Fully Generalized State Model of Optimized Third-Order Dynamical Systems of Class *C*

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Abstract: - State models of piecewise-linear (PWL) autonomous dynamical systems used as prototypes in practical realizations of electronic chaotic oscillators are expressed in more generalized form that includes all previous models as special cases with the same optimum eigenvalue sensitivities.

Key-Words: Dynamical systems, piecewise-linear, modeling, sensitivity optimization

1 Introduction

Recently published new state models of piecewise-linear (PWL) dynamical systems of Class *C* have been derived as prototypes for their circuit realization having minimized eigenvalue sensitivities. The same optimization conditions obtained by using linear topological conjugacy [1] as well as the direct derivation [2] give the possibility to reach the minimum sum of relative eigenvalue sensitivity squares with respect to the change of individual state matrix parameters. As shown in this contribution the detailed analysis applied previously only to the most important case of complex conjugate eigenvalues in all regions of PWL feedback function [3], can be extended for any type of eigenvalues utilizing one common form of the optimized state matrix. A procedure based on the block-decomposed state matrix starting from the second-order subsystem is automatically applied for higher-order system [4], [5].

Let the third-order autonomous PWL dynamical system of Class *C* be described by the general state matrix form

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{b}\,h(\mathbf{w}^T\mathbf{x}) \;, \qquad (1)$$

where $\mathbf{A} \in R^{3\times3}$, $\mathbf{x}, \mathbf{b}, \mathbf{w} \in R^3$. The elementary memoryless PWL feedback function (Fig. 1)

$$h(\mathbf{w}^T\mathbf{x}) = \tfrac{1}{2}\left(\left|\mathbf{w}^T\mathbf{x}+1\right| - \left|\mathbf{w}^T\mathbf{x}-1\right|\right) \qquad (2)$$

determines the inner region D_0 and the outer ones D_{+1} (D_{-1}), respectively [2]. Dynamical behavior of this system is determined by two characteristic polynomials associated with the individual regions, i.e.

$$D_0: \quad P(s) = (s-\mu_1)(s-\mu_2)(s-\mu_3) = \det(s\mathbf{1}-\mathbf{A}_0) = s^3 - p_1 s^2 + p_2 s - p_3 \;, \qquad (3)$$

$$D_{+1}, D_{-1}: \quad Q(s) = (s-\nu_1)(s-\nu_2)(s-\nu_3) = \det(s\mathbf{1}-\mathbf{A}) = s^3 - q_1 s^2 + q_2 s - q_3 \;, \qquad (4)$$

where $\mathbf{1}$ is the unity matrix. Their roots represent the eigenvalues of the corresponding state matrices, which are mutually related by the fundamental expression [5]

$$\mathbf{A}_0 = \mathbf{A} + \mathbf{b}\mathbf{w}^{\mathrm{T}} \;. \qquad (5)$$

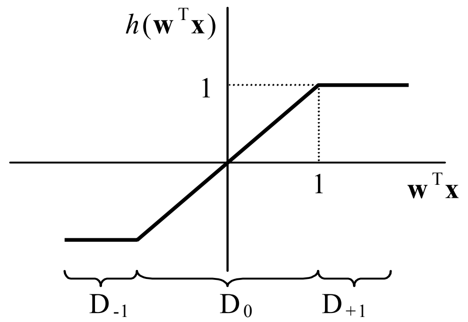

Fig. 1: Simple PWL feedback function.

2 General block-decomposed form of the state model

Consider general form of vectors in the basic matrix formula (1) $\mathbf{b}^{\mathrm{T}} = [b_1 \;\; b_2 \;\; b_3]$ and

$\mathbf{w}^T = [w_1 \ \ w_2 \ \ w_3]$. Suppose the eigenvalues μ_3 and ν_3 represent the real roots of the corresponding characteristic polynomials in eqns (3) and (4), while both remaining pairs μ_1, μ_2 and ν_1, ν_2 can be either real or complex conjugated. To utilize the results for the second-order systems [3], a general third-order model with block-decomposed state matrix containing second-order submatrix can be designed in two basic forms:

(i) *Upper block-triangular form*

$$\mathbf{A} = \left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline 0 & 0 & a_{33} \end{array}\right], \tag{6a}$$

(ii) *Lower block-triangular form*

$$\mathbf{A} = \left[\begin{array}{cc|c} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right]. \tag{6b}$$

In both these forms the parameter a_{33} evidently determines directly the real eigenvalue ν_3, while the pair ν_1, ν_2 is defined as eigenvalues of the second-order submatrix, i.e. by the parameters $a_{11}, a_{12}, a_{21}, a_{22}$. As the remaining two nonzero parameters have no influence to the state matrix eigenvalues, all optimization procedure derived for the second-order subsystem [5] can be completely extended also to any case of the third-order system (6a,b). Then the corresponding state matrix $\mathbf{A_0}$ associated with the inner region must evidently be also block-triangular decomposed but in the complementary form to (6a,b), i.e. generally

$$\text{(i)} \quad \mathbf{A}_0 = \left[\begin{array}{cc|c} a_{11}^{(0)} & a_{12}^{(0)} & 0 \\ a_{21}^{(0)} & a_{22}^{(0)} & 0 \\ \hline a_{31}^{(0)} & a_{32}^{(0)} & a_{33}^{(0)} \end{array}\right], \tag{7a}$$

$$\text{(ii)} \quad \mathbf{A}_0 = \left[\begin{array}{cc|c} a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} \\ a_{21}^{(0)} & a_{22}^{(0)} & a_{23}^{(0)} \\ \hline 0 & 0 & a_{33}^{(0)} \end{array}\right]. \tag{7b}$$

Similarly as in (6a,b) the parameter $a_{33}^{(0)}$ here evidently determine directly the real eigenvalue μ_3, while the pair μ_1, μ_2 is defined as eigenvalues of the second-order submatrix, i.e. by the parameters $a_{11}^{(0)}, a_{12}^{(0)}, a_{21}^{(0)}, a_{22}^{(0)}$ so that the remaining two nonzero parameters have again no influence to this state matrix eigenvalues. Substituting all needed parameters to basic relation (5) both these general forms (7a,b) can be expressed in details as follows

$$\text{(i)} \quad \mathbf{A}_0 = \left[\begin{array}{cc|c} a_{11}+b_1w_1 & a_{12}+b_1w_2 & a_{13}+b_1w_3 \\ a_{21}+b_2w_1 & a_{22}+b_2w_2 & a_{23}+b_2w_3 \\ \hline b_3w_1 & b_3w_2 & a_{33}+b_3w_3 \end{array}\right], \tag{8a}$$

$$\text{(ii)} \quad \mathbf{A}_0 = \left[\begin{array}{cc|c} a_{11}+b_1w_1 & a_{12}+b_1w_2 & b_1w_3 \\ a_{21}+b_2w_1 & a_{22}+b_2w_2 & b_2w_3 \\ \hline a_{31}+b_3w_1 & a_{32}+b_3w_2 & a_{33}+b_3w_3 \end{array}\right]. \tag{8b}$$

Finally, summarize general design conditions for both basic forms introduced. From (6a,b) follows directly one common condition

$$a_{33} = \nu_3 \tag{9}$$

and two different ones

$$\text{(i)} \quad a_{31} = a_{32} = 0, \tag{10a}$$

$$\text{(ii)} \quad a_{13} = a_{23} = 0. \tag{10b}$$

Comparing eqns (7a,b) and (8a,b), again one common condition is obtained for both forms

$$a_{33} + b_3w_3 = \mu_3 \tag{11}$$

and two different sets

$$\text{(i)} \quad a_{13} + b_1w_3 = a_{23} + b_2w_3 = 0, \tag{12a}$$

$$\text{(ii)} \quad a_{31} + b_3w_1 = a_{32} + b_3w_2 = 0. \tag{12b}$$

In the next part the optimization procedure for the second-order subsystem in the generalized form is used simultaneously with the corresponding third-order optimized state model design.

3 General optimized form of the state model

For the second-order subsystem the requirement for low eigenvalue sensitivity measure, i.e. $min\left(\sum S_r^{\,2}(\lambda_k, a_{ij})\right)$ leads to the generalized conditions [3]

$$a_{11} = a_{22} = \frac{1}{2}\left(\lambda_1 + \lambda_2\right), \tag{13a}$$

$$a_{12}\,a_{21} = \frac{1}{4}\left(\lambda_1 - \lambda_2\right)^2, \tag{13b}$$

where λ_1, λ_2 represents eigenvalues ν_1, ν_2 of state matrix $\mathbf{A}$ in outer region and μ_1, μ_2 of state matrix $\mathbf{A}_0$ in inner region. The function $S_r\ (\lambda_k, a_{ij})$ is relative sensitivity of eigenvalue λ_k with respect to state matrix parameter a_{ij}. The conditions (13a,b) are valid for both real and complex conjugate eigenvalues and, moreover, eqn (13b) contains one degree of freedom that is utilized just for optimized design of PWL systems in accordance with [3]. Choosing for simplicity [3]

$$w_1 = w_3 = 1 \tag{14}$$

the corresponding state matrices for both basic forms of the third-order system can be written with respect to conditions (9) to (13) as

(i) $$\mathbf{A} = \left[\begin{array}{cc|c} \frac{1}{2}(\nu_1+\nu_2) & \frac{1}{2}(\nu_1-\nu_2)K & -b_1 \\ \frac{1}{2}(\nu_1-\nu_2)K^{-1} & \frac{1}{2}(\nu_1+\nu_2) & -b_2 \\ \hline 0 & 0 & \nu_3 \end{array}\right], \tag{15a}$$

$$\mathbf{A}_0 = \left[\begin{array}{cc|c} \frac{1}{2}(\mu_1+\mu_2) & \frac{1}{2}(\mu_1-\mu_2)K_0 & 0 \\ \frac{1}{2}(\mu_1-\mu_2)K_0^{-1} & \frac{1}{2}(\mu_1+\mu_2) & 0 \\ \hline b_3 & b_3 w_2 & \mu_3 \end{array}\right], \tag{15b}$$

(ii) $$\mathbf{A} = \left[\begin{array}{cc|c} \frac{1}{2}(\nu_1+\nu_2) & \frac{1}{2}(\nu_1-\nu_2)K & 0 \\ \frac{1}{2}(\nu_1-\nu_2)K^{-1} & \frac{1}{2}(\nu_1+\nu_2) & 0 \\ \hline -b_3 & -b_3 w_2 & \nu_3 \end{array}\right], \tag{16a}$$

$$\mathbf{A}_0 = \left[\begin{array}{cc|c} \frac{1}{2}(\mu_1+\mu_2) & \frac{1}{2}(\mu_1-\mu_2)K_0 & b_1 \\ \frac{1}{2}(\mu_1-\mu_2)K_0^{-1} & \frac{1}{2}(\mu_1+\mu_2) & b_2 \\ \hline 0 & 0 & \mu_3 \end{array}\right]. \tag{16b}$$

To ensure all state matrix parameters are real numbers the additional optimization coefficients K, K_0 must represent either real or imaginary numbers according to the type of individual eigenvalue pairs. Its ratio is possible to express by the generalized optimization procedure [3] including the remaining parameters b_1, b_3 and b_2, w_2. Comparing (8a,b) and (15a,b), (16a,b) the following general relations can be derived

$$b_1 = b_2 w_2 = \tfrac{1}{2}\left[(\mu_1+\mu_2)-(\nu_1+\nu_2)\right], \tag{17a}$$

$$b_3 = (\mu_3 - \nu_3), \tag{17b}$$

$$b_1 w_2 = \tfrac{1}{2}\left[(\mu_1-\mu_2)K_0 - (\nu_1-\nu_2)K\right], \tag{18a}$$

$$b_2 = \tfrac{1}{2}\left[(\mu_1-\mu_2)K_0^{-1} - (\nu_1-\nu_2)K^{-1}\right], \tag{18b}$$

and then

$$w_2 = \frac{(\mu_1-\mu_2)K_0 - (\nu_1-\nu_2)K}{(\mu_1+\mu_2)-(\nu_1+\nu_2)} = \frac{(\mu_1+\mu_2)-(\nu_1+\nu_2)}{(\mu_1-\mu_2)K_0^{-1} - (\nu_1-\nu_2)K^{-1}}, \tag{19a,b}$$

Parameter w_2 is given by two formulas so that comparing (19a) and (19b) the ratio of the optimization coefficients K_0 and K can be expressed as the real root of the formally the same quadratic equation as in special case [3]

$$\left(\frac{K_0}{K}\right)^2 - 2(M+1)\left(\frac{K_0}{K}\right) + 1 = 0, \tag{20a}$$

$$\frac{K_0}{K} = 1 + M \pm \sqrt{M(M+2)}, \tag{20b}$$

where the auxiliary parameter M has the general form

$$M = 2\frac{(\mu_1-\nu_1)(\nu_2-\mu_2)}{(\mu_1-\mu_2)(\nu_1-\nu_2)}. \tag{21}$$

4 Conclusion

The generalized optimization condition for the second-order PWL dynamical systems is utilized for the design of their state models with low eigenvalue sensitivities and also extended for the optimized synthesis of the third-order systems. The result corresponds to the previously derived model with complex conjugate eigenvalues [3]. They can be directly used for real eigenvalues and rewritten for other two combined cases having both real and complex conjugate eigenvalues. It is also in accordance with [5] where this optimization problem is extended by the similar way for higher-order systems.

All these results are important for the practical realization of the autonomous chaotic dynamical systems (oscillators) in the form of simple electronic circuits having optimized eigenvalue sensitivity properties. Both forms of block-decomposed model leads to simple design formulas and give the possibility to adjust separately their parameters. It has been also proved numerically by simulation and also verified by the starting laboratory experiments.

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Appendix - Summary of general formulas for optimized third-order state model design

1. Detailed general matrix form:

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right] \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} h(w_1 x + w_2 y + w_3 z) \quad \text{(A-1)}$$

2. Chosen parameters:

$$w_1 = w_3 = 1 \quad \text{(A-2)}$$

3. Second-order subsystem:

$$a_{11} = a_{22} = \frac{1}{2}(\nu_1 + \nu_2) \quad \text{(A-3)}$$

$$a_{12} = \frac{1}{2}(\nu_1 - \nu_2)K \quad \text{(A-4)}$$

$$a_{12} = \frac{1}{2}(\nu_1 - \nu_2)K^{-1} \quad \text{(A-5)}$$

$$b_1 = \tfrac{1}{2}\left[(\mu_1 + \mu_2) - (\nu_1 + \nu_2)\right] \quad \text{(A-6)}$$

$$b_2 = \tfrac{1}{2}\left[(\mu_1 - \mu_2)K_0^{-1} - (\nu_1 - \nu_2)K^{-1}\right] \quad \text{(A-7)}$$

$$w_2 = \frac{(\mu_1 - \mu_2)K_0 - (\nu_1 - \nu_2)K}{(\mu_1 + \mu_2) - (\nu_1 + \nu_2)} \quad \text{(A-8)}$$

$$\frac{K_0}{K} = 1 + M \pm \sqrt{M(M+2)} \quad \text{(A-9)}$$

$$M = 2\frac{(\mu_1 - \nu_1)(\nu_2 - \mu_2)}{(\mu_1 - \mu_2)(\nu_1 - \nu_2)} \quad \text{(A-10)}$$

4. Third-order system:

$$a_{33} = \nu_3 \quad \text{(A-11)}$$

$$b_3 = (\mu_3 - \nu_3) \quad \text{(A-12)}$$

(i) *Upper block-triangular form*

$$a_{31} = a_{32} = 0 \quad \text{(A-13a)}$$

$$a_{13} = -b_1 \quad \text{(A-13b)}$$

$$a_{23} = -b_2 \quad \text{(A-13c)}$$

(ii) *Lower block-triangular form*

$$a_{13} = a_{23} = 0 \quad \text{(A-14a)}$$

$$a_{31} = -b_3 \quad \text{(A-14b)}$$

$$a_{32} = -b_3 w_2 \quad \text{(A-14c)}$$

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