An Optimal Control of Four Wheel Steering Vehicles

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Abstract: - Four wheel steering vehicles are being used increasingly due to high performance and stability that they bring to the vehicles. This paper deals with a novel high performance four wheel steered vehicle model which is optimally controlled during a lane change manoeuvre in high speeds. Simulation results reveal the effectiveness of the proposed model and controller.

Key-Words: - Optimal control, Four wheel steering, Body side slip angle, Yaw rate.

1 Introduction

Safety in driving a vehicle with high speed is extremely crucial. Due to the large yaw rate and sideslip angle of a vehicle body moving in high speed, turning of a vehicle cannot be well controlled by only steering the front wheels. The four-wheel-steering (4WS) technique is thus developed over the past decade. A great number of studies have been made on various control strategies for 4WS vehicles since the first 4WS system was reported. Four wheel steering is a relatively new technology that improves manoeuvrability in cars, trucks and trailers. In standard two wheel steering vehicles, the rear set of wheels are always directed forward therefore and do not play an active role in controlling the steering. In four wheel steering systems, the rear wheels can turn left and right. To keep the driving controls as simple as possible, a computer is used to control the rear wheels. Yet, a few archival publications in current literature dealt with the dynamics of the 4WS vehicle/driver closed-loop system with nonlinear properties of the lateral tyre forces taken into account [1-3]. The nonlinear behavior, the stability of the vehicle and the nonlinear effects on vehicle dynamics, hence, require a systematical analysis.

In the present work, a new methodology of mathematically modelling for the 4WS vehicle–driver system during turning is developed.

In this paper by optimizing a cost function regards to state variables, the control law is determined. The optimal control used in this paper is based on the Riccatti differential equations [4,5].

The paper is organized as follows: Section 2 describes a complete model of vehicle. In Section 3 the design procedure of optimal control is considered. Simulation results of the system are provided in Section 4. Finally, the paper is concluded in Section 5.

2 Vehicle modelling

There are usually two well known approaches in modelling a vehicle dynamics: single track and two track. The first model is presuming the two front and the two rear wheels as two wheels and hence it is sometimes called the bicycle model. This model has many simplifications and is not valid for accelerations above *0.3g*. The second model (the reduced nonlinear two track model) considers much more nonlinearities and hence gives a much more precise result. In this model, each tyre has forces in direction of the wheel plane and perpendicular to it which are called F_L and F_S respectively. We may introduce two coordinate systems as:

- *"CoG"* (Center of Gravity) for the chassis coordinate system
- *"In"* for the fixed inertial system

The reduced model should contain only those state variables which are essential for vehicle dynamic control. These are the vehicle speed V_{cog}, the vehicle body side slip angle β, and yaw rate $\dot{\psi}$. Now the vehicle speed can be transformed to fixed inertial coordinate system:

$$\begin{bmatrix}\dot{x}_{in}\\ \dot{y}_{in}\end{bmatrix}=V_{cog}\begin{bmatrix}\cos(\beta+\psi)\\ \sin(\beta+\psi)\end{bmatrix} \tag{1}$$

By differentiating the equation (1) we will have:

$$\begin{aligned}\begin{bmatrix}\ddot{x}_{in}\\ \ddot{y}_{in}\end{bmatrix}&=V_{cog}\,(\dot{\beta}+\dot{\psi})\begin{bmatrix}-\sin(\beta+\psi)\\ \cos(\beta+\psi)\end{bmatrix}\\ &+\dot{V}_{cog}\begin{bmatrix}\cos(\beta+\psi)\\ \sin(\beta+\psi)\end{bmatrix}\end{aligned} \tag{2}$$

These accelerations are now transformed from the inertial into the *CoG* coordinate system.

$$\begin{aligned}\begin{bmatrix}\ddot{x}_{cog}\\ \ddot{y}_{cog}\end{bmatrix}&=\begin{bmatrix}\cos\psi & \sin\psi\\ -\sin\psi & \cos\psi\end{bmatrix}\begin{bmatrix}\ddot{x}_{in}\\ \ddot{y}_{in}\end{bmatrix}\\ &=V_{cog}\,(\dot{\beta}+\dot{\psi})\begin{bmatrix}-\sin\beta\\ \cos\beta\end{bmatrix}+\dot{V}_{cog}\begin{bmatrix}\cos\beta\\ \sin\beta\end{bmatrix}\end{aligned} \tag{3}$$

By neglecting gravitational forces F_{gx} and F_{gy}, rolling resistance F_r, lateral wind force and the wind velocity V_{windy}, the complete equation for horizontal translatory motion are then given by:

$$\begin{aligned}&\mathrm{V_{cog}}.(\dot{\beta}+\dot{\psi})\begin{bmatrix}-\sin\beta\\ \cos\beta\end{bmatrix}+\dot{V}_{cog}\begin{bmatrix}\cos\beta\\ \sin\beta\end{bmatrix}\\ &=\frac{1}{M_{cog}}\begin{bmatrix}F_{xfl}+F_{xfr}+F_{xrl}+F_{xrr}+F_{windx}\\ F_{yfl}+F_{yfr}+F_{yrl}+F_{yrr}\end{bmatrix}\end{aligned} \tag{4}$$

which yields:

$$\begin{aligned}\dot{V}_{cog}&=\frac{\cos\beta}{M_{cog}}(F_{xfl}+F_{xfr}+F_{xrl}+F_{xrr}\\ &-C_{aerx}A_l\frac{\rho}{2}V^2{}_{cog})\\ &+\frac{1}{M_{cog}}(F_{yfl}+F_{yfr}+F_{yrl}+F_{yrr})\sin\beta\end{aligned} \tag{5}$$

$$\begin{aligned}\dot{\beta}&=\frac{1}{M_{cog}V_{cog}\cos\beta}(Fyfl+Fyfr+Fyrl\\ &+Fyrr-M_{cog}\dot{V}_{cog}\sin\beta)-\dot{\psi}\end{aligned} \tag{6}$$

$$\begin{aligned}J_z\ddot{\psi}&=(F_{yfr}+F_{yfl}).(L_f-n_{lf})\\ &-(F_{yrr}+F_{yrl}).(L_r+n_{lr})+(F_{xrr}-F_{xrl})\frac{b_r}{2}\\ &+F_{xfr}(\frac{b_f}{2}-n_{sfr}\sin\delta wf)-F_{xfl}(\frac{b_f}{2}+n_{sfl}\sin\delta wf)\\ &+F_{xrr}(\frac{b_r}{2}-n_{srr}\sin\delta wr)-F_{xrl}(\frac{b_r}{2}+n_{srl}\sin\delta wr)\end{aligned} \tag{7}$$

by substitution of :

$$\begin{aligned}F_{xij}&=F_{lij}\cos\delta_{wi}-F_{sij}\sin\delta_{wi}\\ F_{yij}&=F_{sij}\cos\delta_{wi}+F_{lij}\sin\delta_{wi}\end{aligned} \tag{8}$$

in eq. 5 to 7 we will have the state space variables in terms of longitudinal and lateral forces and other vehicle parameters.

The longitudinal forces F_{lij} are regarded as control inputs (by assuming a vehicle with four electrical driving motors on four wheels). The wheel lateral forces F_{sij} are now approximated to be proportional to the tyre side slip angle α_{ij} [6,7].

$$F_{sfl}=C_{fl}.\alpha_{fl}=C_{fl}(\delta_{wf}-\beta-\frac{l_f\dot{\psi}}{V_{cog}}) \tag{9}$$

$$F_{sfr}=C_{fr}.\alpha_{fr}=C_{fr}(\delta_{wf}-\beta-\frac{l_f\dot{\psi}}{V_{cog}}) \tag{10}$$

$$F_{srl}=C_{rl}.\alpha_{rl}=C_{rl}(\delta_{wr}-\beta+\frac{l_r\dot{\psi}}{V_{cog}}) \tag{١١}$$

$$F_{srr}=C_{rr}.\alpha_{rr}=C_{rr}(\delta_{wr}-\beta+\frac{l_r\dot{\psi}}{V_{cog}}) \tag{12}$$

Hence the wheel turn angle and the longitudinal wheel forces F_{lij} are utilized as control inputs for vehicle dynamic control by steering. So the state space variables of the reduced nonlinear four wheels steered two track model become:

$$\begin{aligned}f_1=\dot{V}_{cog}&=\frac{1}{M_{cog}}\{F_{lfl}\cos(\beta-\delta_{wf})\\ &+F_{lfr}\cos(\beta-\delta_{wf})+F_{lfr}\cos(\beta-\delta_{wr})\\ &+F_{lrr}\cos(\beta-\delta_{wr})-C_{aer}.A_l.\frac{\rho}{2}.V^2{}_{cog}\cos\beta\\ &+[C_{fl}\delta_{wf}-C_{fl}\beta-C_{fl}\frac{l_f\dot{\psi}}{V_{cog}}]\sin(\beta-\delta_{wf})\\ &+[C_{fr}\delta_{wf}-C_{fr}\beta-C_{fr}\frac{l_f\dot{\psi}}{V_{cog}}]\sin(\beta-\delta_{wf})\\ &+[C_{rl}\delta_{wr}-C_{rl}\beta-C_{rl}\frac{l_r\dot{\psi}}{V_{cog}}]\sin(\beta-\delta_{wr})\\ &+[C_{rr}\delta_{wr}-C_{rr}\beta-C_{rr}\frac{l_r\dot{\psi}}{V_{cog}}]\sin(\beta-\delta_{wr})\}\end{aligned} \tag{13}$$

$$
\begin{aligned}
f_2 = \dot{\beta} = {} & \frac{1}{M_{cog}.V_{cog}}.\{Sin(\beta-\delta_{wf})[-F_{lfl}-F_{lfr}] \\
& + Sin(\beta-\delta_{wr})[-F_{lrl}-F_{lrr}] \\
& + \cos(\beta-\delta_{wf})[C_{fr}+C_{fl}].[\delta_{wf}-\beta-\frac{l_f\dot{\psi}}{V_{cog}}] \\
& + \cos(\beta-\delta_{wr})[C_{rr}+C_{rl}].[\delta_{wr}-\beta-\frac{l_r\dot{\psi}}{V_{cog}}]
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
f_3 = \dot{\psi} = {} & \frac{1}{2J_z.V_{cog}}.\{\begin{pmatrix} C_{fl}\delta_{wf}V_{cog}b_f \\ -C_{fl}\beta V_{cog}b_f \\ -C_{fl}l_f\dot{\psi}b_f \\ +2F_{lfl}V_{cog}l_f \end{pmatrix}\sin\delta_{wf} \\
& +\begin{pmatrix} 2C_{fl}\delta_{wf}l_fV_{cog}-2C_{fl}\beta V_{cog}l_f \\ -2C_{fl}l_f{}^2\dot{\psi}-F_{lfl}b_fV_{cog} \end{pmatrix}\cos\delta_{wf} \\
& +\begin{pmatrix} -C_{fr}\delta_{wf}V_{cog}b_f+C_{fr}\beta V_{cog}b_f \\ +C_{fr}l_f\dot{\psi}b_f+2F_{lfr}V_{cog}l_f \end{pmatrix}\sin\delta_{wf} \\
& +\begin{pmatrix} 2C_{fr}\delta_{wf}l_fV_{cog}-2C_{fr}\beta V_{cog}l_f \\ -2C_{fr}l_f{}^2\dot{\psi}+F_{lfr}b_fV_{cog} \end{pmatrix}\cos\delta_{wf} \\
& +\begin{pmatrix} -C_{rr}\delta_{wr}V_{cog}b_r+C_{rr}\beta V_{cog}b_r \\ -C_{rr}l_r\dot{\psi}b_r-2F_{lrr}V_{cog}l_r \end{pmatrix}\sin\delta_{wr} \\
& +\begin{pmatrix} -2C_{rr}\delta_{wr}l_rV_{cog}+2C_{rr}\beta V_{cog}l_r \\ -2C_{rr}l_r{}^2\dot{\psi}+F_{lrr}b_rV_{cog} \end{pmatrix}\cos\delta_{wr} \\
& +\begin{pmatrix} C_{rl}\delta_{wr}V_{cog}b_r-C_{rl}\beta V_{cog}b_r \\ +C_{rl}l_r\dot{\psi}b_r-2F_{lrl}V_{cog}l_r \end{pmatrix}\sin\delta_{wr} \\
& +\begin{pmatrix} -2C_{rl}\delta_{wr}l_rV_{cog}+2C_{rl}\beta V_{cog}l_r \\ -2C_{rl}l_r{}^2\dot{\psi}-F_{lrr}b_rV_{cog} \end{pmatrix}\cos\delta_{wr} \\
& +\beta\begin{pmatrix} 2n_{lf}V_{cog}(C_{fl}+C_{fr}) \\ +2n_{lr}V_{cog}(C_{rl}+C_{rr}) \end{pmatrix} \\
& -2\delta_{wf}n_{lf}V_{cog}(C_{fl}+C_{fr}) \\
& -2\delta_{wr}n_{lr}V_{cog}(C_{rl}+C_{rr})\}.
\end{aligned}
\tag{15}
$$

3 Controller Design

In the state space form the reduced nonlinear two track model can be written as:

$$
\begin{aligned}
\dot{\underline{x}} &= \underline{A}(\underline{x},\underline{u})\underline{x}+\underline{B}(\underline{x},\underline{u})\underline{u} \\
\underline{y} &= \underline{C}(\underline{x},\underline{u})\underline{x}
\end{aligned}
\tag{16}
$$

The state vector is:

$$\underline{x} = [V_{cog} \quad \beta \quad \dot{\psi}]^T \tag{17}$$

While the control output is:

$$\underline{y} = [V_{cog} \quad \dot{\psi}]^T \tag{18}$$

This nonlinear state space equation has to be optimally controlled. Regarding this will we would need to express our state variables in terms of Taylor series around an actual operating point. Since the non linear state space equations are rather complicated a first order Taylor series would be appropriate.

The state space equations are rewritten as:

$$
\begin{aligned}
\dot{\underline{x}} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \dot{V}_{cog} \\ \dot{\beta} \\ \ddot{\psi} \end{bmatrix} = \underline{f}(\underline{x},\underline{u}) = \begin{bmatrix} \underline{f}_1(\underline{x},\underline{u}) \\ \underline{f}_2(\underline{x},\underline{u}) \\ \underline{f}_3(\underline{x},\underline{u}) \end{bmatrix} \\
\underline{y} &= \underline{c}(\underline{x},\underline{u})\ \underline{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\underline{x}
\end{aligned}
\tag{19}
$$

where:

$$
\begin{aligned}
\underline{f}(\underline{x},\underline{u}) = {} & \underline{f}(\underline{x}_0,\underline{u}_0) + \left.\frac{\partial \underline{f}(\underline{x},\underline{u})}{\partial \underline{x}}\right|_{\substack{\underline{x}=\underline{x}_0 \\ \underline{u}=\underline{u}_0}}.(\underline{x}-\underline{x}_0) \\
& + \left.\frac{\partial \underline{f}(\underline{x},\underline{u})}{\partial \underline{u}}\right|_{\substack{\underline{x}=\underline{x}_0 \\ \underline{u}=\underline{u}_0}}.(\underline{u}-\underline{u}_0)
\end{aligned}
\tag{20}
$$

In this equation $\frac{\partial \underline{f}(\underline{x},\underline{u})}{\partial \underline{x}}$ and $\frac{\partial \underline{f}(\underline{x},\underline{u})}{\partial \underline{u}}$ are Jaccobian matrices which are defined as:

$$
\frac{\partial \underline{f}(\underline{x},\underline{u})}{\partial \underline{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}
\tag{21}
$$

$$
\frac{\partial \underline{f}(\underline{x},\underline{u})}{\partial \underline{u}} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial u_3} & \frac{\partial f_1}{\partial u_4} & \frac{\partial f_1}{\partial u_5} & \frac{\partial f_1}{\partial u_6} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_3} & \frac{\partial f_2}{\partial u_4} & \frac{\partial f_2}{\partial u_5} & \frac{\partial f_2}{\partial u_6} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_3}{\partial u_3} & \frac{\partial f_3}{\partial u_4} & \frac{\partial f_3}{\partial u_5} & \frac{\partial f_3}{\partial u_6} \end{bmatrix}
\tag{22}
$$

While the input vector $\underline{u}$ is:

$$
\begin{aligned}
\underline{u} &= \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{bmatrix}^T \\
&= \begin{bmatrix} F_{lfl} & F_{lfr} & F_{lrl} & F_{lrr} & \delta_{wf} & \delta_{wr} \end{bmatrix}^T
\end{aligned}
\tag{23}
$$

Now by proper determination of operating point and also the destination state all needed state space parameters are determined.
The state-space representation of the system in Equation 1 can be written as:

$$\dot{\underline{x}} = A\,\underline{x} + B\,\underline{u} \tag{24}$$

The LQR problem is to find the optimal gain matrix such that the state-feedback law minimizes the quadratic cost function.

$$J = \int_0^{\infty} (x^T F \ x + u^T G\,u) \tag{25}$$

The matrices F and G are referred to as the weighting matrices on the state and the input respectively. A smaller F increased the relative weighting on the input matrix. This should decrease the magnitude of the input necessary to maintain control. In order to insure that all the states go to zero as time goes to infinity, F must be chosen to be a positive-definite matrix. G is also chosen as a positive-definite matrix to insure the control is finite. The weighting matrices are chosen based on matlab simulations and driving simulator tests such that

$$F = \begin{bmatrix} 10^5 & 0 & 0 \\ 0 & 10^3 & 0 \\ 0 & 0 & 3\times10^3 \end{bmatrix}$$

$$G = \begin{bmatrix} 0.15 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100 \end{bmatrix} \tag{26}$$

constant gain optimal control is

$$u^*(t) = \delta_d(t) = -G^{-1}B^T P\,x(t) \tag{27}$$

where, P is the steady-state solution to the matrix differential Riccatti equation of the form :

$$\dot{P} = -A^T P - PA + PBG^{-1}B^T P - F \tag{28}$$

The boundary condition at terminal time is zero, such that:

$$P(t_f) = 0 \tag{29}$$

4 Simulation Results

In this section, the closed-loop responses using the system parameters shown in Table will be presented.

Table Vehicle Parameters

Vehicle mass M_{cog}	*1600 kg*
Mass moment of Inertia J_z	*2300 kg.m²*
Distance from CoG to front axle l_f	*1.2m*
Distance from CoG to rear axle l_r	*1m*
Distance between wheels on front and rear axles b_f, b_r	*1.25m*
Front and rear tire longitudinal casters: n_{lf}, n_{lr}	*0.05m*
Effective vehicle surface A_l	*1.5m²*

In fig. 1-3 we can see the results of the model operating in a lane change manoeuvre in a highway. The vehicle is changing the position from the primary velocity of *25 m/s* and body side slip of 6 degrees while the yaw rate is *0.15 rad/sec* to the final state that the velocity has increased to 35 m/s, the body side slip angle of *4.5* degrees and the yaw rate of *0.136 rad/sec*. The inputs of the system are 4 longitudinal forces applied to the 4 wheels of the vehicle of which the first one F_{lfl} is depicted in fig.6 as well as the two steering angles of the front and rear wheels shown in figs. 4,5.

5 Conclusion

In this paper, a three degree of freedom model for a four wheel steering system was considered and an optimal controller was designed to control the vehicle during its lane change manoeuvres in highways. Simulation results reveal that the model has a very good and effective performance.

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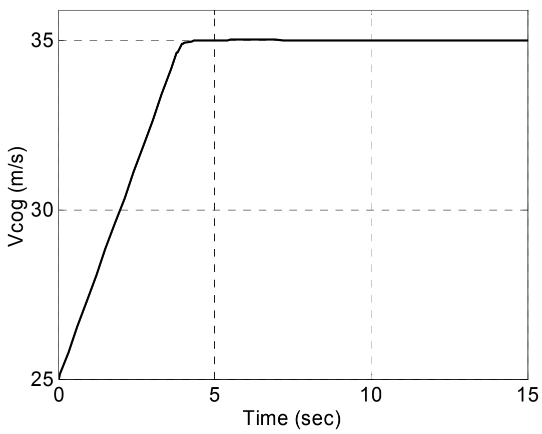

Fig. 1. Vehicle velocity during lane change maneuver

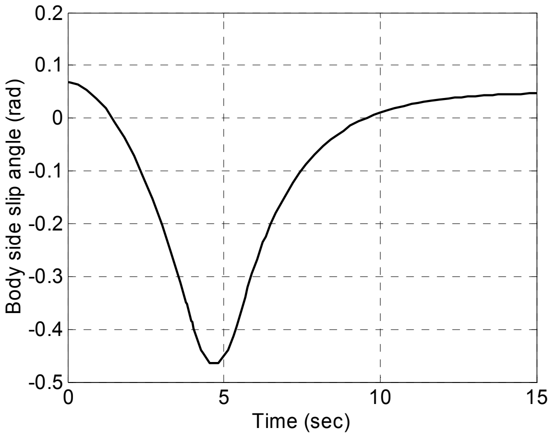

Fig. 2. Vehicle body side slip angle during lane change maneuver

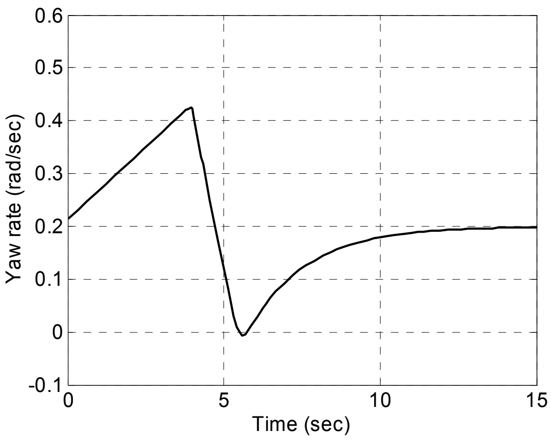

Fig. 3. Yaw rate during lane change maneuver

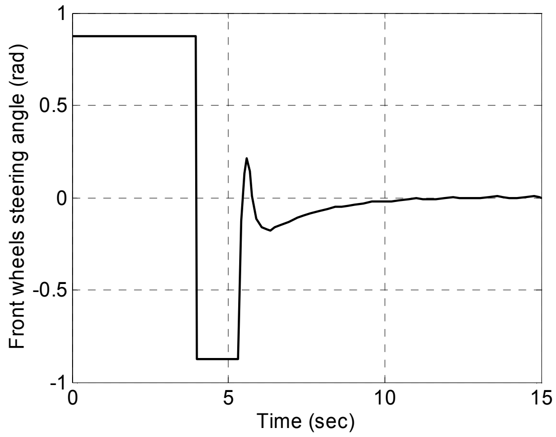

Fig. 4. Front wheels steering angle during lane change maneuver

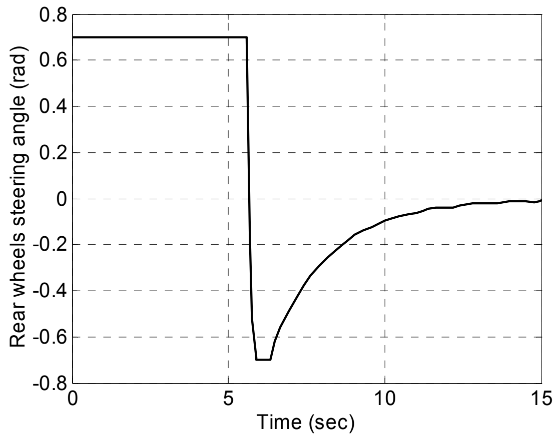

Fig. 5. Rear wheels steering angle during lane change maneuver

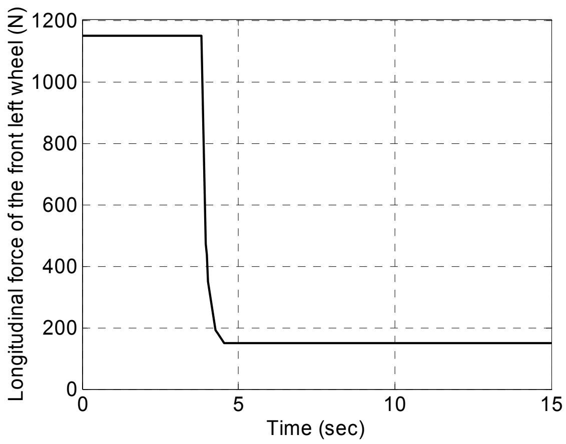

Fig. 6. Front left wheel longitudinal force during lane change maneuver