On Realization of 2D Filters by Roesser Model

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Abstract: A constructive procedure for realization of two-dimensional (2D) general filters is proposed which may lead to a Roesser local state-space model with lower order than the existing realization procedures. An illustrative example shows that minimal realization, which cannot be reached by the existing algorithms, may also be obtained for a certain class of 2D filters by our method.

Keywords: Realization; Roesser local state-space model; 2D filters.

1 Introduction

One of the fundamental issues in two-dimensional (2D) systems theory is the realization of a given transfer function by a certain kind of 2D local state-space model, typically by Roesser model or Fornasini-Marchesini second (FM-II) model (see, e.g., [3]-[14]). Unlike the one-dimensional (1D) case, it is not always possible to obtain a minimal state-space realization for an arbitrary filter in the 2D case. Minimal realization can only be reached for some particular categories of 2D systems, e.g., continued fraction expandable systems, all-pole, and all-zero filters [1–12]. Therefore, it is desirable to obtain a state-space realization with as low order as possible, and ideally with the minimal one.

In this paper, we are concerned with the realization problem for single-input single-output (SISO) systems by Roesser model. In Section 2 some preliminaries are briefly presented. In Section 3, a new constructive realization procedure is proposed which may lead to a Roesser local state-space model with lower order than the existing realization procedures [1, 2]. In Section 4, an illustrative example is provided to show the details and effectiveness of the proposed procedure.

2 Preliminaries

The 2D Roesser model for a 2D SISO system is described by [13]

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = A \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Bu(i,j) \quad (1a)$$

$$y(i,j) = C \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Du(i,j) \quad (1b)$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

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$x^h(i,j) \in R^{n_1}$ is the horizontal state vector; $x^v(i,j) \in R^{n_2}$ is the vertical state vector, $u(i,j)$ is the input, $y(i,j)$ is the output, and A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, A, B, C, D are all real matrices of appropriate dimensions. The transfer function of (1) is

$$G(z_1, z_2) = CZ\,(I - AZ)^{-1}\,B + D \qquad (2)$$

with $Z = \mathrm{diag}\{z_1 I_{n_1}, z_2 I_{n_2}\}$.

Let $G(z_1, z_2)$ be a general 2D transfer function expressed as follows.

$$\begin{aligned} G(z_1, z_2) =& \frac{n(z_1, z_2)}{d(z_1, z_2)} \\ =& \frac{n_{10}z_1 + n_{01}z_2 + \cdots + n_{mn}z_1^m z_2^n}{1 + d_{10}z_1 + d_{01}z_2 + \cdots + d_{mn}z_1^m z_2^n} \end{aligned} \qquad (3)$$

As it is ready to get $D = G(0,0)$ from (2), we assume here without loss of generality that the transfer function $G(z_1, z_2)$ under investigation is strictly causal, which means that $n(0,\,0) = 0$. It is also assumed without loss of generality that $d(0,0) = 1$. The realization problem now becomes how to find the real matrices A, B, C such that

$$G(z_1, z_2) = CZ(I - AZ)^{-1}B \qquad (4)$$

Currently, there are mainly two techniques leading to the realization of a general 2D transfer function with Roesser Model [1, 2]. Both of them dealt with the considered 2D transfer function (3) as a transfer function of one parameter (for example, z_1) over the field of rational functions in the other parameter (i.e., z_2).

The first realization technique [1] reduces essentially to an implementation method for a 2D rational transfer function using some delay elements z_1 and z_2. The circuit implementation corresponds directly to a Roesser model. The order of the realization is $m+2n$ (or $n+2m$).

The second technique [2] is based on two-step realization. The first-level realization provides a 1D system with matrices having entries on the field of rational functions in one variable, and the second-level realization gives the system matrices in Roesser's form. The order of the realization is also $m+2n$ (or $n+2m$).

For the transfer function of a particular system, there may be some terms absent in the polynomials $d(z_1, z_2)$ and $n(z_1, z_2)$. That is to say, some power products $z_1^h z_2^k$ may have zero coefficient in the polynomials. It is natural to consider that in such cases it may be possible to have a realization with lower order. In the next section, a new realization constructive procedure will be proposed for 2D general filters, by which a Roesser state-space model with lower order may be achieved.

3 New Constructive Realization Procedure

Consider a general 2D transfer function

$$G(z_1, z_2) = \frac{n(z_1, z_2)}{d(z_1, z_2)} \qquad (5)$$

where it is assumed without loss of generality that $d(0,0) = 1$ and $n(0,0) = 0$. A realization of $G(z_1, z_2)$ can be constructively obtained by the following procedure.

Step 1. Collect all the power products $z_1^h z_2^k$ with non-zero coefficients occurring in both $d(z_1, z_2)$ and $n(z_1, z_2)$, and construct the column vectors $\tilde{\Psi}_1$, $\tilde{\Psi}_2$ and $\tilde{\Psi}_{12}$ by putting the collected power products z_1^h, z_2^k into $\tilde{\Psi}_1$, $\tilde{\Psi}_2$ according to the **descending** total degree lexicographic order and $z_1^h z_2^k$ into $\tilde{\Psi}_{12}$ according to the **ascending** total degree lexicographic order, respectively. It is assumed, without loss of generality, that the order of z_2 is higher than z_1 in this paper. Let $\Psi_\delta = z_\delta^{-1}\tilde{\Psi}_\delta$, $\delta = 1, 2$ and n_δ, n_{12} be the dimensions of the vectors Ψ_δ and $\tilde{\Psi}_{12}$, respectively, i.e., $\Psi_\delta \in R^{n_\delta}$, $\tilde{\Psi}_{12} \in R^{n_{12}}$, $\delta = 1, 2$. Denote the jth element of Ψ_δ by $\Psi_\delta(j)$, $j = 1, \cdots, n_\delta$. Note that $\Psi_1(1) = z_1^{h_1}$ and $\Psi_2(1) = z_2^{k_1}$ have the highest order among the elements of Ψ_1 and Ψ_2, respectively. In the case that there is no collected power product to put into, e.g., Ψ_δ, we denote it as an empty vector by $\Psi_\delta = \emptyset$ and set the dimension of Ψ_δ to zero, i.e., $n_\delta = 0$.

Step 2. Fill all absent power products z_1^h, $0 \leq h < h_1$ into Ψ_1 following the descending total degree lexicographic order. Carry out the same operation for Ψ_2. Thus the dimensions of the vectors Ψ_δ, $\delta = 1,\ 2$ are $n_1 = h_1 + 1$ and $n_2 = k_1 + 1$, respectively. In the case when Ψ_δ is empty, however, do not carry out the filling operation.

Step 3. $j = 0$. If $\tilde{\Psi}_{12} \neq \emptyset$, proceed to Step 4. Otherwise, go to Step 6.

Step 4. $j = j + 1$.

If $j > n_{12}$, go to Step 6.

Otherwise, for the jth entry of $\tilde{\Psi}_{12}$, say $\tilde{\Psi}_{12}(j) = z_1^h z_2^k$, verify whether there exists $1 \leq j_1 \leq n_1$ or $1 \leq j_2 \leq n_2$ such that (a) or (b) is satisfied.

(a) $z_1^{-1}\tilde{\Psi}_{12}(j) = z_1\Psi_1(j_1)$ or $z_1^{-1}\tilde{\Psi}_{12}(j) = z_2\Psi_2(j_2)$

(b) $z_2^{-1}\tilde{\Psi}_{12}(j) = z_1\Psi_1(j_1)$ or $z_2^{-1}\tilde{\Psi}_{12}(j) = z_2\Psi_2(j_2)$

- If yes, then insert $z_1^{-1}\tilde{\Psi}_{12}(j) = z_1^{h-1}z_2^k$ into Ψ_1 and set $n_1 = n_1+1$ when (a) is satisfied, or insert $z_2^{-1}\tilde{\Psi}_{12}(j) = z_1^h z_2^{k-1}$ into Ψ_2 and set $n_2 = n_2 + 1$ when (b) is satisfied, according to the descending total degree lexicographic order.

  For the case that both the conditions (a) and (b) are satisfied, insert $z_1^{-1}\tilde{\Psi}_{12}(j)$ into Ψ_1 at an appropriate position and set $n_1 = n_1 + 1$ when $h \geq k$, or insert $z_2^{-1}\tilde{\Psi}_{12}(j)$ into Ψ_2 at an appropriate position and set $n_2 = n_2 + 1$ when $h < k$.

  Repeat Step 4.
- If no, go to Step 5.

Step 5. Insert $z_1^{-1}\tilde{\Psi}_{12}(j)$ into Ψ_1 and set $n_1 = n_1+1$ when $h \geq k$, or insert $z_2^{-1}\tilde{\Psi}_{12}(j)$ into Ψ_2 and set $n_2 = n_2 + 1$ when $h < k$, according to the descending total degree lexicographic order. Meanwhile, also insert the corresponding power product into $\tilde{\Psi}_{12}$ as the $(j+1)$th element $\tilde{\Psi}_{12}(j+1)$ and set $n_{12} = n_{12} + 1$. (Note that we do not follow the total degree lexicographic order here.)

Return to Step 4.

Step 6. Denote $\Psi = [\Psi_1^T \ \ \Psi_2^T]^T$. Let $\tilde{d}(z_1, z_2) = 1 - d(z_1, z_2)$ and express $\tilde{d}(z_1, z_2)$ and $n(z_1, z_2)$ in the forms of

$$\tilde{d}(z_1, z_2) = D_{HT} Z \Psi \qquad (6)$$

$$n(z_1, z_2) = N_{HT} Z \Psi \qquad (7)$$

where $Z = \mathrm{diag}\{z_1 I_{n_1}, z_2 I_{n_2}\}$, and D_{HT}, $N_{HT} \in R^{1\times(n_1+n_2)}$ are row vectors whose entries are the corresponding coefficients of $\tilde{d}(z_1, z_2)$ and $n(z_1, z_2)$, respectively.

Step 7. Let $\tau \in R^{n_1+n_2}$. Set the initial value of $\tau(i) = 0$ for $i = 1, \cdots, n_1+n_2$. Then for $i = 1, \cdots, n_1+n_2$, let $\tau(i) = h$ if there exists a $\Psi(h)$ such that

$$\Psi(i) = z_1\Psi(h) \text{ with } 1 \leq h \leq n_1, \quad (8)$$

or, in the case that (8) is not satisfied,

$$\Psi(i) = z_2\Psi(h) \text{ with } n_1+1 \leq h \leq n_1+n_2. \quad (9)$$

Construct the matrix $A_0 \in R^{(n_1+n_2)\times(n_1+n_2)}$ and the column vector $B \in R^{n_1+n_2}$ by the following method. Set initially $A_0(i, j) = 0$ and $B(i) = 0$, $i, j = 1, \ldots, n_1 + n_2$. For $i = 1, \ldots, n_1 + n_2$, let $A(i, h) = 1$ if $\tau(i) = h \neq 0$, and let $B(i) = 1$ if $\Psi(i) = 1$.

Similar to the basic idea of [14], we have that

$$\Psi\frac{1}{d(z_1, z_2)} = (I - AZ)^{-1}B \quad (10)$$

with $A = A_0 + BD_{HT}$.

It follows from (7) and (10) that

$$\begin{aligned} G(z_1, z_2) &= \frac{n(z_1, z_2)}{d(z_1, z_2)} = N_{HT}Z\Psi\frac{1}{d(z_1, z_2)} \\ &= CZ(I - AZ)^{-1}B \quad (11) \end{aligned}$$

with $C \triangleq N_{HT}$.

That is, (A, B, C) gives a realization for $G(z_1, z_2)$.

4 An Illustrative Example

A simple example is shown here to illustrate some details of the procedure.

Example 1. Consider the transfer function

$$\begin{aligned} G(z_1, z_2) =& \frac{n(z_1, z_2)}{d(z_1, z_2)} \\ =& \frac{n_{01}z_2 + n_{11}z_1z_2 + n_{12}z_1z_2^2}{1 + d_{01}z_2 + d_{11}z_1z_2 + d_{12}z_1z_2^2}. \end{aligned}$$

- As the power products in $d(z_1, z_2)$ and $n(z_1, z_2)$ with non-zero coefficients are $\{z_2,\ z_1z_2,\ z_1z_2^2\}$, we have that

  $$\tilde{\Psi}_1 = \emptyset, \quad \tilde{\Psi}_2 = [z_2], \quad \tilde{\Psi}_{12} = [z_1z_2,\ z_1z_2^2]^T$$

  and $\Psi_1 = z_1^{-1}\tilde{\Psi}_1 = \emptyset$, $\Psi_2 = z_2^{-1}\tilde{\Psi}_2 = [1]$ with $n_1 = 0$, $n_2 = 1$.

- First, consider the entry $\tilde{\Psi}_{12}(1) = z_1z_2$. Because $z_1^{-1}\tilde{\Psi}_{12}(1) = z_2\Psi_2(1)$, insert $z_1^{-1}\tilde{\Psi}_{12}(1) = z_2$ into Ψ_1 to get an updated Ψ_1 as $\Psi_1 = [z_2]$ with $n_1 = 0 + 1 = 1$.

  Then, consider the entry $\tilde{\Psi}_{12}(2) = z_1z_2^2$. Because $z_2^{-1}\tilde{\Psi}_{12}(2) = z_1\Psi_1(1)$, insert $z_2^{-1}\tilde{\Psi}_{12}(2) = z_1z_2$ into Ψ_2 to get an updated Ψ_2 as $\Psi_2 = [z_1z_2 \quad 1]^T$ with $n_2 = 1 + 1 = 2$.

- Denote

  $$\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ \hline z_1z_2 \\ 1 \end{bmatrix}$$

  Next, let $\tilde{d}(z_1, z_2) = 1 - d(z_1, z_2)$ and express $\tilde{d}(z_1, z_2)$ and $n(z_1, z_2)$ in the forms of

$$\begin{aligned}\tilde{d}(z_1,z_2) &= -d_{01}z_2 - d_{11}z_1z_2 - d_{12}z_1z_2^2\\ &= \begin{bmatrix}-d_{11} & -d_{12} & -d_{01}\end{bmatrix}\begin{bmatrix}z_1 & 0 & 0\\ 0 & z_2 & 0\\ 0 & 0 & z_2\end{bmatrix}\begin{bmatrix}z_2\\ z_1z_2\\ 1\end{bmatrix}\\ &\triangleq D_{HT}Z\Psi\end{aligned}$$

$$\begin{aligned}n(z_1,z_2) &= n_{01}z_2 + n_{11}z_1z_2 + n_{12}z_1z_2^2\\ &= \begin{bmatrix}n_{11} & n_{12} & n_{01}\end{bmatrix}\begin{bmatrix}z_1 & 0 & 0\\ 0 & z_2 & 0\\ 0 & 0 & z_2\end{bmatrix}\begin{bmatrix}z_2\\ z_1z_2\\ 1\end{bmatrix}\\ &\triangleq N_{HT}Z\Psi.\end{aligned}$$

- Let $\tau \in R^3$. Set the initial value of $\tau(i) = 0$ for $i = 1, 2, 3$. Since $\Psi(1) = z_2\Psi(3)$, $\Psi(2) = z_1\Psi(1)$ and $\Psi(3) = 1$, we have that $\tau(1) = 3$, $\tau(2) = 1$ and $\tau(3) = 0$.

  Based on the values of $\tau(i)$, we can now construct the matrix A_0 as follows.

$$A_0 = \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

  Construct the column vector B as

$$B = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$$

  because $\Psi(3) = 1$.

  The matrix A can now be obtained as

$$A = A_0 + D_{HT}B = \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ -d_{11} & -d_{12} & -d_{01}\end{bmatrix}$$

It is easy to verify that

$$\begin{aligned}&(I - AZ)^{-1}B\\ &= \begin{bmatrix}1 & 0 & -z_2\\ -z_1 & 1 & 0\\ d_{11}z_1 & d_{12}z_2 & 1 + d_{01}z_2\end{bmatrix}^{-1}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\\ &= \Psi\frac{1}{d(z_1,z_2)}.\end{aligned}$$

Then it is straightforward to get that

$$G(z_1,z_2) = CZ(I - AZ)^{-1}B$$

with $C = N_{HT}$, and (A, B, C) giving a realization for $G(z_1, z_2)$.

Remark 1. *As shown in Example 1, the order of the realization by the proposed constructive procedure is 3, which is lower than the one ($m + 2n = 5$) obtained by existing procedures [1, 2]. It is obvious that the order given by our procedure closely relates to the number of the terms absent in $d(z_1, z_2)$ and $n(z_1, z_2)$, and generally the more the absent terms are, the lower the obtained order may be. It should be noted, however, that if no or few terms are absent, then the order of the realization due to our procedure may be higher than that obtained by the procedures of [1, 2].*

Remark 2. *It is observed that the Roesser model obtained in Example 1 is in fact a minimal realization of the considered transfer function. It is ready to see that if for an $(m + n)$-order transfer function, the dimension of the constructed vector Ψ is $n + m$ as well, i.e., $n_1 + n_2 = m + n$, then the minimal realization of Roesser model can be achieved.*

It should be mentioned that $G(z_1, z_2)$ used in Example 1 does not satisfy the conditions required by the methods of [3–12], and thus the minimal realization cannot be reached by these methods.

5 Conclusions

A constructive realization procedure has been proposed for the 2D general filters. Based on the procedure, a Roesser model with lower order than the existing procedures may be obtained. An example was presented to illustrate the effectiveness of the proposed realization procedure.

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