Synthesis of the inverted pendulum on New systhem theory

Andrea Fialová
Center of Applied Cybernetics
University Of West Bohemia, Czech Republic

ABSTRACT
The design of the control law (dynamical feedback control or only feedback control) is a natural method of cybernetics. However, some results are disputatious, mainly from point of view viability, causality etc. A rather dissimilar approach to design of the control law is presented in this paper. This method is grounded in system theory, where the system definition is based on observations nature. Thereafter the close loop system must be a system as whole and must fulfill the causality law. In this paper is presented method to design of the feedback control law for inverted pendulum which is connected with car.

KEY WORDS
Stabilization, Lyapunov function, non-linear system, system theory, discrete-time system, continuous-time system

1 Introduction

The synthesis of the control design for linear systems fall into often solved problem, but for non-linear systems are shortly discussed. One of them consists in approximative linearizing of the non-linear system to be stabilized about an operating point, and then linear feedback control methods are used to design a controller. This approach is successful in case of a system trajectory is restricted to a small neighborhood about the chosen operating point. The other methods are based on a transformation of a non-linear system into a suitable form. This way is presented in for example (see [6]), which method made by using of a transformation into "controllable like" canonical form.

The paper deals with the stabilization method which has a similar idea as the method mentioned in (see [4]). The main idea is in that the close loop system must be a system as whole and must fulfill the causality law. This we can easily guarantee on discrete-time. Than the continuous-time system is taken as limits of discrete-time (see [1]) and (see [2]).

2 Problem formulation

Consider a subsystem Σ_1

$$\Sigma_1 : \quad \begin{array}{l} \dot{x}(t) = F\left(x(t), u(t)\right) \\ y(t) = h(x(t)) \end{array} \tag{1}$$

where $F : R^{n+1} \to R^n$, $h : R^n \to R$ ($F, h \subset C^\infty$), x we call as a state of the subsystem Σ_1, y as output of the subsystem Σ_1 and u as input of the subsystem Σ_1. Further we assume that y and u are measurable.

Our aim is to design a subsystem Σ_2

$$\Sigma_2 : \quad \begin{array}{l} \dot{z}(t) = G\left(z(t), x(t)\right) \\ u(t) = \Phi\left(z(t), x(t)\right) \end{array} \tag{2}$$

where $G : R^{s+n} \to R^s$, $\Phi : R^{s+n} \to R$ ($G, \Phi \subset C^\infty$), z is a state of the subsystem Σ_2 and u is output of the subsystem Σ_2 and x is input of the subsystem Σ_2.

So that a system Σ, which is created from subsystem Σ_1 and Σ_2, fulfill the next conditions:

- Σ is a system, i.e. causal system,
- Σ is asymptotically stable.

3 Problem solution

The main idea, that the close loop system must be a system as whole and that must fulfill the causality law, is easily guarantee on discrete-time. Because this problem is as the first solved for the discrete-time system (see [1]) and consequently continuous-time system is taken as limits of discrete-time (see [2]).

And because we use with the non-linear systems, where working are more complicated we need a original continuous-time subsystem Σ_1 in special form (concretely in canonical form "controlable-like canonical form"):

$$\Sigma_1 : \begin{array}{lll} \dot{x}_i(t) & = x_{i+1}(t) & i = 1, \dots n-1 \\ \dot{x}_n(t) & = f_n(x(t)) + g_n(x(t))u(t) & \\ y(t) & = x_1(t) & \end{array} \tag{3}$$

In case where the original subsystem isn't in this form, we can use a state space transformation (see [6]).

In following sections we limit on situation, regarding query, when order of the subsystem Σ_2 is equal zero ($s = 0$), i.e. the equation (2) is $\Sigma_2 : u(t) = \Phi\left(x(t)\right)$.

Now we must compute a discrete-time system. This we obtain after application proces. For example, the derivation is made account of

$$\frac{dx}{dt} = \lim_{h \to 0} \frac{x_{t+h} - x_t}{h} \tag{4}$$

where h is a sample time. For this discrete-time system:

$$\begin{aligned}\Sigma_1 :\ x_i(t+h) &= x_i(t) + h\, x_{i+1}(t) \\ & \qquad i = 1, \dots\ n-1 \\ x_n(t+h) &= x_n(t) + h\, f_n(x(t)) + \\ & +h\, g_n(x(t)) u(t) \\ y(t) &= x_1(t)\end{aligned} \tag{5}$$

we compute a subsystem Σ_2:

$$\Sigma_2 :\ u(t) \quad = \Phi_D(x(t)) \tag{6}$$

using by method based on metrical equivalence (see [4]). Hence we assume that the system Σ (the close loop system Σ_1 with unknown control law Σ_2)

$$\begin{aligned}\Sigma :\ x_i(t+h) &= x_i(t) + h\, x_{i+1}(t) \\ & \qquad i = 1, \dots\ n-1 \\ x_n(t+h) &= x_n(t) + h\, f_n(x(t)) + \\ & +h\, g_n(x(t)) \Phi_D(x(t)) \\ y(t) &= x_1(t)\end{aligned} \tag{7}$$

is state equivalent with a system Σ^*, which we choose as asymptotically stable and **causal** system. This system can be non-linear and is in the other special form:

$$\begin{aligned}\Sigma^* : x_i^*(t+h) &= \alpha_i(x_i^*(t))\, x_i^*(t) \\ y^*(t) &= \textstyle\sum_1^n x_i^*\end{aligned} \tag{8}$$

with the Lyapunov function $V(x(t)) = \sum_{i=1}^n x_i^*(t)^2$ and $\triangle V(x(t)) = \sum_{i=1}^n [(\alpha_i(x_i^*(t))^2 - 1) x_i^*(t)^2]$. The non-linear function $\alpha_i(\cdot)$ are smooth and non-zero. For this system is important that the Lyapunov function is always in this special form. The proof and note for using this form we can see in [1].

Further from the state space equivalence ($\triangle^k y(t) = \triangle^k y^*(t)$ for $k = 0, \dots\ n-1$) of the systems (7) and (8) we compute a state transformation $x(t) = T^*(x^*)$. Then we obtain, from a relationship $\triangle^n y(t) = \triangle^n y^*(t)$, a non-linear function $\Phi_D(x)$ as:

$$\begin{aligned}\Phi_D(x) = (h^n\, g_n(x))^{-1} \Big[& h^n\, f_n(x) + x_1 + \\ + \tbinom{n}{1} h\, x_2 + \dots + & \tbinom{n}{n-1} h^{n-1}\, x_n + \\ - [\textstyle\sum_{i=1}^n \alpha_i^n(x_i^*(k))\, x_i^*(k)]\, & |_{x^* = T^{*-1}(x)} \Big]\end{aligned} \tag{9}$$

The proof you can see in [1] or [2].

Now the continuous-time system is regarded as limits of suitable set of the time. This limit proces is called the continualisation (see [5]) and (see [2]). Therefore the continuous-time control law we obtain from:

$$\Phi_C(x) = \lim_{h \to 0} h^n\, \Phi_D(x) \tag{10}$$

The proof of this is in [2].

4 Inverted pendulum

In this section which is a mainly part of this paper is presented results on the well-know example Inverted pendulum with the car. Employ a physical principle and we write this equations:

$$m \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(s(t) + l \sin(\varphi(t)) \right) = H(t) \tag{11}$$

$$m \frac{\mathrm{d}^2}{\mathrm{d}t^2} (l \cos(\varphi(t))) = V(t) - mg \tag{12}$$

$$\begin{aligned} J \tfrac{\mathrm{d}^2 \varphi(t)}{\mathrm{d}t^2} = \quad & V(t) l \sin(\varphi(t)) - \\ & -H(t) l \cos(\varphi(t))\end{aligned} \tag{13}$$

where significance of the variables are visible from picture 1.

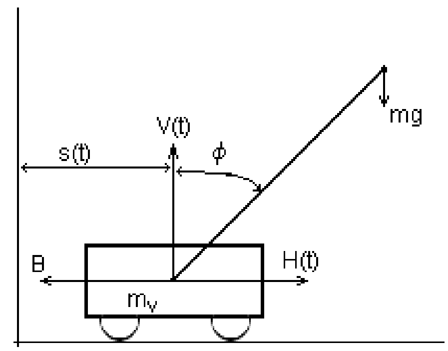

Fig. 1: Model inverted pendulum on the cart

From consequential modification we obtain the following equations:

$$\begin{aligned}\ddot{\varphi}(J + ml^2) \quad & +D\dot{\varphi} - mgl \sin(\varphi) + \\ & +ml \cos(\varphi) \ddot{s} = 0\end{aligned} \tag{14}$$

Now we must compute the further equations which describe a behavior of the car with engine. It is a more complicated, for this we would need a non-linear differential equation, order five. Hence we use a identification method for determination of this model, factually a AR model.

$$A(q^{-1}) z(t) = B(q^{-1}) u(t) + e(t) \tag{15}$$

where $A(\cdot)$ and $B(\cdot)$ are a relevant polynomials.

The result is in [3] and it can be write in form:

$$\frac{Z(z)}{U(z)} = \frac{0.00050676 z^2 + 0.001144 z + 0.0001227}{z^3 - 1.994 z^2 + 1.048 z - 0.05329} \tag{16}$$

with the sample time $h = 0.01s$. This is typical for the identification that the result is in discrete-time.

Thank to here presented method we cannot compute the continuous-time equations but contrariwise we apply a relationship (4) to the equation (14) and after manipulation we obtain a state space subsystem Σ_1:

$$\begin{aligned} x_1(t+h) &= x_2(t) \\ x_2(t+h) &= x_3(t) \\ x_3(t+h) &= 0.05329x_1(t) - 1.048x_2(t)+ \\ &\quad +1.994x_3(t) + u \\ x_4(t+h) &= x_4(t) + 0.01x_5(t) \\ x_5(t+h) &= 0.9961x_5(t)+ \\ &\quad +0.89143\sin(x_4)- \\ &\quad -0.09084\cos(x_4)\,x_3 \\ y_1(t) &= 0.0001227x_1(t)+ \\ &\quad +0.001144x_2(t)+ \\ &\quad +0.00050676x_3(t) \\ y_2(t) &= x_4(t) \end{aligned} \tag{17}$$

where $y_1(t) = s(t)$ and $y_2(t) = x_4(t) = \varphi(t)$, $D = 0.012kg\,m\,s^{-1}$, $m = 0.063\,kg$, $l = 0.44m$, $J = 0.0184924\,kg$. We can see that this is a multi-output system, but for a design of a control law we use only a output $y_2(t)$.

Complication is a relative degree of the subsystem Σ_1. Because for the design a control law we use the subsystem $\Sigma_{1_{red}}$, order 3 in form:

$$\Sigma_{1_{red}}: \begin{aligned} \dot{\bar{x}}_1 &= \bar{x}_2 \\ \dot{\bar{x}}_2 &= \bar{x}_3 \\ \dot{\bar{x}}_3 &= 2613.7\sin(\bar{x}_1) - 114.2014\bar{x}_2+ \\ &\quad +8.9143\cos(\bar{x}_1)\,\bar{x}_2 + 5060.7\cos(\bar{x}_1)\,u \\ \bar{y} &= \bar{x}_1 \end{aligned} \tag{18}$$

where the state space transformation $\bar{x} = \bar{T}(x)$ is equal:

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ \diamond \end{bmatrix} \tag{19}$$

($\diamond = -0.3895x_5 + 8.9143\sin(x_4) - 0.9084\cos(x_4)\,x_3$)

Therefore the discrete-time system of this subsystem (18), after application a (19), is in form:

$$\Sigma_{1_{red}}: \begin{aligned} \bar{x}_1(t+h) &= \bar{x}_1(t) + 0.01\,\bar{x}_2(t) \\ \bar{x}_2(t+h) &= \bar{x}_2(t) + 0.01\,\bar{x}_3 \\ \bar{x}_3(t+h) &= \bar{x}_3(t) + 26.137\sin(\bar{x}_1)- \\ &\quad -1.142014\bar{x}_2+ \\ &\quad +0.089143\cos(\bar{x}_1)\,\bar{x}_2+ \\ &\quad +50.607\cos(\bar{x}_1)\,u \\ \bar{y} &= \bar{x}_1 \end{aligned} \tag{20}$$

And now we can use this method, i.e. we apply a unknown control law $\Phi_D(\bar{x})$. Then this close loop system:

$$\Sigma_{1_{red}}: \begin{aligned} \bar{x}_1(t+h) &= \bar{x}_1(t) + 0.01\,\bar{x}_2(t) \\ \bar{x}_2(t+h) &= \bar{x}_2(t) + 0.01\,\bar{x}_3 \\ \bar{x}_3(t+h) &= \bar{x}_3(t) + 26.137\sin(\bar{x}_1)- \\ &\quad -1.142014\bar{x}_2+ \\ &\quad +0.089143\cos(\bar{x}_1)\,\bar{x}_2+ \\ &\quad +50.607\cos(\bar{x}_1)\,\Phi_D(\bar{x}) \\ \bar{y} &= \bar{x}_1 \end{aligned} \tag{21}$$

we use for design a control law. So that we choose the system Σ^*, which described a requested behaviour of a close loop system, for example:

$$\Sigma^*: \begin{aligned} x_1^*(t+h) &= 0.4x_1^*(t) \\ x_2^*(t+h) &= 0.5x_2^*(t) \\ x_3^*(t+h) &= 0.6x_3^*(t) \\ y^*(t) &= x_1^*(t) + x_2^*(t) + x_3^*(t) \end{aligned} \tag{22}$$

The state space transformation we obtain from relationship $\triangle^k\bar{y}(t) = \triangle^k y^*(t)$ for $k = 0,\,1,\,2$ and after manipulation we can write state space transformation equations as:

$$\begin{aligned} x_1^* &= 10\bar{x}_1 + 0.4481\bar{x}_2 + 0.005\bar{x}_3 \\ x_2^* &= -24\bar{x}_1 - 0.9962\bar{x}_2 + 0.01\bar{x}_3 \\ x_3^* &= 15\bar{x}_1 + 0.5481\bar{x}_5 + 0.005\bar{x}_3 \end{aligned} \tag{23}$$

Now we substitute from (21) and (22) (with using (23)) to the relationship (9) and after manipulation we obtain a control law in discrete-time in form:

$$\begin{aligned} \Phi_D(\bar{x}) &= (50.607\cos(\bar{x}_1))^{-1}\Big[0.88\bar{x}_1+ \\ &\quad +1.1626\bar{x}_2 - 0.9998\bar{x}_3- \\ &\quad -26.137\sin(\bar{x}_1)- \\ &\quad -0.089\cos(\bar{x}_1)\,\bar{x}_2\Big] \end{aligned} \tag{24}$$

And as the last we substitute the previous relationship (24) into (10) with applying (19) and compute a control law for continuous-time system in original coordinates. Then this control law is equal:

$$\begin{aligned} \Phi_C(x) &= (5060.7\cos(x_4))^{-1}\Big[0.1517x_5- \\ &\quad -3.4721\sin(x_4)+ \\ &\quad +266.6967\cos(x_4)\,x_3+ \\ &\quad +8.9143\cos(x_4)\,x_5\Big] \end{aligned} \tag{25}$$

This control law (25) we use for the simulation of this problem now. But for the simulation we need the whole system continuous-time or discrete-time. This simulation results are presented for the continuous-time system, where the equation (16) is use in macro (Matlab) `d2c`. Now we can use this macro because we know that the whole system (close loop) is a causal system!

In the figure 2 we can showed simulation of the position of the car $s(t)$. In the next figure 3 we can showed simulation the amplitude of the phase $\varphi(t)$.

5 Conclusion

The design of the control law (dynamical feedback control or feedback control) is a natural method of cybernetics. A rather dissimilar approach to design of the control law is presented in this paper. This method is grounded in new system theory, where the system definition is based on observations nature. Thereafter the close loop system must be a system as whole and must fulfill the causality law.

In this paper is presented method to design of the feedback control law for problem of the inverted pendulum with the car. This way adventitious control is part of whole system, with respect a causal law.

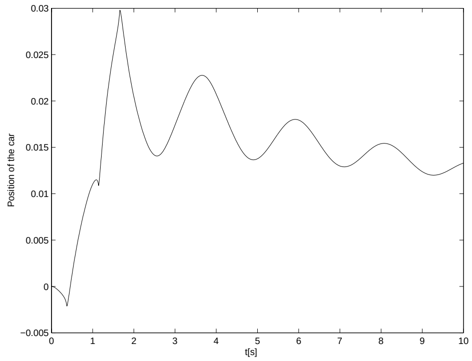

Fig 2.: Course of the position of the car $s(t) = y_1(t)$

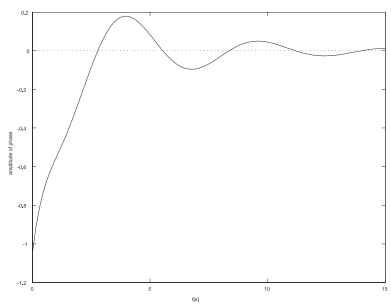

Fig 3.: Course of the amplitude of the phase $\varphi(t) = y_2(t)$

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