and τ_n). The parameters χ_i are usually assigned in percentage with respect to their maximum value χ_N. Moreover, the values of χ_N are limited in the practice (a typical value for χ_N is 0.05). As a result, each of the above defined quality indices can be essentially considered as function of σ_i, ζ and α only (or of σ_i, τ_n and β). Such functions can be numerically evaluated and, for a easier use of them, can be tabled and/or diagrammed.
To have an idea, in Figures 6,7,8,9 the diagrams relative to $T_s^{'}$,S',E',T' are reported for the case N=2, $\chi_1 = 0.025$, $\chi_2 = 0.05$, $\zeta = 0.10:0.15:2.5$, $\alpha = 0$ and with the normalized control levels chosen as follows:

- $\sigma_1 = 0.8\,, \sigma_2 = 1.25$, i.e. the intermediate level less than the nominal level;
- $\sigma_1 = 1.25\,, \sigma_2 = 1.50$, i.e. the intermediate level greater than the nominal level.

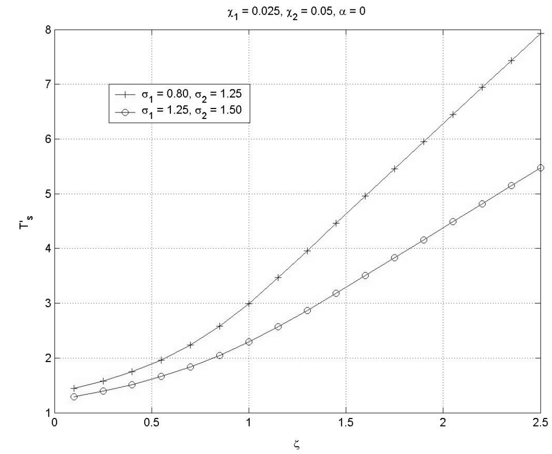

Fig. 6: The normalized rising time

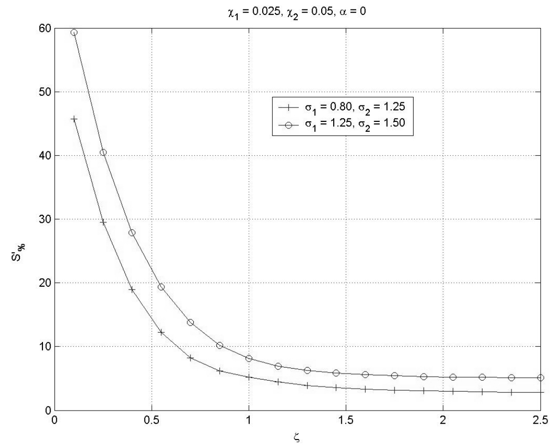

Fig. 7: The normalized overshoot/undershoot

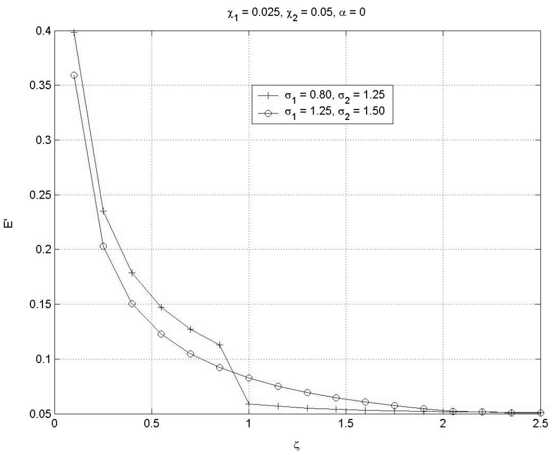

Fig. 8: The maximum absolute steady-state error

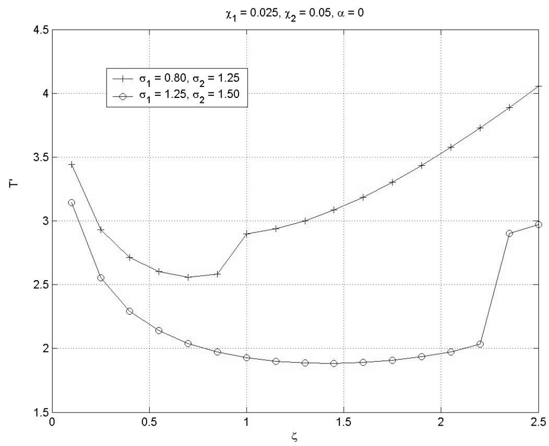

Fig. 9: The control signal period

Remark 1. From the diagrams of T_s',*S'*,*E'*,*f'* that are not reported results that the finite-valued control does not guarantee high performance for plants with excessive delay (i.e. α< -0.25) and/or with relatively low damping coefficient (i.e. ζ<0.25). ♦

Remark 2. In principle, by virtue of conditions (19)-(20) the synthesis problem can be solved by using, of the available control levels, the maximum and the minimum levels only. The intermediate levels are not indispensable for the reference trajectory tracking, but they are useful for alleviating the average switching frequency. This result is particularly evident when at least one of the intermediate levels is greater than the nominal control level: in this case the maximum level serves to reduce the rising time. ♦

For a given plant α and ζ, for a given number of control levels *N* and a given maximum threshold χ_N, the diagrams of T_s',*S'*,*E'*,*T'* allow to determine the interval of the values of σ_i (as intersection of various intervals) that, when not empty, solve the Problem. If the Problem does not admit a solution and we are far from the maximum allowable switching frequency, the value of χ_N can be reduced compatibly with the output transducer precision. In the hypothesis that it is not possible to attain a solution in this way it is necessary to be satisfied with the values of T_s',*S'*,*E'*,*T'* obtained by the most onerous values of σ_i and χ_N.

4 Example

Consider the problem of controlling the temperature of an industrial refrigerator where a body has been placed inside. The linear model of the plant is the following:

$$\begin{bmatrix}\dot{\theta}_f\\ \dot{\theta}_b\\ \dot{q}\end{bmatrix} = \begin{bmatrix}-5.4E-3 & 0.4E-3 & 0.2E-3\\ 2E-3 & -2E-3 & 0\\ 0 & 0 & -0.5\end{bmatrix}\begin{bmatrix}\theta_f\\ \theta_b\\ q\end{bmatrix} + \\ + \begin{bmatrix}0\\0\\0.5\end{bmatrix}u + \begin{bmatrix}5E-3\\0\\0\end{bmatrix}\theta_e \qquad (21)$$

$$\theta_f(0) = \theta_b(0) = \theta_e\ ,\ q(0) = 0 \qquad (22)$$

where: θ_f, θ_b and θ_e respectively denote the refrigerator, the body and the environmental temperatures; *q* is the subtracted heat power and *u* denotes the control input.

If the reference refrigerator temperature is $\theta_r = 0°C$, and the environmental temperature (disturbance) is $\theta_e = 20°C$, design a finite discrete valued controller with *N*=2 control levels (non-null levels), in order to satisfy the following set of requirements:

1. the rise time has to be $T_s \leq 250$ sec ;
2. the undershoot has to be $S_{\%} \leq 7\%$;
3. the absolute maximum steady state error that is tolerated is $E \leq 1.5°C$;
4. the control signal period $T \geq 60$ sec .

The transfer function between *u* and θ_f is, including the compressor (actuator) dynamics:

$$W_{u\to\theta_f}(s) = \frac{0.1E-3s+0.2E-6}{s^3+0.5074s^2+3.71E-3s+5.0E-6} \qquad (23)$$

According to the synthesis procedure described in Section 3, the first step consists of reducing the system model to the second order. In this case, by neglecting the actuator high order dynamics it is:

$$W_{u\to\theta_f}(s) \cong \frac{0.2E-3s+0.4E-6}{s^2+7.4E-3s+0.1E-4}\ . \qquad (24)$$

The corresponding normalised transfer function is:

$$W'_{u'\to\theta'_f}(s') = \frac{\alpha s'+1}{1+2\zeta s'+s'^2} \qquad (25)$$

where $\alpha \cong 1.58$, $\zeta \cong 1.17$. By interpolating the diagrams and/or the tables relative to the normalized closed loop system, with the values of α and ζ above, $\chi_1 = 0.025$, $\chi_2 = 0.05$ and for the cases:

- $\sigma_1 = 0.80$, $\sigma_2 = 1.0$,
- $\sigma_1 = 1.25$, $\sigma_2 = 1.50$,

the estimated (and denormalised) values of the rising time (in seconds), of the undershoot (in percentage), of the maximum absolute regulation error (in Celsius degrees) and of the control period (in seconds), are:

$T_s \cong \{355, 233\}$ sec ,
$S_{\%} \cong \{2.52\%, 5.04\%\}$,
$E \cong \{0.998, 0.995\}°C$,
$T \cong \{139, 78.7\}$ sec .

From the above results it is clear that all of the control system requirements are satisfied for the case $\sigma_1 = 1.25$, $\sigma_2 = 1.50$. In terms of denormalised control parameters, the solution has been found for: $U_1 = -625, U_2 = -750$ and $\varepsilon_1 = 0.50, \varepsilon_2 = 1.0$.
The simulation values of the quality indices for the (real) multi-valued control system are sufficiently close to the estimated values above, which confirms the validity of the proposed method.

5 Conclusions

In this paper the synthesis of a finite discrete valued control system has been presented. The design of the oversupply factors and the thresholds values passes through a direct synthesis approach based on simulation results. A normalisation procedure has been also presented that allows reducing the number of parameters to vary and therefore the number of simulations required to evaluate the closed loop system performance

6 Appendix

By multiplying both members of the relationship:

$$Y(s) = W(s)U(s) \quad \text{(A1)}$$

by $\sqrt{a_2}$ and substituting s with $\sqrt{a_2}s'$ it is:

$$\sqrt{a_2}V(\sqrt{a_2}s') = W(\sqrt{a_2}s')\sqrt{a_2}U(\sqrt{a_2}s') \quad \text{(A2)}$$
$$\mathcal{L}\left(v(t'/\sqrt{a_2})\right) = W(\sqrt{a_2}s')\mathcal{L}\left(u(t'/\sqrt{a_2})\right). \quad \text{(A3)}$$

By dividing both members of (A3) by the static gain of system (3):

$$\frac{\mathcal{L}\left(v(t'/\sqrt{a_2})\right)}{b_1/a_2} = \frac{W(\sqrt{a_2}s')}{b_1/a_2}\mathcal{L}\left(u(t'/\sqrt{a_2})\right) \quad \text{(A4)}$$

then

$$\mathcal{L}\left(v'(t'/\sqrt{a_2})\right) = \frac{W(\sqrt{a_2}s')}{b_1/a_2}\mathcal{L}\left(u(t'/\sqrt{a_2})\right), \quad \text{(A5)}$$

hence the normalised transfer function is:

$$W'(s') = \frac{W(\sqrt{a_2}s')}{b_1/a_2} = \frac{\frac{b_0\sqrt{a_2}}{b_1}s'+1}{s'^2 + \frac{a_1\sqrt{a_2}}{a_2}s'+1}. \quad \text{(A6)}$$

If the following parameters are introduced:

$$\alpha = \frac{b_0\sqrt{a_2}}{b_1} \quad \text{(A7)}$$

$$\zeta = \frac{a_1}{2\sqrt{a_2}}, \quad \text{(A8)}$$

the (A6) becomes:

$$W'(s') = \frac{1+s'\alpha}{1+2\zeta s'+s'^2}, \quad \text{(A9)}$$

which is the final synthetic representation of the normalized plant model.

References:

[1] C.T. Chen, *System and signal analysis*, Oxford University Press, 2nd Edition, New York, pp. 532-538, 1994.

[2] G. Celentano, R. Iervolino, “Direct synthesis of relay controllers,” *Internal Report*, Dipartimento di Informatica e Sistemistica, Università degli Studi di Napoli, Oct. 2003.

[3] U. Itkis, *Control systems of variable structure*. Wiley, New York, 1976.

[4] H. Khalil, *Nonlinear systems.* Prentice Hall, NJ, 2nd Ed., 1996.

[5] S. Nikitin, Piecewise smooth stabilizing extension. *Proceedings of the 2nd European Control Conference,* Groningen, The Netherlands, 50-52, 1993.

[6] S. Nikitin, *Global controllability and stabilization of nonlinear systems.* World Scientific Publishing Co. Pte Ltd, ISBN 981-02-1779-X, 1994.

[7] S. Sastry, *Nonlinear systems: analysis, stability and control*. Springer-Verlag, New York, 1999.

[8] E.D. Sontag, *Mathematical control theory. Deterministic finite dimensional systems*. Springer-Verlag New York, 1990.

[9] V. Utkin, *Sliding modes in optimisation and control.* Springer-Verlag, New York, 1992.