A geometric approach of generalized linear systems

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Abstract: -
A geometric approach of generalized linear systems as pairs of linear maps defined modulo a subspace is presented. This study permit us, to obtain conditions for controllability of the system.

Key-Words: - pair of linear maps, similarity, feedback, derivative feedback, generalized linear systems, controllability.

1 Introduction

The equivalence relation between pairs of matrices representing linear systems under feedback equivalence has been largely studied during the last three decades (see [2], [4], for example). With respect generalized linear systems, the study has experimented a great deal of interest in recent years where feedback and derivative feedback has been considered.

Here we present a geometrical approach consisting in to associate to the system a pair of linear maps defined modulo a subspace in order to obtain a collection of structural invariants that permit to deduce controllability conditions.

Let X be a complex finite dimensional vector space. In the paper, we tackle the problem to classify pairs of linear maps defined modulo a subspace and coinciding over this subspace:

$$\begin{array}{ll} f: X \longrightarrow X/W, & g: X \longrightarrow X/W, \\ W \subset X, & f_{|W} = g_{|W}. \end{array} \tag{1}$$

We will write this kind of maps as a couple $(f,g): X \longrightarrow X/W$, and we will refer simply, as a pair of linear maps.

Our aim is to provide a classification in relation to a natural generalization of the usual similarity of pairs of endomorphisms. The classification of pair of matrices (A,B) under feedback equivalence as linear map defined modulo a subspace $f: X \longrightarrow X/W$ is presented in [2].We recall that $f: X \longrightarrow X/W$ and $f': X' \longrightarrow X'/W'$ are equivalent and we will note $f \sim f'$, if and only if there exist an isomorphism $\varphi: X \longrightarrow X'$ with $\varphi(W) = W'$ such that $f' \circ \varphi = \tilde{\varphi} \circ f$, where $\tilde{\varphi}$ is the induced isomorphism $\tilde{\varphi} X/W \longrightarrow X'/W'$.

These results can be applied to study triples of matrices (E,A,B) representing generalized linear systems $E\dot{x} = Ax + Bu$, under feedback and derivative feedback, as well as to obtain conditions for controllability.

2 Pairs of linear maps defined modulo a subspace

Our aim is to classify pairs of linear maps $(f,g): X \longrightarrow X/W$, where X is a finite dimensional vector space, W is a linear subspace verifying $f_{|W} = g_{|W}$. We will refer to such a map simply as a pair of linear maps defined modulo a subspace.

The key to solve classification problem will be to reduce to classifying two associated pairs of linear maps defined as follows:

Definition 2.1. *Let $(f,g): X \longrightarrow X/W$ be a pair of linear maps defined modulo a subspace. We consider the following pairs of linear maps* *induced in a natural way by (f,g):*

$$\begin{array}{rl} (\dot{f},\dot{g}):W & \longrightarrow W_1 \\ w & \longrightarrow f(w)=g(w) \\ (f_1,g_1):X/W & \longrightarrow X_1/W_1 \\ \pi(x) & \longrightarrow \pi_1(f,g)(x) \end{array} \tag{2}$$

where $W_1 = f(W) = g(W)$, $X_1 = X/W$ and $\pi : X \longrightarrow X/W$ and $\pi_1 : X/W \longrightarrow X_1/W_1$ the canonical projections.

Then we have the following commutative diagrams:

$$\begin{array}{ccc} W & \xrightarrow{\dot{f}} & W_1 \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X_1 \\ \downarrow & & \downarrow \\ X/W & \xrightarrow{f_1} & X_2 \end{array} \quad , \quad \begin{array}{ccc} W & \xrightarrow{\dot{g}} & W_1 \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & X_1 \\ \downarrow & & \downarrow \\ X/W & \xrightarrow{g_1} & X_2 \end{array} \tag{3}$$

(where $X_1 = X/W$, and $X_2 = X_1/W_1$)

Notice that the maps $\dot{f} = \dot{g}$ are exhaustive and in the case where $f_{1|W_1} = g_{1|W_1}$ then (f_1, g_1) is a pair of linear maps with $\dim X_1 \leq \dim X$ and $\dim X_1 = \dim X$ if and only if $W = \{0\}$.

3 Equivalence relation

In order to define an equivalence between two pairs of linear maps $(f,g) : X \longrightarrow X/W$, $(f',g') : X' \longrightarrow X'/W'$ of this kind, we consider the pairs of isomorphisms $(\varphi,\psi) : X \longrightarrow X'$ where the maps induced in a natural way

$$\begin{array}{c} (\dot{\varphi},\dot{\psi}) : W \longrightarrow W' \\ (\tilde{\varphi},\tilde{\psi}) : X/W \longrightarrow X'/W' \end{array} \tag{4}$$

verify $\dot{\varphi} = \dot{\psi}$ and $\tilde{\varphi} = \tilde{\psi}$. We denote by $\mathcal{H}(W)$ the group of such pairs of isomorphisms, obviously we must suppose $\dim X = \dim X'$ and $\dim W = \dim W'$. From now on, these dimensions will be denoted by $n+m$ and m respectively.

Definition 3.1. *Let $(f,g) : X \longrightarrow X/W$, $(f',g') : X' \longrightarrow X'/W'$ be two pairs of linear maps. We say that they are equivalent, (written $(f,g) \sim (f',g')$), if there is $(\varphi,\psi) \in \mathcal{H}(W)$ such that*

$$\begin{array}{c} f' \circ \varphi = \tilde{\varphi} \circ f \\ g' \circ \psi = \tilde{\psi} \circ g \end{array} \tag{5}$$

and we will write simply as

$$(f',g') \circ (\varphi,\psi) = \tilde{\varphi} \circ (f,g) \tag{6}$$

In particular, if $W = \{0\}$ then $W' = \{0\}$ and $(f,g) \sim (f',g')$ is the simultaneous equivalence of pairs of maps.

Notice that, if $(f,g) \sim (f',g')$ then $\tilde{\varphi} = \tilde{\psi}$ induces

$$\begin{array}{ll} (\varphi_1,\psi_1) & : X_1 \longrightarrow X'_1 \\ (\dot{\varphi}_1,\dot{\psi}_1) & : W_1 \longrightarrow W'_1 \\ (\tilde{\varphi}_1,\tilde{\psi}_1) & : X_1/W_1 \longrightarrow X'_1/W'_1 \end{array} \tag{7}$$

verifying $\dot{\varphi}_1 = \dot{\psi}_1$ and $\tilde{\varphi}_1 = \tilde{\psi}_1$.

Remark 3.1. Let $(f,g) : X \longrightarrow X/W$ and $(f',g') : X' \longrightarrow X'/W'$ be two equivalent pairs of maps, then $f \sim f'$ and $g \sim g'$. (The equivalence is as a maps defined modulo a subspace defined at the introduction).

Proposition 3.1. ([2], Theorem I.3.2). *Let $(f,g) : X \longrightarrow X/W$ and $(f',g') : X' \longrightarrow X'/W'$ be two equivalent pairs of linear maps. Then*

i) $f_1 \sim f'_1$ and $\operatorname{rank} \dot{f} = \operatorname{rank} \dot{f}'$,

ii) $g_1 \sim g'$ and $\operatorname{rank} \dot{g} = \operatorname{rank} \dot{g}'$.

Theorem 3.1. *Let $(f,g) : X \longrightarrow X/W$ and $(f',g') : X' \longrightarrow X'/W'$ be two pairs of linear maps. Suppose (f_1,g_1) is a pair then, $(f,g) \sim (f',g')$ if and only if $(f_1,g_1) \sim (f'_1,g'_1)$ and $\operatorname{rank} \dot{f} = \operatorname{rank} \dot{f}'$.*

Proof. Suppose $(f,g) \sim (f',g')$. Proposition 3.1 ensures the commutativity of the two three-dimensional diagrams.

$$\begin{array}{ccccccc} W & \longrightarrow & X & \xrightarrow{f} & X_1 & & \\ {\scriptstyle\dot{\varphi}}\downarrow & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle\tilde{\varphi}}\downarrow & & \\ W' & \longrightarrow & X & \xrightarrow{f'} & X'_1 & & \\ {\scriptstyle\dot{f}}\searrow & & {\scriptstyle f}\searrow & & {\scriptstyle f_1}\searrow & & \\ {\scriptstyle\dot{f}'}\searrow & & {\scriptstyle f}\searrow & & {\scriptstyle\dot{f}'_1}\searrow & & \\ & W_1 & \longrightarrow & X_1 & \xrightarrow{f} & X_2 & \\ & {\scriptstyle\dot{\varphi}_1}\downarrow & & {\scriptstyle\varphi_1}\downarrow & & {\scriptstyle\tilde{\varphi}_1}\downarrow & \\ & W'_1 & \longrightarrow & X_1 & \xrightarrow{f'_1} & X'_2 & \end{array} \tag{8}$$

$$
\begin{array}{ccccccccc}
W & \longrightarrow & X & \xrightarrow{g} & X_1 & & & & \\
\dot{\psi}\downarrow & & \psi\downarrow & & \tilde{\psi}\downarrow & & & & \\
W' & \longrightarrow & X & \xrightarrow{g'} & X'_1 & & & & \\
\dot{g}\searrow & & g\searrow & & g_1\searrow & & & & \\
\dot{g}'\searrow & & \dot{g}\searrow & & \dot{g}'_1\searrow & & & & \\
& W_1 & & \longrightarrow & X_1 & \xrightarrow{g} & X_2 & & \\
& \dot{\psi}_1\downarrow & & & \psi_1\downarrow & & \tilde{\psi}_1\downarrow & & \\
& W'_1 & & \longrightarrow & X_1 & \xrightarrow{g'_1} & X'_2 & &
\end{array}
\tag{9}
$$

where $\dot{\varphi}$ $\tilde{\varphi}$, $\dot{\psi}$, and $\tilde{\psi}$ are the isomorphisms induced according (3).

Now, it suffices to observe that $\dot{\varphi}_1 = \dot{\psi}_1$ and $\tilde{\varphi}_1 = \tilde{\psi}_1$.

For the converse, we consider sections $\sigma : X_1 \longrightarrow X$ and $\sigma' : X'_1 \longrightarrow X'$, $(\pi \circ \sigma = I_{X_1}, \pi' \circ \sigma' = I_{X'_1})$.

We have the following decompositions

$$
\begin{aligned}
X &= W \oplus \sigma X_1, \\
X' &= W' \oplus \sigma' X'_1.
\end{aligned}
\tag{10}
$$

We define $(\varphi, \psi) : X \longrightarrow X'$ as

$$
\begin{gathered}
(\varphi, \psi)(\sigma x_1) = \sigma'(\varphi_1, \psi_1)(x_1), \quad \forall x_1 \in X_1, \\
(\varphi, \psi)(w) = \alpha(w), \quad \forall w \in W.
\end{gathered}
\tag{11}
$$

where α is the isomorphism that join with the isomorphism β make commutative the following diagram.

$$
\begin{array}{ccc}
W & \overset{\dot{f}=\dot{g}}{\longrightarrow} & W_1 \\
\alpha\downarrow & & \beta\downarrow \\
W' & \overset{\dot{f}'=\dot{g}'}{\longrightarrow} & W'_1
\end{array}
\tag{12}
$$

Note that, the existence of maps α, β is ensured because rank $\dot{f}$ = rank $\dot{f}'$ and (f, g) and (f', g') are pairs of linear maps. $\square$

Let now, $(f, g) : X \longrightarrow X/W$ a pair of linear maps. Suppose that $f_{1|W_1} = g_{1|W_1}$, then we can consider the pair of linear maps $(f_1, g_1) : X_1 \longrightarrow X_1/W_1$ Then we have the following commutative diagram:

$$
\begin{array}{ccc}
W_1 & \overset{(\dot{f}_1,\dot{g}_1)}{\longrightarrow} & W_2 \\
\downarrow & & \downarrow \\
X_1 & \overset{(f_1,g_1)}{\longrightarrow} & X_1/W_1 = X_2 \\
\downarrow & & \downarrow \\
X_1/W_1 & \overset{(f_2,g_2)}{\longrightarrow} & X_2/W_2 = X_3
\end{array}
\tag{13}
$$

Inductively, we can consider the collection of pairs of linear maps defined modulo a subspace $(f, g), (f_1, g_1), \dots, (f_i, g_i)$ for all i such that $f_{|W} = g_{|W}, \dots, f_{i|W_i} = g_{i|W_i}$. We have three possibilities

i) there exists $s \in \mathbb{N}$ such that $f_{s-1|W_{s-1}} = g_{s-1|W_{s-1}} = X_s$. Then $X_{s+1} = \{0\}$ and $(f_{s+\ell}, g_{s+\ell}) = (0, 0)$, for all $\ell \geq 1$

ii) there exists $s \in \mathbb{N}$ such that $f_{s|W_s} = g_{s|W_s} = \{0\} = W_{s+1}$. Then (f_{s+1}, g_{s+1}) is a pair of endomorphisms.

iii) there exists $s \in \mathbb{N}$ such that $f_{s+1|W_{s+1}} \neq g_{s+1|W_{s+1}}$.

With the same notations, it is not difficult to prove the following Theorem.

Theorem 3.2. *Let $(f, g) : X \longrightarrow X/W$ and $(f', g') : X' \longrightarrow X'/W'$ equivalent pairs of linear maps. Then $(f, g) \sim (f', g')$ if and only if the following conditions*

i) $s = s'$,

ii) $\dim W_i = \dim W'_i$ *for all* i,

iii) $(f_i, g_i) \sim (f'_i, g'_i)$ *for all* $i = 1, \cdots, s$,

hold.

4 Matrix representation of pairs of linear maps defined modulo a subspace

Let $(f, g) : X \longrightarrow X/W$ a pair of linear maps. In order to obtain a matrix representation we consider pairs of bases of X adapted to W, that is to say bases $(b_f = \{e_1, \dots, e_n, e_{n+1}, \dots, e_{n+m}\}, b_g = \{\bar{e}_1, \dots, \bar{e}_n, e_{n+1}, \dots, e_{n+m}\})$ such that $\{e_{n+1}, \dots, e_{n+m}\}$ is a base for W, and $e_i - \bar{e}_i \in W$ for all $i = 1, \cdots, n$. Consequently $\{e_1 + W, \dots, e_n + W\} = \{\bar{e}_1 + W, \dots, \bar{e}_n + W\}$ is a base for X/W.

The matrices $\mathbf{A}_f$ and $\mathbf{A}_g$ of the linear maps f and g in this pair of bases are in the form

$$\mathbf{A}_f = \begin{pmatrix} A_1 & B \end{pmatrix}, \quad \mathbf{A}_g = \begin{pmatrix} A_2 & B \end{pmatrix}, \qquad (14)$$

with $A_1, A_2 \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$.

We will write simply as a triple of matrices (A_1, A_2, B).

Proposition 4.1. *Let (A_1, A_2, B) and (A'_1, A'_2, B') be two triples of matrices corresponding to the matrix representation of two equivalent pairs of maps (f, g) and (f', g') respectively. Then there exist invertible matrices $P \in Gl(n;\mathbb{C})$ $Q \in Gl(m;\mathbb{C})$ and rectangular matrices $F_1, F_2 \in M_{m\times n}(\mathbb{C})$ such that the following equality holds.*

$$\begin{pmatrix} A_1 & A_2 & B \end{pmatrix} = P^{-1} \begin{pmatrix} A'_1 & A'_2 & B' \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ F_1 & F_2 & Q \end{pmatrix}. \qquad (15)$$

Remark 4.1. Let (A_1, A_2, B) and (A'_1, A'_2, B') equivalent triples. Then the pairs of matrices (A_1, B) and (A_2, B) are feedback equivalent to (A'_1, B') and (A'_2, B') respectively. The converse is not true.

Proposition 4.2. *Let (f, g) be a pair of linear maps where* $\dim W = 1$*, $s = n$ ($(f_s, g_s) = (0, 0)$). Then there exists a pair of adapted bases (b_f, b_g) such that*

$$\begin{aligned}
\mathbb{A}_1 &= \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \\
\mathbb{A}_2 &= \begin{pmatrix} a_{1,1} & 1 & \dots & 0 & 0 \\ a_{2,1} & a_{2,2} & \dots & 0 & 0 \\ & \ddots & \ddots & & \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \qquad (16) \\
\mathbb{B} &= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.
\end{aligned}$$

Proof. We have that $W_s = X_s$. We consider the sections

$$\begin{aligned}
&\sigma'_s : W_s \longrightarrow X_{s-1}, \text{ such that } X_{s-1} = W_{s-1} \oplus \sigma'_s W_s, \\
&\sigma'_{s-1} : X_{s-1} \longrightarrow X_{s-2}, \text{ such that } X_{s-2} = W_{s-2} \oplus \sigma'_{s-1} X_{s-1}, \\
&\vdots \\
&\sigma'_1 : X_1 \longrightarrow X, \text{ such that } X = W \oplus \sigma'_1 X_1,
\end{aligned}$$

calling $\sigma_i = \sigma'_1 \circ \dots \circ \sigma'_i$ we have $X = W \oplus \sigma_1 W_1 \oplus \dots \oplus \sigma_s W_s$ and $f : \sigma_i(W_i) \longrightarrow \pi\sigma_{i+1}(W_{i+1})$.

We consider $b_f = \{w_1, \dots, w_{n+1}\}$, where $w_1 \in \sigma_s W_s$,

$$\begin{aligned}
w_2 &= f^{-1}\pi w_1 \in \sigma_{s-1} W_{s-1}, \\
&\dots \\
w_n &= f^{-1}\pi w_{n-1} \in \sigma_1 W_1, \\
w_{n+1} &= f^{-1}\pi w_n \in W.
\end{aligned}$$

So, taking into account that $f_{i|W_i} = g_{i|W_i}$,

$$\begin{aligned}
g(w_1) &= a_{1,1}w_1 + a_{2,1}w_2 + \dots + a_{n-1,1}w_{n-1} + a_{n,1}w_n, \\
g(w_2) &= w_1 + a_{2,2}w_2 \dots + a_{n-1,2}w_{n-1} + a_{n,2}w_n, \\
&\vdots \\
g(w_n) &= w_{n-1,n-1} + a_{n,n-1}w_n, \\
g(w_{n+1}) &= w_n.
\end{aligned}$$

Finally, taking $b_g = \{\bar{w}_1, \cdots, \bar{w}_n, w_{n+1}\}$ with $\bar{w}_i = w_i - a_{n,i}w_{n+1}$ for all $i = 1, \cdots, n$ we have that (A_1, A_2, B) has the desired form. □

Remark 4.2. Numbers a_{ij} in matrix $\mathbb{A}_2$ characterize the equivalence class of triples of matrices verifying proposition (4.2).

Corollary 4.1. *Let (A_1, A_2, B), $A_1, A_2 \in M_n(\mathbb{C})$, $B \in M_{n\times 1}(\mathbb{C})$ be a triple of matrices representing a pair of linear maps $(f, g) : X \longrightarrow X/W$. Then, $(A_1, A_2, B) \sim (\mathbb{A}_1, \mathbb{A}_2, \mathbb{B})$ if and only if*

$$\begin{aligned}
&\mathit{rank}\,(B) = 1, \\
&\mathit{rank}\begin{pmatrix} B & A_1B \end{pmatrix} = 2, \\
&\mathit{rank}\begin{pmatrix} B & A_2B \end{pmatrix} = 2, \\
&\mathit{rank}\begin{pmatrix} B & (A_1 - A_2)B \end{pmatrix} = 1, \\
&\mathit{rank}\begin{pmatrix} B & A_1B & A_1^2B \end{pmatrix} = 3, \\
&\mathit{rank}\begin{pmatrix} B & A_2B & A_2^2B \end{pmatrix} = 3, \\
&\mathit{rank}\begin{pmatrix} B & A_1B & (A_1^2 - A_2^2)B \end{pmatrix} = 2, \\
&\vdots \\
&\mathit{rank}\begin{pmatrix} B & A_1B & \dots & A_1^{n-1}B \end{pmatrix} = n, \\
&\mathit{rank}\begin{pmatrix} B & A_2B & \dots & A_2^{n-1}B \end{pmatrix} = n, \\
&\mathit{rank}\begin{pmatrix} B & A_1B & \dots & A_1^{n-2} & (A_1^{n-1} - A_2^{n-1})B \end{pmatrix} = n - 1.
\end{aligned} \qquad (17)$$

Proof. We observe that if $(A_1, A_2, B) \sim (A'_1, A'_2, B')$, then (A_1, A_2, B) verifies (17) if and only if (A'_1, A'_2, B') verifies.

So, suppose $(A_1, A_2, B) \sim (\mathbb{A}_1, \mathbb{A}_2, \mathbb{B})$, it suffices to compute

$$\begin{array}{l} \operatorname{rank} \mathbb{B} = 1, \\ \operatorname{rank} \begin{pmatrix} \mathbb{B} & \mathbb{A}_1\mathbb{B} \end{pmatrix} = 2, \\ \vdots \\ \operatorname{rank} \begin{pmatrix} \mathbb{B} & \mathbb{A}_1\mathbb{B} & \cdots & \mathbb{A}_1^{n-2}\mathbb{B} & (\mathbb{A}_1^{n-1} - \mathbb{A}_2^{n-1})\mathbb{B} \end{pmatrix} = n - 1. \end{array}$$

For the converse, if (A_1, A_2, B) is a triple verifying (17), it is not difficult to prove that $\dim W = 1$, $s = n$ and $f_{i|W_i} = g_{i|W_i}$. □

Example 4.1. Let (A_1, A_2, B) with

$$A_1 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

So,

$$\begin{array}{l} \operatorname{rank}(B) = 1, \\ \operatorname{rank} \begin{pmatrix} B & A_1B \end{pmatrix} = 2, \\ \operatorname{rank} \begin{pmatrix} B & A_2B \end{pmatrix} = 2, \\ \operatorname{rank} \begin{pmatrix} B & (A_1 - A_2)B \end{pmatrix} = 1, \\ \operatorname{rank} \begin{pmatrix} B & A_1B & A_1^2B \end{pmatrix} = 3, \\ \operatorname{rank} \begin{pmatrix} B & A_2B & A_2^2B \end{pmatrix} = 3, \\ \operatorname{rank} \begin{pmatrix} B & A_1B & (A_1^2 - A_2^2)B \end{pmatrix} = 2. \end{array}$$

Then, there exists a pair of basis (b_f, b_g) such that the triple is equivalent to $(\mathbb{A}_1, \mathbb{A}_2, \mathbb{B})$. In fact

$$\mathbb{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbb{A}_2 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbb{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Corollary 4.2. *Let (A_1, A_2, B) be a triple verifying (17). If rank $\begin{pmatrix} B & A_1 - A_2 \end{pmatrix} = 1$, then $(A_1, A_2, B) \sim \mathbb{A}_1, \mathbb{A}_2, \mathbb{B})$ with $\mathbb{A}_2 = \mathbb{A}_1$.*

Example 4.2. Let (A_1, A_2, B) with

$$A_1 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{7}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{3}{2} \end{pmatrix}, \ A_2 = \begin{pmatrix} 2 & -\frac{1}{2} & 2 \\ 2 & \frac{1}{2} & 5 \\ -1 & \frac{1}{2} & -3 \end{pmatrix}, \ B = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$$

So,

$$\begin{array}{l} \operatorname{rank} \begin{pmatrix} B & A_1 - A_2 \end{pmatrix} = 1, \\ \operatorname{rank}(B) = 1, \\ \operatorname{rank} \begin{pmatrix} B & A_1B \end{pmatrix} = 2, \\ \operatorname{rank} \begin{pmatrix} B & A_2B \end{pmatrix} = 2, \\ \operatorname{rank} \begin{pmatrix} B & (A_1 - A_2)B \end{pmatrix} = 1, \\ \operatorname{rank} \begin{pmatrix} B & A_1B & A_1^2B \end{pmatrix} = 3, \\ \operatorname{rank} \begin{pmatrix} B & A_2B & A_2^2B \end{pmatrix} = 3, \\ \operatorname{rank} \begin{pmatrix} B & A_1B & (A_1^2 - A_2^2)B \end{pmatrix} = 2. \end{array}$$

Then $(A_1, A_2, B) \sim (\mathbb{A}_1, \mathbb{A}_1, \mathbb{B})$.

5 Application to generalized linear systems

A generalized linear system is described by the following state space equation

$$E\dot{x} = Ax + Bu, \tag{18}$$

where E and A are n-square complex matrices and B a rectangular complex matrix in adequate size. We can represent it by a triple of matrices (E, A, B). Using theorem (3.1), we obtain sufficient conditions for controllability of this kind of systems.

Let $E\dot{x} = Ax + Bu$ be a generalized linear system, the standard transformations that can be applied are

1. basis change in the state space: $(E, A, B) \to (P^{-1}EP, P^{-1}AP, P^{-1}B)$,
2. basis change in the input space: $(E, A, B) \to (E, A, BQ)$,
3. feedback: $(E, A, B) \to (E, A + BF_2, B)$,
4. and derivative feedback: $(E, A, B) \to (E + BF_1, A, B)$.

We get the following definition of equivalence for generalized linear systems

Definition 5.1. *Two generalized linear systems (E, A, B), (E', A', B'), are equivalent if and only if there exist matrices $P \in Gl(n; \mathbb{C})$, $Q \in Gl(m; \mathbb{C})$ and $F_1, F_2 \in M_{m\times n}(\mathbb{C})$ such that these equalities*

$$\begin{aligned} E' &= P^{-1}EP + P^{-1}BF_1 \\ A' &= P^{-1}AP + P^{-1}BF_2 \\ B' &= P^{-1}BQ \end{aligned} \tag{19}$$

hold.

It is straightforward that this relation is an equivalence relation.

Then a generalized linear system can be interpreted as a matrix representation of a pair of linear maps defined modulo a subspace, and given two equivalent systems we can consider as two equivalent pairs of linear maps.

Notice that (19) have the following matrix expression

$$\begin{pmatrix} E' & A' & B' \end{pmatrix} = P^{-1} \begin{pmatrix} E & A & B \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ F_1 & F_2 & Q \end{pmatrix}$$

We are interested in to study the controllability of generalized linear systems. For that we remember the following Proposition (see [3]).

Proposition 5.1. *The generalized linear system (E, A, B), is controllable if and only if*

$$\begin{array}{l} \operatorname{rank} \begin{pmatrix} E & B \end{pmatrix} = n \\ \operatorname{rank} \begin{pmatrix} sE - A & B \end{pmatrix} = n \quad \forall s \in \mathbb{C} \end{array}$$

Remark that, the controllability of generalized linear systems is invariant under equivalence relation considered. In fact we have the following proposition (see [3]).

Proposition 5.2. *The rank of the matrix $\begin{pmatrix} sE - A & B \end{pmatrix}$ as well the rank of the matrix $\begin{pmatrix} E & B \end{pmatrix}$ are invariant under equivalence defined above.*

Proposition 5.3. *Let (E, A, B) be a generalized linear system. Let (f, g) a pair of linear maps such that the matrix representation with respect the canonical basis of $\mathbb{C}^n \times \mathbb{C}^m$ is (E, A, B). Suppose that the triple has the form of proposition 4.2.Then the generalized linear system is controllable if and only if $a_{i,i} \neq 0$ for all $i = 1, \cdots, n-1$.*

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