Control Theory for A Class

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Abstract: - In this paper, the VH^∞ control theory on an infinite dimensional algebra to itself is investigated. In order to establish the VH^∞ control theory, the concept of a meromorphic mapping on an infinite dimensional algebra to itself and the theory of VH^p spaces are presented.

Key words: - Infinite dimensional algebra, vector modulus, control theory, VH^∞ space, model-matching problem, optimal controller.

1 Introduction

The theory of H^p-spaces and the H^∞ control theory on one-dimensional space with range in finite dimensional spaces have been summarized by C. G. Hu and C. C. Yang, and B. A. Francis and J. C. Doyle in [8] and [2,3] respectively. In 1993, B. V. Keulen [9] extended the classical H^∞ control theory to its range in the infinite dimensional Hilbert space. In 2002, C. G. Hu and L. X. Ma [7] extended the result of Keulen [9] to the locally convex space containing the Hilbert space. In this article, new VH^∞ control theory on infinite dimensional algebras is established. For this aim, a meromorphic mapping on an infinite dimensional algebra without appearance in [1,10] is defined firstly, and the theory of VH^p-spaces on an infinite dimensional Fréchet algebra to itself is obtained. It follows that the VH^∞ control theory on an infinite dimensional algebra to itself is presented by using the above concepts and theory.

The paper is organized as follows. In Section 2 meromorphic mappings and properties on the infinite dimensional Fréchet algebra to itself are introduced. Section 3 contains the concept and properties of VH^p-spaces on the infinite dimensional Fréchet algebra. In Section 4 the optimal solutions and the infimal model-matching error of the VH^∞ control theory are presented.

2 Meromorphic mappings

Let $\mathcal{S}$ be the sequence space of all complex variables. Here $s = (s_1, s_2, \ldots, s_i, \ldots) \in \mathcal{S}$ and $s_i \in \mathbb{C}_i$(complex plane) for any i. If $s = (s_1,s_2, \ldots,s_i, \ldots) \in \mathcal{S}$, then the quasinorm over $\mathcal{S}$ is defined by

$$\|s\| = \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^i \frac{|s_i|}{1+|s_i|}.$$

The multiplication of s and w in $\mathcal{S}$ can be defined by

$$sw = (s_1w_1, s_2w_2, \ldots, s_iw_i, \ldots),$$

where $w = (w_1, \ldots, w_i, \ldots) \in \mathcal{S}$.

Obviously from the definition of the multiplication we may derive

$$\|sw\| \le \|s\|\|w\|,$$

$$s^\kappa = (s_1^\kappa, \ldots, s_i^\kappa, \ldots)$$

for any $\kappa > 0$. Thus $\mathcal{S}$ is a Fréchet algebra.

Assume for convenience sake, that $L = \prod_{i=1}^{\infty} L_i$ is a manifold over $\mathcal{S}$, where each $L_i \subset \mathbb{C}_i$ is a simple path, and that $f(t) = (f_1(t), \ldots, f_i(t), \ldots) : L \to \mathcal{S}$, and $t = (t_1, t_2, \ldots, t_i, \ldots) \in \mathcal{S}$. Let $\mathfrak{D}_s = \prod_{i=1}^{\infty} \mathfrak{D}_{si}$ be a domain over $\mathcal{S}$. Here $\mathfrak{D}_{si}$ is a domain over $\mathbb{C}_i$.

Lemma 2.1. *A mapping $f : \mathfrak{D}_s \to \mathcal{S}$ is holomorphic if and only if f can be denoted by*

$$f(s) = (f_1(s_1), f_2(s_2), \ldots, f_i(s_i), \ldots) \in \mathcal{S},$$

where $s = (s_1, s_2, \ldots, s_i, \ldots) \in \mathcal{S}$ and $f_i(s_i) : \mathbb{C}_i \to \mathbb{C}_i$ is a holomorphic function.

Proof. If f is a holomorphic mapping in $\mathfrak{D}_s$, then for any fixed $s_0 = (s_{01}, s_{02}, \ldots, s_{0i}, \ldots) \in \mathfrak{D}_s$, there exists a neighborhood $U(s_0)\subset\mathfrak{D}_s$ such that

$$\begin{aligned} f(s) &= \textstyle\sum_{k=0}^{\infty}\text{ß}_k(s-s_0)^k \\ &= \textstyle\sum_{k=0}^{\infty}(\text{ß}_{k1},\text{ß}_{k2},\ldots,\text{ß}_{ki},\ldots) \\ &\quad ((s_1-s_{01})^k,\ldots,(s_i-s_{0i})^k,\ldots)) \\ &= \big(\textstyle\sum_{k=0}^{\infty}\text{ß}_{k1}(s_1-s_{01})^k,\ldots, \\ &\quad \textstyle\sum_{k=0}^{\infty}\text{ß}_{ki}(s_i-s_{0i})^k,\ldots\big) \\ &= \big(f_{01}(s_1-s_{01}),f_{02}(s_2-s_{02}),\ldots, \\ &\quad f_{0i}(s_i-s_{0i}),\ldots\big) \end{aligned}$$

for $s\in U(s_0)$, where $\text{ß}_k=(\text{ß}_{k1},\text{ß}_{k2},\ldots,\text{ß}_{ki},\ldots)\in\mathcal{S}$ and

$$f_{0i}(s_i-s_{0i})=\sum_{k=0}^{\infty}\text{ß}_{ki}(s_i-s_{0i})^k.$$

Analytic continuation and the Uniqueness Theorem of holomorphic mappings in complex analysis yield that $f(s)$ can be written as

$$f(s)=\big(f_1(s_1),f_2(s_2),\ldots,f_i(s_i),\ldots\big)\in\mathcal{S}.$$

This is just required conclusion.

Conversely, because the above each step is invertible, f is holomorphic in $\mathfrak{D}_s$. This proof is ended. □

A meromorphic mapping on $\mathcal{S}$ without appearance in [1,10] may be defined as follows:

Definition 2.1. *A mapping f on $\mathcal{S}$ is called meromorphic if its each component $f_i(s_i)$ is a meromorphic mapping of s_i over $\mathbb{C}_i$ for each i.*

Remark. Using the similar method to Definition 2.1 we may define a meromorphic mapping on a domain $\mathfrak{D}_s=\prod_{i=1}^{\infty}\mathfrak{D}_{si}$, where $\mathfrak{D}_{si}$ is a domain over $\mathbb{C}_i$ for each i.

Let

$$\Big\{s(\imath)^{(0)}\Big\}_{\imath=1}^{\infty}=\Big\{(s_{\imath1}^{(0)},\ldots,s_{\imath i}^{(0)},\ldots)\Big\}_{\imath=1}^{\infty} \tag{2.1}$$

be an increasing sequence of distinct complex elements tending to the infinity $\infty=(\infty,\ldots,\infty,\ldots)$. From the above definition for convenience sake, without loss generality we may assume that every f_i is a meromorphic function of s_i which can be written for any i as

$$\sum_{\imath=1}^{\infty}\sum_{k=-m_{\imath i}}^{\infty}\text{ß}_{\imath ki}(s_i-s_{\imath i}^{(0)})^k,$$

where the following conditions are satisfied:

$(\alpha)\{s_{\imath i}^{(0)}\}$ for any $\imath,i$ has no any finite limit point;

$(\beta)-\infty<\inf\{-m_{\imath i}\}$.

Under the preceding conditions $s(\imath)^{(0)}$ is called a pole of f.

On a meromorphic mapping $f(s)$ on $\mathcal{S}$ there is the following conclusion.

Lemma 2.2. *Let $\{s(\imath)^{(0)}\}_{\imath=1}^{\infty}$ satisfy (α)–(β), and let*

$$\Big\{\hbar_{(\imath)}(s-s(\imath)^{(0)})\Big\}_{\imath=1}^{\infty}=\Big\{(h_{\imath1}(s_1-s_{\imath1}^{(0)}),\ldots,h_{\imath i}(s_i-s_{\imath i}^{(0)}),\ldots)\Big\}_{\imath=1}^{\infty}$$

be a sequence. Here

$$h_{\imath i}(s_i-s_{\imath i}^{(0)})=\sum_{k=-m_{\imath i}}^{-1}\text{ß}_{\imath ki}(s_i-s_{\imath i}^{(0)})^k$$

and $\text{ß}_{\imath(-m_{\imath i})i}\neq0$. Then there exists a meromorphic mapping

$$f(s)=\sum_{\imath=1}^{\infty}\sum_{k=-m_\imath}^{\infty}\text{ß}_{\imath k}(s-s(\imath)^{(0)})^k,$$

such that its poles coincide with (2.1)*, and its principal part at the pole $s(\imath)^{(0)}$ equals $\hbar_{(\imath)}$, for each $\imath=0,1,2,\ldots$ and $\text{ß}_{\imath k}=(\text{ß}_{\imath k1},\ldots,\text{ß}_{\imath ki},\ldots)\in\mathcal{S}$.*

Proof. In the proof of Lemma 2.1 we replace the power series $\sum_{k=0}^{\infty}\text{ß}_{ki}(s_i-s_{0i})^k$ by the Laurent series $\sum_{k=-m_{\imath i}}^{\infty}\text{ß}_{\imath ki}(s_i-s_{\imath i}^{(0)})^k$. And using the famous Mittag-Leffler's theorem in complex analysis for each component we may obtain required result. This proof is finished. □

Next, integrals can be defined on a manifold L in $\mathcal{S}$ as follows.

Definition 2.2. *Let L_i be any closed rectifiable Jordan curve contained in a simply connected subdomain of a domain $\mathfrak{G}_i$ in $\mathbb{C}_i$ and $\mathsf{L}=\prod_{i=1}^{\infty}\mathsf{L}_i$. The positive direction of L can be defined by the positive direction of L_i for each i. Let $\mathfrak{D}_i^+$ be the interior of L_i and $\mathfrak{D}^+=\prod_{i=1}^{\infty}\mathfrak{D}_i^+$. Then $\mathfrak{D}^+$ is called the interior of L. Let $\overline{\mathfrak{D}}^+=\prod_{i=1}^{\infty}\overline{\mathfrak{D}}_i^+$.*

$$\int_{\mathsf{L}}f(s)ds:=\left(\int_{\mathsf{L}_1}f_1(s_1)ds_1,...,\int_{\mathsf{L}_i}f_i(s_i)ds_i,...\right)\in\mathcal{S},$$

where $f=(f_1,...,f_i,...)\in\mathcal{S}$.

It follows that the most main theorems can be derived from the definition of integral and corresponding to classical theorems in complex analysis respectively. For examples, Cauchy's integral theorem and

Cauchy's integral formula on $\mathcal{S}$ are as follows.

LEMMA 2.3. *Let $f(s)$ be a single $\mathcal{S}$-valued holomorphic mapping on $\mathfrak{G}$. Then*

(A)(Cauchy's integral theorem).

$$\int_{\mathsf{L}} f(s)ds = 0,$$

where $\mathfrak{G} = \prod_{i=1}^{\infty} \mathfrak{G}_i$;

(B)(Cauchy's integral formula).

$$\frac{1}{2\pi j}\int_{\mathsf{L}} f(t)(t-s)^{-1}dt = f(s),$$

where $s \in \mathfrak{D}^+ \subset \mathfrak{G}$, *and* $(t-s)^{-1}$ *exists.*

DEFINITION 2.3. *A subset E_0 of E is called a base-real subspace of E and $\bar{u}$ is called the conjugate element of u if following conditions hold:*

a) *E is a vector space on $\mathbb{C}$.*

b) *E_0 is a vector subspace of E on $\mathbb{R}$.*

c) *For every $u \in E$, there exists an $\bar{u}(\in E)$ such that $u + \bar{u} \in E_0$ and $j(u - \bar{u}) \in E_0$ satisfying a unique decomposition $u = \xi + j\eta$ for $\xi, \eta \in E_0$.*

d) *$E_0 \cap jE_0 = \{0\}$, where 0 is the zero element.*

LEMMA 2.4. *If E is a complex vector space, then there exists a base-real subspace E_0 such that $E = E_0 + jE_0$, i.e. a complex vector space can be represented by a direct sum of two spaces which are generated by some real vector space.*

Proof. For any $x_0(\neq 0) \in E$, let $\mathfrak{M}_0 = \mathbb{R}x_0 = \{\varsigma x_0 : \varsigma \in \mathbb{R}\}$. Then $\mathfrak{M}_0$ is a vector subspace of E on the restricted number field $\mathbb{R}$, and $\mathfrak{M}_0 \cap j\mathfrak{M}_0 = \{0\}$. Setting $\mathbb{C}\mathfrak{M}_0 = \{s_0 m_0 : s_0 \in \mathbb{C}, m_0 \in \mathfrak{M}_0\}$, we have that $\mathbb{C}\mathfrak{M}_0$ is a complex vector subspace of E. For any $x_1 \in E\backslash\mathbb{C}\mathfrak{M}_0$ we know that $\mathfrak{M}_1 = \mathfrak{M}_0 + \mathbb{R}x_1$ is also a vector subspace on a restricted number field $\mathbb{R}$ of E and $\mathfrak{M}_1 \cap j\mathfrak{M}_1 = \{0\}$. By induction we obtain a sequence $\{\mathfrak{M}_n\}$ of vector subspaces with $\mathfrak{M}_n \cap j\mathfrak{M}_n = \{0\}$. Assume that $\mathfrak{M}$ is the family of all vector subspaces on $\mathbb{R}$, that $\mathfrak{M}' = \{\mathfrak{M}^0 \in \mathfrak{M} : \mathfrak{M}^0 \cap j\mathfrak{M}^0 = \{0\}\}$ (clearly, $\mathfrak{M}'$ is nonempty), and that $\{\mathfrak{M}_\jmath\}_{\jmath\in J}$ is the family of totally ordered subsets of $\mathfrak{M}'$, where J is an indexing set. Consequently, $E_{\mathfrak{M}} = \cup_{\jmath\in J}\mathfrak{M}_\jmath$ is a vector subspace on $\mathbb{R}$ and a supremum of $\{\mathfrak{M}_\jmath\}_{\jmath\in J}$. Further we have

$$\begin{aligned} E_{\mathfrak{M}} \cap jE_{\mathfrak{M}} &= \bigcup_{\jmath\in J}\mathfrak{M}_\jmath \cap j\bigcup_{\jmath\in J}\mathfrak{M}_\jmath \\ &= \bigcup_{\jmath\in J}\mathfrak{M}_\jmath \cap j\bigcup_{l\in J}\mathfrak{M}_l \\ &= \bigcup_{\jmath\in J}\bigcup_{l\in J}(\mathfrak{M}_\jmath \cap j\mathfrak{M}_l). \end{aligned}$$

Since $\{\mathfrak{M}_\jmath\}_{\jmath\in J}$ is a family of totally ordered subsets with $\mathfrak{M}_\jmath \cap j\mathfrak{M}_\jmath = \{0\}$ for any $\jmath \in J$ and $\mathfrak{M}_\jmath \cap j\mathfrak{M}_l = \{0\}$ for any $\jmath, l \in J$, we get $E_{\mathfrak{M}} \cap jE_{\mathfrak{M}} = \{0\}$. Now Zorn's lemma yields that $\mathfrak{M}'$ has a maximum element E_0.

Next we shall show that E_0 is the required subspace. Firstly, $\mathbb{C}E_0 = E$, here $\mathbb{C}E_0$ is the smallest complex subspace containing E_0. In fact, if $\mathbb{C}E_0 \neq E$, then there exists an $x \in E\backslash\mathbb{C}E_0$. It follows that $E_0' = E_0 + \mathbb{R}x$ is a vector subspace on $\mathbb{R}$ containing E_0 with $E_0' \cap jE_0' = \{0\}$. This is in contradiction with the maximality of E_0. Obviously, $\mathbb{C}E_0 = E_0 + jE_0$. Since $E_0 \cap jE_0 = \{0\}$, $E = E_0 + jE_0$, i.e. E_0 is a base real subspace of E. This proof is finished. □

Suppose that $\mathfrak{e}$ is an idempotent element in $\mathcal{S}$. Theorem 5.3.2 in [5] may be extended to the Fréchet algebra $\mathcal{S}$ containing the Banach algebra. Then using the result after extending we can get

$$\exp(\operatorname{Log} s) = s,$$

$$\exp(s + 2\pi j\mathfrak{e}) = \exp s \quad \text{for} \quad s \in \mathcal{S},$$

$$\exp(s + 2\pi jn\mathfrak{e}) = \exp s,$$

for $n = 0, \pm1, \pm2, \ldots$, where

$$\exp s = \sum_{i=1}^{\infty} s^i/i!,$$

and

$$\operatorname{Log}[\exp s] = s + 2\pi jn\mathfrak{e} = s' + j(s'' + 2\pi n\mathfrak{e}) \tag{2.2}$$

for any integer n and $s \in \mathcal{S}$, where $\mathcal{S} = \mathcal{S}_0 + j\mathcal{S}_0$, $\mathcal{S}_0$ is a base-real Fréchet algebra, and $s', s'' \in \mathcal{S}_0$. From (2.2) we can define the argument of $\exp s$ being $s'' + 2\pi jn\mathfrak{e}$. It follows that

$$\begin{aligned} \operatorname{Log}s = (\log|s_1|, \ldots, \log|s_i|, \ldots) \\ + j(\operatorname{Arg}s_1, \ldots, \operatorname{Arg}s_i, \ldots), \end{aligned} \tag{2.3}$$

and that the argument $\operatorname{Arg}s$ of s is

$$(\operatorname{Arg}s_1, \ldots, \operatorname{Arg}s_i, \ldots).$$

3 The VH^p-space

Let $|f(s)| = (|f_1(s_1)|, \ldots, |f_i(s_i)|, \ldots)$ be the vector modulus of f. For any $a, b \in \mathcal{S}$, $a \leq (<) b$ is $a_i \leq (<) b_i$ for each i. Let $\mathbb{C}_i^+ = \{s_i \in \mathbb{C}_i : \Re s_i > 0\}$ and $\mathcal{S}^+ = \prod_{i=1}^{\infty} \mathbb{C}_i^+$. The set $VH^p(\mathcal{S}^+)$ consists of all holomorphic mappings $f : \mathcal{S}^+ \to \mathcal{S}$ satisfying

$$\sup_{\Re s>0}\left\{\int_{\mathfrak{I}} |f(\xi + j\omega)|^p d\omega\right\}^{\frac{1}{p}} < \infty,$$

where $\mathfrak{I} = \prod_{j=1}^{\infty}\{(-\infty, \infty)\}$, $s = \xi + j\omega \in \mathcal{S}^+$, and $0 < p < \infty$. The set $VH^\infty(\mathcal{S}^+)$ consists of all holomorphic mappings $f : \mathcal{S}^+ \to \mathcal{S}$ satisfying

$$\sup_{\Re s>0}\{|f(\xi + j\omega)|\} < \infty.$$

LEMMA 3.1. *If $f \in VH^p(\mathcal{S}^+)$, then there exists a constant element c such that*

$$|f(s)| \leq c\xi^{-\frac{1}{p}}, s = \xi + j\omega \in \mathcal{S}^+.$$

Proof. Let

$$\mathfrak{L} = \prod_{i=1}^{\infty}\{(0, 2\pi)\}$$

and

$$\mathfrak{R} = \prod_{i=1}^{\infty}\{(0, r_i)\}.$$

Let $\mathfrak{X} = \prod_{i=1}^{\infty}\{(\xi_i - r_i, \xi_i + r_i)\}$ and $\mathfrak{Y} = \prod_{i=1}^{\infty}\{(\omega_i - r_i, \omega_i + r_i)\}$. Because $|f(s)|^p$ is a subharmonic mapping (see [4]),

$$\begin{aligned} &|f(s)|^p \\ \leq &\frac{1}{2\pi}\int_{\mathfrak{L}} |f(s + \rho e^{j\theta})|^p d\theta, 0 < \rho < \xi \in \mathfrak{X}. \end{aligned}$$

Product the two sides of the above formula by ρ and integrate on $\mathfrak{R}$ with respect to ρ. It follows that there exists a constant element $\mathfrak{m} \in \mathcal{S}^+$ using hypothesis such that

$$\begin{aligned} &\frac{r^2}{2}|f(s)|^p \\ \leq &\frac{1}{2\pi}\int_{\mathfrak{R}}\int_{\mathfrak{L}} |f(s + \rho\, e^{j\theta})|^p \rho d\theta d\rho \\ \leq &\frac{1}{2\pi}\int_{\mathfrak{X}}\int_{\mathfrak{Y}} |f(\xi + j\omega)|^p d\xi d\omega \\ \leq &\frac{1}{2\pi}\int_{\mathfrak{X}} \mathfrak{m} d\xi = \frac{\mathfrak{m}r}{\pi}. \end{aligned}$$

Therefore $|f(s)|^p \leq \frac{c^p}{r}$ where $c^p = \frac{2\mathfrak{m}}{\pi}$. Lemma 3.1 is proved as $r \to \xi$. □

LEMMA 3.2. *If $f \in VH^p(\mathcal{S}^+)(p \geq 1)$ and $\Delta s \in \mathcal{S}^+$, then*

$$\begin{aligned} &f(s + \Delta s) \\ = &\frac{1}{\pi}\int_{\mathfrak{I}} \xi f(jt + \Delta s)[\xi^2 + (\omega - t)^2]^{-1} dt, \\ &s = \xi + j\omega \in \mathcal{S}^+. \end{aligned}$$

Proof. Take $\mathsf{L}_r = \prod_{i=1}^{\infty} \mathsf{L}_{r_i}$. Here $\mathsf{L}_{r_i} \subset \mathbb{C}_i$ is a closed lune path consisting of a line $\xi_i = \Delta s_i (> 0)$ and a circular arc with the center at the origin and radius r_i sufficiently large in the right-half plane $\mathbb{C}_i^+$. Let $r_i \cos\theta_{0i} = \Delta s_i$. Since

$$\exp j\theta = (\exp j\theta_1, \ldots, \exp j\theta_i, \ldots),$$

$$\cos\theta_0 = (\cos\theta_{01}, \ldots, \theta_{0i}, \ldots).$$

So $r\cos\theta_0 = \Delta s \in \mathcal{S}^+$. From Lemma 2.3 we derive

$$f(s + \Delta s) = \frac{1}{2\pi i}\int_{\mathsf{L}_r} f(\eta)[\eta - (s + \Delta s)]^{-1} d\eta,$$

where $s + \Delta s$ is in the interior of L_r. Because $\Delta s - \overline{s}$ lies the exterior of L_r, Lemma 2.3 yields

$$\frac{1}{2\pi i}\int_{\mathsf{L}_r} f(\eta)[\eta - (-\overline{s} + \Delta s)]^{-1} d\eta = 0.$$

It follows that

$$\begin{aligned} &|f(s + \Delta s)| \\ = &\frac{1}{2\pi i}\int_{\mathsf{L}_r} f(\eta)\{[\eta \\ &-(s + \Delta s)]^{-1} - [\eta - (-\overline{s} + \Delta s)]^{-1}\} d\eta \\ = &\frac{1}{\pi j}\int_{\mathsf{L}_r} \xi f(\eta)[(\eta - \Delta s - j\omega)^2 - \xi^2]^{-1} d\eta \\ = &\frac{1}{\pi}\int_{\mathfrak{T}} \xi f(jt + \Delta s)[\xi^2 + (\omega - t)^2]^{-1} dt \\ &+ \frac{1}{\pi}\int_{\mathfrak{H}} \xi re^{j\theta} f(re^{j\theta})\{[re^{j\theta} \\ &-\Delta s - j\omega]^2 - \xi^2\}^{-1} d\theta \\ = &I_1 + I_2, \end{aligned}$$

where

$$\mathfrak{T} = \prod_{i=1}^{\infty}(-r_i \sin\theta_{0i}, r_i \sin\theta_{0i}),$$

and

$$\mathfrak{H} = \prod_{i=1}^{\infty}(-\theta_{0i}, \theta_{0i}).$$

Obviously

$$\lim_{r\to\infty} I_1 = \frac{1}{\pi}\int_{\mathfrak{I}} \xi f(jt + \Delta s)[\xi^2 + (\omega - t)^2]^{-1} dt.$$

Next we show $\lim_{R\to\infty} I_2 = 0$.

Lemma 3.1 implies

$$|f(re^{j\theta})| \leq c(r\cos\theta)^{-\frac{1}{p}}.$$

It follows that

$$\begin{aligned} &\left|\xi re^{j\theta}[(re^{j\theta} - \Delta s - j\omega)^2 - \xi^2]^{-1}\right| \\ = &\xi r|re^{j\theta} - \Delta s - j\omega + \xi|^{-1} \\ &|re^{j\theta} - \Delta s - j\omega - \xi|^{-1} \\ \leq &\xi r(|r - \Delta s - \omega - \xi|^{-1})^2 \end{aligned}$$

for r sufficiently large. Thus

$$|I_2| \leq \frac{1}{\pi}\int_{\mathfrak{O}} c(r\cos\theta)^{-\frac{1}{p}}$$

$$\xi r(|r-\Delta s-\omega-\xi|^{-1})^2 d\theta,$$

where $\mathfrak{D}=\prod\{(-\frac{\pi}{2},\frac{\pi}{2})\}$. Because the integral

$$\int_{\mathfrak{D}}(\cos\theta)^{-\frac{1}{p}}d\theta$$

converges and

$$\lim_{r\to\infty}\xi r^{1-\frac{1}{p}}(r-\Delta s-\omega-\xi)^{-1}=0,$$

$$\lim_{r\to\infty}I_2=0 \text{ if } p\geq 1.$$

Combining the above, letting $r\to\infty$ we obtain required conclusion. □

Lemma 3.3. *It $f\in VH^p(\mathcal{S}^+)$ with $1\leq p$, then f is written as*

$$f(s)=\frac{1}{\pi}\int_{\mathfrak{J}}\xi f(jt)[\xi^2+(\omega-t)^2]^{-1}dt,$$

where $s\in\mathcal{S}^+, f(jt)\in VL^p(\mathfrak{J})$.

Proof The following two cases are discussed.

$\alpha)$ $p>1$

Since $f\in VH^p(\mathcal{S}^+)$, there is an $\flat>0$ such that $\int_{\mathfrak{J}}|f(jt+\Delta s)|^p dt\leq\flat$, where $\Delta s>0$ is any element in $\mathcal{S}$. It follows that $f(jt+\Delta s)$ is weak convergence to $f(jt)\in VL^p(\mathfrak{J})$. Lemma 3.2 yields $f(s+\Delta s)=\frac{1}{\pi}\int_{\mathfrak{J}}\xi f(jt+\Delta s)[\xi^2+(\omega-t)^2]^{-1}dt$. Setting $\Delta s\to 0$ in the above formula we get the result of Lemma 3.3.

$\beta)$ $p=1$

From Lemma 2.3 we derive $\int_{\mathfrak{J}}f(jt+\Delta s)(jt+\overline{s})^{-1}dt=0$ for any $\Delta s\in\mathcal{S}^+$. The mapping $f(jt+\Delta s)dt$ is weak* convergence to $d\mu(t)$, where $\mu(t)$ is a measure satisfying $\int_{\mathfrak{J}}|d\mu(t)|<\infty$ if $\Delta s\to 0$. It follows that for any $0<\xi\in\mathcal{S}$, $\int_{\mathfrak{J}}(jt+\xi-j\omega)^{-1}d\mu(t)=0$. Letting $\omega=0$ in the above formula we have $\int_{\mathfrak{J}}(jt+\xi)^{-1}d\mu(t)=0$. Finding the Fréchet derivatives of each order we obtain $\int_{\mathfrak{J}}(jt+\xi)^{-n}d\mu(t)=0$ for $n=0,1,\ldots$. Specially there are $\int_{\mathfrak{J}}(jt+I)^{-n}d\mu(t)=0$ for $n=0,1,\ldots$ as $\xi=I$, where I is the unit element in $\mathcal{S}$. Let $dv(\tau)=(jt-I)^{-1}d\mu(t)$. The conformal mapping $w=(s-I)(s+I)^{-1}$ implies

$$\begin{aligned}&\int_{\mathfrak{L}}e^{jn\tau}dv(\tau)\\ =\ &\int_{\mathfrak{J}}(jt-I)^{n-1}(jt+I)^{-n}d\mu(t)\\ =\ &\int_{\mathfrak{J}}[(jt+I)-2I]^{n-1}(jt+I)^{-n}d\mu(t)\\ =\ &\sum_{k=1}^{n}\left[a_k\int_{\mathfrak{J}}(jt+I)^{-k}d\mu(t)\right]=0,\end{aligned}$$

where $a_k\in\mathcal{S}$ is a constant element for each k. From Riesz's theorem (see [6]) and the absolute continuity of $v(\tau)$ we derive that $\mu(t)$ is also absolutely continuous and $f(jt)\in VL^1(\mathfrak{J})$ and that $d\mu(t)=f(jt)dt$. Hence Lemma 3.2 yields

$$f(s)=\frac{1}{\pi}\int_{\mathfrak{J}}\xi f(jt)[\xi^2+(\omega-t)^2]^{-1}dt.$$

This proof is finished. □

Lemma 3.4. *Assume that $F(s)\in VH^p(\mathcal{S}^+)$, and that $f(w)=F(s)$, then $f(w)\in VH^p(\mathfrak{D}_s)$, where $w=(s-I)(s+I)^{-1}$, $\mathfrak{D}_s=(\mathfrak{D}_{s1},\ldots,\mathfrak{D}_{si},\ldots)$ and $\mathfrak{D}_{si}$ is a unit disk in $\mathbb{C}_i$ for each i.*

Proof. The following two cases are discussed.

$\alpha)$ $p\geq 1$

From Lemma 3.3 we derive

$$F(s)=\frac{1}{\pi}\int_{\mathfrak{J}}\xi F(jt)[\xi^2+(\omega-t)^2]^{-1}dt,\ s\in\mathcal{S}^+,$$

where $F(jt)\in VL^p(\mathfrak{J})$. Using $F(jt)=f(e^{jt})$, we can get

$$F(s)=\frac{1}{2\pi}(I-r^2)\int_{\mathfrak{L}}f(e^{j\tau})[I+r^2 -2r\cos(\varphi-\tau)]^{-1}d\tau.$$

So

$$\int_{\mathfrak{L}}|f(e^{j\tau})|^p d\tau=\int_{\mathfrak{J}}2|F(jt)|^p(I+t^2)^{-1}dt \leq 2\int_{\mathfrak{J}}|F(jt)|^p dt<\infty.$$

Hence $f(w)\in VH^p(\mathfrak{D}_s)$.

$\beta)$ $0<p<1$.

Lemma 3.1 yields $|F(s)|\leq c\xi^{-\frac{1}{p}}$. Particularly $|F(s)|$ and $|F(s)|^p$ are bounded on the half space $\prod_{i=1}^{\infty}\{s_i:\Re s_i\geq\Delta s_i>0\}$. Thus $|F(s+\Delta s)|^p$ is a subharmonic mapping on $\mathcal{S}^+$. It follows that

$$|F(s+\Delta s)|^p \leq\frac{1}{\pi}\int_{\mathfrak{J}}\xi|F(jt+\Delta s)|^p|\xi^2+(\omega-t)^2|^{-1}dt.$$

Since $\int_{\mathfrak{J}}|F(jt+\Delta s)|^p dt\leq\mathfrak{m}'$ for any $\Delta s>0$, there is a measure μ such that $\int_{\mathfrak{J}}|d\mu(t)|<\infty$ and

$$|F(s)|^p\leq\frac{1}{\pi}\int_{\mathfrak{J}}\xi|\xi^2+(\omega-t)^2|^{-1}d\mu(t).$$

Let $d\nu(\tau)=-2(I+t^2)^{-1}d\mu(t)$. Then

$$|f(re^{j\varphi})|^p\leq\frac{1}{2\pi}\int_{\mathfrak{L}}(I-r^2)|I+r^2 -2r\cos(\varphi-\tau)|^{-1}d\nu(\tau).$$

Fubini's theorem yields

$$\int_{\mathfrak{L}}|f(re^{j\varphi})|^p d\varphi\leq 2\int_{\mathfrak{J}}|I+t^2|^{-1}|d\mu(t)|<\infty$$

for any $r > I$, so $f(w) \in VH^p(\mathfrak{D}_s)$. □

Let $æ(t,s) = (jts - I)[(jt-s)(I+t^2)]^{-1}$.

LEMMA 3.5. *If $F(s) \in VH^p(\mathcal{S}^+)$, then*

$$F(s) = F_{[i]}(s)F_{[o]}(s),$$

where

$$F_{[i]}(s) = e^{j\gamma}B(s)\exp\left\{\frac{1}{\pi}\int_{\mathfrak{I}} æ(t,s)d\sigma(t)\right\}e^{j\epsilon s}$$

is called the inner mapping of F, $\gamma \in \mathfrak{I}, B(s)$ is the the Blaschke product of $F, \sigma(t)$ is a singular measure, $\int_{\mathfrak{I}}(I+t^2)^{-1}d\sigma(t) > -\infty$, $\epsilon(\in \mathcal{S}) > 0$ and the mapping

$$F_{[o]}(s) = \exp\left\{\frac{1}{\pi}\int_{\mathfrak{I}} æ(t,s)\log|F(jt)|dt\right\}$$

is called the outer mapping of F.

Proof. By using the transform $w = (s-I)(s+I)^{-1}$ and Lemma 3.4 we obtain $f(w) \in VH^p(\mathfrak{D}_s)$, where $f(w) = F(s)$. Theorem 2.2.8 in [8] implies $f(w) = f_{[i]}(w)f_{[o]}(w)$, where

$$f_{[i]}(w) = e^{j\gamma}B_f(w)\exp\left(-\frac{1}{2\pi}\int_{\mathfrak{L}} \mho(\tau,w)d\nu(\tau)\right),$$

$$B_f(w) = \prod_n œ_{[n]}(w),$$

$$œ_{[n]}(w) = (|w_{[n]}|(w_{[n]}-w)[w_{[n]}(I-\overline{w}_{[n]}w)]^{-1},$$

$$f_{[o]}(w) = \exp\left(\frac{1}{2\pi}\int_{\mathfrak{L}} \mho(\tau,w)\log|f(e^{j\tau})|d\tau\right),$$

$\gamma \in \mathfrak{I}$ and $\nu \geq 0$ is a singular measure on $\mathfrak{L}$,

$$\mho(\tau,w) = (e^{j\tau}+w)(e^{j\tau}-w)^{-1}.$$

It follows that

$$f_{[i]}(w) = e^{j\gamma}B_f(w)\exp\left(\frac{1}{\pi}\int_{\mathfrak{I}} æ(t,s)d\sigma(t)\right)e^{-\epsilon s},$$

$$f_{[o]}(w) = \exp\left(\frac{1}{\pi}\int_{\mathfrak{I}} æ(t,s)\log|F(jt)|dt\right),$$

where $\epsilon = \frac{1}{2\pi}[\nu(0)+\nu(2\pi)]$, and $d\nu(\tau) = -2(I+t^2)^{-1}d\sigma(t)$.

Let $B(s) = B_f(w), F_{[i]}(s) = f_{[i]}(w)$, and $F_{[o]}(s) = f_{[o]}(w)$ via $w = (s-I)(s+I)^{-1}$. Then $F(s) = F_{[i]}(s)F_{[o]}(s)$. By using Theorem 2.2.8 in [8] we can check that $F_{[i]}(s)$ and $F_{[o]}(s)$ are an inner mapping and an outer mapping respectively.

This ends the proof. □

LEMMA 3.6. *A mapping $f \in VH^\infty(\mathcal{S}^+)$ is outer if and only if fVH^2 is dense in VH^2*

Proof. Let $\pounds$ is a shift operator on VH^2, i.e. $\pounds(f) = sf(s)$. Assume that Υ is a closure of fVH^2 in VH^2. Obviously, Υ is an invariant subspace with respect to $\pounds$. By using Beurling's Theorem in [4], we know that there is an inner mapping g such that $\Upsilon = gVH^2$. Since $f \in \Upsilon$, the mapping f can be represented as $f = gh$, where $h \in VH^2$.

Necessity. If f is an outer mapping, then $g \equiv$ *const* by $f = gh$. It follows that $\Upsilon = VH^2$, namely, fVH^2 is dense in VH^2.

Sufficiency. Use proof by contradiction. Suppose that f is not outer, and that $f = f_{[i]}f_{[o]}$, then $f_{[i]} \not\equiv$ *const*. We can check that $f_{[i]}VH^2$ is an invariant subspace with respect to $\pounds$ and $f_{[i]}VH^2 \supset fVH^2$. Thus fVH^2 is not dense in VH^2. This is in contradiction with the hypothesis. This contradiction shows the sufficiency.

Therefore the result of this lemma holds. □

4 The VH^∞-control theory

In this section, we replace $\mathcal{S}$ by the bounded sequence space l^∞. The subset of VH^∞ consisting of all elements with every component being real-rational function, is denoted by VRH^∞.

The space $(l^\infty)^{n\times m}$ consists of all $n\times m$ complex matrices with each element being in l^∞. If $f(s) \in (l^\infty)^{n\times m}$, then f can be written as

$$\begin{aligned} f &= \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ f_{21} & \cdots & f_{2n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{pmatrix} \\ &= (\mathfrak{f}_1, \ldots, \mathfrak{f}_i, \ldots), \end{aligned}$$

where

$$\mathfrak{f}_i = \begin{pmatrix} f_{11i} & \ldots & f_{1ni} \\ f_{21i} & \ldots & f_{2ni} \\ \vdots & \ddots & \vdots \\ f_{m1i} & \ldots & f_{mni} \end{pmatrix}.$$

Let ϱ_i be the maximal singular value of $\mathfrak{f}_i$. Then $(\varrho_1, \ldots, \varrho_i, \ldots)$ is called the maximal singular value-vector of f. Let VL^∞ be a space consisting of all mapping matrices $f(j\omega)$ with

$$\sup_\omega \bar{\sigma}[f(j\omega)] < \infty.$$

Here $\bar{\sigma}[f(j\omega)]$ is its maximal singular value-vector for any fixed ω. The vector norm of $f \in VL^\infty$ is defined by

$$\|f\|_\infty = \sup_\omega \bar{\sigma}[f(j\omega)].$$

The space VRL^∞ consists of all real-rational mapping matrices in VL^∞.

The space VL^2 consists of all mapping matrices $\{x(j\omega)\}$ which are in $(l^\infty)^n$ and satisfy

$$\int_{\mathfrak{I}} x^*(j\omega)x(j\omega)d\omega < \infty,$$

where x^* is the complex-conjugate transpose of x.

The space VH^∞ consists of all holomorphic mapping matrices $\{f(s)\}$ satisfying

$$\sup\{\bar{\sigma}[f(s)] : \Re s > 0\} < \infty. \tag{4.1}$$

The space VRH^∞ consists of all real-rational mapping matrices in VH^∞. Define

$$\|f\|_\infty = (\|\mathfrak{f}_1\|_\infty, \ldots, \|\mathfrak{f}_i\|_\infty, \ldots)$$

if $f \in VH^\infty$. It is the vector norm of f. Here $\|\mathfrak{f}_i\|_\infty$ is in the sense of the norm of the classical H^∞ control. Obviously $\|f\|_\infty < \infty$ if and only if formula (4.1) holds.

We call f to be strong proper if $f(\infty) < \infty$, and strictly strong proper if $f(\infty) = 0$. We call f to be stable if $f \in VRH^\infty$ and f has no poles in the domain $\bar{\mathcal{S}}^+(= \prod_{i=1}^\infty \bar{\mathcal{S}}_i^+)$, where $\bar{\mathcal{S}}_i^+ = \{s_i : \Re s_i \geq 0\}$.

From the above definitions and the corresponding conclusion in [7] we derive $f \in VRH^\infty$ if and only if f is strong proper and stable.

Three transfer matrices

$$T^{[\ell]} = (T_1^{[\ell]}, \ldots, T_i^{[\ell]}, \ldots), \ell = 1, 2, 3$$

are controllers. Similar to the classical method in [2], we define the transfer mapping matrix

$$G(s) := \begin{bmatrix} T^{[1]}(s) & T^{[2]}(s) \\ T^{[3]}(s) & 0 \end{bmatrix},$$

$$K(s) = -Q(s),$$

where $T^{[\ell]} \in VH^\infty$ for $\ell = 1, 2, 3$ are given. Let

$$T^{[\ell]} = \begin{bmatrix} T_{\ell 1} & \cdots & T_{\ell i} & \cdots \end{bmatrix}$$

for $\ell = 1, 2, 3$. Then G can be written as

$$\left[\begin{bmatrix} T_{11} & T_{21} \\ T_{31} & 0 \end{bmatrix} \cdots \begin{bmatrix} T_{1i} & T_{2i} \\ T_{3i} & 0 \end{bmatrix} \cdots \right].$$

In VH^∞ control theory, the model matching problem is to find a strong proper element $Q \in VRH^\infty$ or a matrix $Q \in VRH^\infty$ to minimize $\left\|T^{[1]} - T^{[2]}QT^{[3]}\right\|_\infty$ under the constraint that K stabilize G, Q is the controller to be designed. Let

$$\alpha := \inf\left\{\left\|T^{[1]} - T^{[2]}QT^{[3]}\right\|_\infty\right\}$$

be the infimal model-matching error.

The following linear time invariant system in VRH^∞ is defined by

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t).$$

Completely controllable (c.c.) and completely observable (c.o.) concepts and symbols (A, B) and (A, C) are similar to Definition 2.3 in [7]. The concept of the minimal realization is similar to Definition 2.4 (see [7]). A matrix $A \in VRH^\infty$ is said to be antistable if all the generalized eigenvalue vectors consisting of its all eigenvalues, of A, are in $\mathcal{S}^+$.

From the classical H^∞-control theory we derive the following result.

Lemma 4.1.

(1) *A realization $[A, B, C, 0]$ of a given transfer matrix $G(s) \in VRH^\infty$ is minimal if (A, B) is completely controllable and (A, C) is completely observable respectively.*

(2) *If A is antistable, then the Lyapunov equations*

$$A\mathcal{L}_c + \mathcal{L}_c A^T = BB^T$$
$$A^T\mathcal{L}_o + \mathcal{L}_o A = C^T C$$

have the unique solutions respectively, where

$$\mathcal{L}_c = \int_{\mathsf{L}} e^{-At}BB^Te^{-A^Tt}dt,$$

$$\mathcal{L}_o = \int_{\mathsf{L}} e^{-A^Tt}C^TCe^{-At}dt,$$

where $\mathsf{L} = \prod_{i=1}^\infty \mathsf{L}_i, \mathsf{L}_i = (0, \infty)$.

Lemma 3.5 implies that a mapping T in $(l^\infty)^n \cap VRH^\infty$ is inner if and only if $T(-s)T(s) = I$, and outer if it has no zeros in $\mathcal{S}^+$, that $T(-s)T(s) = I$ if and only if each component $T_j(-s_i)T_i(s_i) = 1$ for any i, that every mapping T in $(l^\infty)^n \cap VRH^\infty$ has a factorization $T = T_{[i]}T_{[o]}$ with $T_{[i]}$ inner and $T_{[o]}$ outer, and $\left\|T_{[i]}(j\omega)\right\|_\infty = I$, and that if $T(j\omega) \neq 0$ for all $\omega \in \bar{\mathsf{L}} = \prod_{i=1}^\infty \bar{\mathsf{L}}_i$, where $\bar{\mathsf{L}}_i = [0, \infty]$ for any i, then $T_{[o]}^{-1}$ exists and $T_{[o]}^{-1} \in VRH^\infty$.

Returning to the model-matching problem, for simplicity, we may assume $T^{[3]} = I$ and bring in an inner-outer factorization of $T^{[2]} : T^{[2]} = T_{[i]}^{[2]}T_{[o]}^{[2]}$. It follows that for Q in VRH^∞ we have

$$\begin{aligned}\|T^{[1]} - T^{[2]}Q\|_\infty &= \|T_{[i]}^{[2]^{-1}}T^{[1]} - T_{[o]}^{[2]}Q\|_\infty \\ &= \|R - X\|_\infty,\end{aligned}$$

where $R = T_{[i]}^{[2]^{-1}}T^{[1]}, X = T_{[o]}^{[2]}Q$.

Let λ^2 be a generalized eigenvalue vector of $\mathcal{L}_c\mathcal{L}_o$ and w the correspondent generalized eigenvector matrix respectively. Define

$$f(s) = [A, w, C, 0],$$

$$g(s) = [-A^T, \lambda^{-1}\mathcal{L}_o w, B^T, 0]$$

and

$$X(s) = R(s) - \lambda f(s)[g(s)]^{-1}.$$

Let $F \in VL^\infty$ and $g \in VL^2$. Then the operator

$$\Lambda_F : \Lambda_F g = Fg$$

is called the Laurent operator. For F in VL^∞, the Hankel operator with symbol F, denoted by Γ_F, maps VH^2 to $VH^{2^\perp}$ and is defined as

$$\Gamma_F := \Pi_1 \Lambda_F | VH^2,$$

where Π_1 is the projection from VL^2 onto $VH^{2^\perp}$.

Let $\{s_i : \Re s_i = 0, \Im s_i \geq 0\} = \Xi_i$ and $\prod_{i=1}^\infty \Xi_i = \Xi$.

Basing on previous results we may obtain the following conclusions.

Theorem 4.1.

(a) *If the ranks of $T^{[2]}$ and $T^{[3]}$ are constant on Ξ, then the optimal Q exists.*

(b) *There exists a closest VRH^∞-mapping $X(s)$ to a given VRL^∞-mapping $R(s)$, and $\|R-X\|_\infty = \|\Gamma_R\|$, where*

$$\|\Gamma_R\| = (\|\Gamma_{1R_1}\|, \ldots, \|\Gamma_{iR_i}\|, \ldots).$$

(c) *The infimal model-matching error α equals $\|\Gamma_R\|$ and the unique optimal X equals*

$$R(s) - \lambda f(s)[g(s)]^{-1}.$$

The optimal controller

$$Q = (Q_1, \ldots, Q_i, \ldots) = \left(T_{[o]}^{[2]}\right)^{-1} X \in VRH^\infty$$

is found via this theorem. Therefore the VH^∞-control theory is solved.

Remark. In this article, all corresponding conclusions hold for arbitrary Fréchet algebras and Banach algebras being isometric isomorphism to $\mathcal{S}$ and l^∞ respectively.

5 Conclusions

- The concept and properties of meromorphic mappings on an infinite dimensional algebra to itself are obtained. These are breakthrough in infinite dimensional complex analysis without appearing in [1,10].
- The concept (2.3) of an argument on infinite dimensional algebras is defined. This is a breakthrough in infinite dimensional geometry.
- The theory of VH^p spaces on infinite dimensional algebras to infinite dimensional algebras is presented.
- The infimal model-matching error and the unique optimal solution of VH^∞ control theory on infinite dimensional algebras to infinite dimensional algebras are established.
- All control theory on finite dimensional spaces can be extended that on infinite dimensional spaces to infinite dimensional spaces by using methods in this paper.

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