Properties and Modeling of Partial Conjunction/Disjunction

JOZO J. DUJMOVIĆ
Department of Computer Science
San Francisco State University
1600 Holloway Ave, San Francisco, CA 94132
U.S.A.
jozo@cs.sfsu.edu

HENRIK LEGIND LARSEN
Department of Computer Science
Aalborg University Esbjerg
Niels Bohrs Vej 8, DK 6700 Esbjerg
DENMARK
legind@cs.aue.auc.dk

Abstract: The partial conjunction/disjunction function (PCD) integrates conjunctive and disjunctive features in a single function. Special cases of this function include the pure conjunction, the pure disjunction, and the arithmetic mean. PCD enables a continuous transition from the pure conjunction to the pure disjunction, using a parameter that specifies a desired level of conjunction (andness) or disjunction (orness). In this paper, we investigate and compare various approaches to organize the PCD function. Our goal is to specify the most important necessary conditions that the PCD should satisfy. The next step would then be to derive the best version of PCD and use it to create other compound continuous logic functions.

Key-Words: - Continuous logic, preferences, andness, orness, partial conjunction/disjunction, andor, orand

1 Introduction

PCD is a mapping $\lambda : [0,1]^n \to [0,1]$, n>1, that has properties similar to logic functions of conjunction and disjunction. The level of similarity is adjustable using a parameter $\alpha \in [0,1]$ called the conjunction degree (andness), it its complement $\omega = 1-\alpha$, that is called the disjunction degree (orness) [3,4]. If $\alpha = 1, \omega = 0$, then the PCD behaves as the pure conjunction. Similarly, if $\omega = 1, \alpha = 0$, then the PCD behaves as the pure disjunction.

PCD is used in variety of decision problems. The main application areas for PCD function are:

(1) *System evaluation*: preference modeling, system comparison, selection, and optimization.
(2) *Classification* (pattern matching): object recognition, and information retrieval (search).

These two areas have different specific requirements that PCD must satisfy. In some cases, PCD is interpreted as a logic connective used to aggregate logic variables and compute the resulting degree of truth [3,4,5,6]. In other cases, PCD is interpreted as the averaging operator [11,12,15,16,17,18].

In this paper, we focus on applications where PCD is interpreted as a logic connective and used to create compound continuous logic functions, such as partial absorption [6], and other more complex logic functions [7]. These functions are frequently used in the area of system evaluation.

Continuous logic models based on PCD are a generalization of the binary Boolean logic. The classic Boolean logic is based on binary values $B = \{0,1\}$ and three basic operations: conjunction $x \wedge y = \min(x, y)$, disjunction $x \vee y = \max(x, y)$, and negation $\bar{x} = 1 - x$. The basic operations are used for making compound functions, such as implication $x \to y = \bar{x} \vee y$, nand $x \barwedge y = \overline{x \wedge y}$, nor $x \,\overline{\vee}\, y = \overline{x \vee y}$, exclusive or $x \oplus y = (\bar{x} \wedge y) \vee (x \wedge \bar{y})$, and equivalence $x \sim y = (\bar{x} \wedge \bar{y}) \vee (x \wedge y)$.

The binary set B can be replaced by the unit interval $I = [0,1]$. If $x \in I$ and $y \in I$ the same logic operations (*min*, *max*, and $x \mapsto 1 - x$) can be used to get the traditional continuous logic.

In this paper, we are interested in continuous logic functions that are suitable for building multiple criteria decision models [1]. Suppose that a criterion for system evaluation consists of m requirements that a system is expected to satisfy. So, we have m input variables $x_1,...,x_m$, $x_i \in I$, $i = 1,...,m$ that reflect the level of satisfaction of m specific requirements. Consequently, x_i is interpreted as the degree of truth in the statement asserting that the i^{th} requirement is completely satisfied. We call $x_1,...,x_m$ *elementary preferences*. Our problem is to create a model that uses elementary preferences to compute the *global preference* $y = L(x_1,...,x_m), y \in I$ that reflects the global satisfaction of requirements. More precisely, the global preference is interpreted as the degree of truth in the statement that a complex system completely satisfies all requirements.

A related problem is to find functions that are suitable for building the decision model $L: I^m \to I$. These functions must be suitable for modeling logic relationships between individual requirements. These relationships include simultaneity, replaceability, various combinations of mandatory, desired, and optional features, etc.

If preferences are interpreted as degrees of truth, then the mathematical logic is a natural environment for creating system evaluation models. The corresponding *continuous preference logic* (CPL) should reflect those aspects of human decision making that include adjustable level of andness, orness, and relative importance (weights) [5,6].

2 Basic Sensitivity Features

The global satisfaction of requirements regularly increases when we increase the satisfaction of component requirements: $\partial y / \partial x_i > 0$. This is the main reason why traditional continuous logic functions that are only combinations of *min*, *max*, and $x \mapsto 1-x$ operators cannot be suitable for building decision models for system evaluation and comparison. With these operators, the condition $\partial y / \partial x_i > 0$ holds only for a small subset of input preferences (those that have extreme values in their groups; for example, in a group of *n*, if $y = \min(x_1,...,x_n) = x_k$, then $\forall x_i > x_k$, $\partial y / \partial x_i = 0$). In the majority of cases, this is not acceptable because it implies the inability to improve a system by improving its components.

This problem can be solved if instead of pure conjunction (*min*) and pure disjunction (*max*), we use a partial conjunction/disjunction $\lambda: I^n \to I$, $n > 1$, that includes *min* and *max* as extreme special cases. Let $x_1,...,x_n$, $(x_i \in I, i = 1,...,n)$ be input preferences, and let $\omega \in I$ be the necessary orness of input preferences. Let $y = \lambda(x_1,...,x_n;\omega)$, $y \in I$ be an output preference computed using the PCD function λ. The basic PCD properties are:

$$\lambda(x_1,...,x_n;0) = x_1 \wedge ... \wedge x_n = \min(x_1,...,x_n)$$

$$\lambda(x_1,...,x_n;1) = x_1 \vee ... \vee x_n = \max(x_1,...,x_n)$$

Since PCD is a mix of conjunctive and disjunctive properties (and includes conjunction and disjunction as special cases), it can be called "*andor*" or "*orand*". More specifically, the name *andor* can be used for $0 < \omega < 0.5$, (α>ω) and *orand* for $0.5 < \omega < 1$, (α<ω). We also use the symbol $\Diamond$ for the andor/orand operator, assuming that it corresponds to specific level of orness (therefore, $y = \lambda(x_1,...,x_n;\omega)$ and $y = x_1 \Diamond x_2 \Diamond ... \Diamond x_n$ are equivalent notations).

If $0 < \omega < 1$, $0 < x_i < 1$, $i = 1,...,n$, then PCD has the following sensitivity features:

$$\frac{\partial}{\partial \omega} \lambda(x_1,...,x_n;\omega) > 0$$

$$\frac{\partial}{\partial x_i} \lambda(x_1,...,x_n;\omega) > 0$$

The last condition means that generally a system can be improved if we improve any of its components.

3 Means as Logic Functions

All means satisfy the fundamental PCD property

$$x_1 \wedge ... \wedge x_n \le \lambda(x_1,...,x_n;\omega) \le x_1 \vee ... \vee x_n .$$

Consequently, some means can be interpreted as logic functions and the PCD can be organized as a mean. Of course, the theory of means offers a wide spectrum of candidate mathematical models [9,14], and the question is which mean is the best material for building the PCD function. Obviously, the most suitable are those means that have adjustable parameters enabling easy adjustment of orness/andness and continuous transition from the pure conjunction to the pure disjunction.

We investigated this problem in [3,4] using a general framework of Losonczi means [13]:

$$\lambda(x_1,...,x_n) = F^{-1}\left(\sum_{i=1}^{n} \Phi_i(x_i) F_i(x_i) \Big/ \sum_{i=1}^{n} \Phi_i(x_i)\right)$$

The mean uses weight functions $\Phi_i : I \to \{0\} \cup R^+$ and a strictly monotone function $F: I \to R$. In a special case of constant weights $\Phi_i = W_i$, $W_i > 0$, $i = 1,...,n$, $W_1 + ... + W_n = 1$, the Losonczi mean reduces to the weighted quasi-arithmetic mean:

$$\lambda(x_1,...,x_n) = F^{-1}\left(\sum_{i=1}^{n} W_i F_i(x_i)\right)$$

Of course, even in this simplified case, there is a spectrum of possible *F* functions.

A variety of means and generate results between the pure conjunction and the pure disjunction [9]. However, the PCD must have an adjustable parameter that enables a continuous transition between the pure conjunction and the pure disjunction. This condition simplifies the selection of the *F* function

The simplest form of the *F* function is the power function $F(x) = x^r$, $r \in R$. This selection yields traditional weighted power means [9,14]:

$$\lambda_p(x_1,\dots,x_n;r)=\begin{cases}\left(\sum_{i=1}^{n}W_i x_i^r\right)^{1/r}, & 0<|r|<+\infty\\ \prod_{i=1}^{n}x_i^{W_i}, & r=0\\ x_1\wedge\dots\wedge x_n, & r=-\infty\\ x_1\vee\dots\vee x_n, & r=+\infty\end{cases}$$

$$\frac{\partial}{\partial x_i}\lambda_p(x_1,\dots,x_n;r)\ge 0,\quad i=1,\dots,n$$

$$\frac{\partial}{\partial r}\left(\sum_{i=1}^{n}W_i x_i^r\right)^{1/r}>0,\quad x_i\ne x_j,\quad i\ne j,\quad |r|<+\infty$$

$$\frac{\partial}{\partial r}\left(\sum_{i=1}^{n}W_i x_i^r\right)^{1/r}=0,\quad x_1=x_2=\dots=x_n$$

The advantage of weighted power means is that their properties are well known in mathematics, because their special cases are harmonic mean ($r=-1$), geometric mean ($r=0$), arithmetic mean ($r=1$), and quadratic mean ($r=2$).

Another option is to use the exponential function $F(x)=e^{rx}, r\in R$ and the weighted exponential mean:

$$\lambda_e(x_1,\dots,x_n;r)=\begin{cases}\frac{1}{r}\ln\left(\sum_{i=1}^{n}W_i e^{rx_i}\right), & 0<|r|<+\infty\\ \sum_{i=1}^{n}W_i x_i, & r=0\\ x_1\wedge\dots\wedge x_n, & r=-\infty\\ x_1\vee\dots\vee x_n, & r=+\infty\end{cases}$$

For these means, the andness and orness are functions of the parameter *r*.

4 Andness and Orness

Let us first consider a case of two variables and the andor function $y=x_1\Diamond x_2$. The andness α is a measure of similarity between the andor function and the pure conjunction. Similarly, the orness ω is a measure of similarity between the andor function and the pure disjunction. They can be defined as follows:

$$\alpha(x_1,x_2)=\frac{(x_1\vee x_2)-(x_1\Diamond x_2)}{(x_1\vee x_2)-(x_1\wedge x_2)},\quad 0\le\alpha(x_1,x_2)\le 1$$

$$\omega(x_1,x_2)=\frac{(x_1\Diamond x_2)-(x_1\wedge x_2)}{(x_1\vee x_2)-(x_1\wedge x_2)},\quad 0\le\omega(x_1,x_2)\le 1$$

$$\alpha(x_1,x_2)+\omega(x_1,x_2)=1,\qquad (x_1\ne x_2)$$

According to this definition, α and ω depend on x_1 and x_2. Their mean values $\bar{\alpha}$ and $\bar{\omega}$ are:

$$\bar{\alpha}=\int_0^1 dx_1\int_0^1\alpha(x_1,x_2)dx_2,\qquad \bar{\omega}=\int_0^1 dx_1\int_0^1\omega(x_1,x_2)dx_2$$

The values of $\bar{\alpha}$ and $\bar{\omega}$ can be obtained using numerical integration. Since

$$\alpha(x_1,x_2)(x_1\wedge x_2)+\omega(x_1,x_2)(x_1\vee x_2)=x_1\Diamond x_2$$

it follows that α and ω can be constant if the andor function is defined in the following linear form:

$$x_1\Diamond x_2=\alpha(x_1\wedge x_2)+\omega(x_1\vee x_2)=\alpha(x_1\wedge x_2)+(1-\alpha)(x_1\vee x_2)$$

This form of andor function is convenient for understanding the concepts of andness and orness. Unfortunately, the linear form does not satisfy some of desired PDF properties and its use is limited.

In system evaluation practice, decision makers specify desired constant levels of andness and orness. Suitable definitions for constant andness and orness, proposed in [3,4], can be based on mean values, as follows:

$$\omega=\frac{\overline{x_1\Diamond x_2}-\overline{x_1\wedge x_2}}{\overline{x_1\vee x_2}-\overline{x_1\wedge x_2}},\qquad \alpha=\frac{\overline{x_1\vee x_2}-\overline{x_1\Diamond x_2}}{\overline{x_1\vee x_2}-\overline{x_1\wedge x_2}}=1-\omega$$

The following example illustrates the computation of andness and orness for the geometric mean $\sqrt{x_1x_2}$, *i.e.*, for *n* = 2 and $W_1=W_2=½$:

$$\overline{x_1\wedge x_2}=\int_0^1 dx_1\int_0^1(x_1\wedge x_2)dx_2=\int_0^1 dx_1\left(\int_0^{x_1}x_2dx_2+\int_{x_1}^1 x_1dx_2\right)$$

$$=\int_0^1\left(x_1-\frac{x_1^2}{2}\right)dx_1=\frac{1}{3}$$

$$\overline{x_1\vee x_2}=\int_0^1 dx_1\int_0^1(x_1\vee x_2)dx_2=\int_0^1 dx_1\left(\int_0^{x_1}x_1dx_2+\int_{x_1}^1 x_2dx_2\right)$$

$$=\int_0^1\left(\frac{1+x_1^2}{2}\right)dx_1=\frac{2}{3}$$

$$\overline{x_1\Diamond x_2}=\overline{\sqrt{x_1x_2}}=\int_0^1 dx_1\int_0^1\sqrt{x_1x_2}\,dx_2=\int_0^1\sqrt{x_1}\,dx_1\int_0^1\sqrt{x_2}\,dx_2=\frac{4}{9}$$

$$\omega=\frac{\overline{x_1\Diamond x_2}-\overline{x_1\wedge x_2}}{\overline{x_1\vee x_2}-\overline{x_1\wedge x_2}}=\frac{\overline{x_1\Diamond x_2}-1/3}{1/3}=3(\overline{x_1\Diamond x_2})-1=\frac{1}{3}$$

$$\alpha=1-\omega=2-3(\overline{x_1\Diamond x_2})=\frac{2}{3}$$

Using a similar approach [3], in the case of *n* variables we have:

$$\overline{x_1\wedge\dots\wedge x_n}=\int_0^1 dx_1\int_0^1 dx_2\dots\int_0^1(x_1\wedge\dots\wedge x_n)dx_n=\frac{1}{n+1}$$

$$\overline{x_1\vee\dots\vee x_n}=\int_0^1 dx_1\int_0^1 dx_2\dots\int_0^1(x_1\vee\dots\vee x_n)dx_n=\frac{n}{n+1}$$

$$\omega=\frac{\overline{x_1\Diamond\dots\Diamond x_n}-\overline{x_1\wedge\dots\wedge x_n}}{\overline{x_1\vee\dots\vee x_n}-\overline{x_1\wedge\dots\wedge x_n}}=\frac{(n+1)(\overline{x_1\Diamond\dots\Diamond x_n})-1}{n-1}=1-\alpha$$

$$\alpha=\frac{\overline{x_1\vee\dots\vee x_n}-\overline{x_1\Diamond\dots\Diamond x_2}}{\overline{x_1\vee\dots\vee x_n}-\overline{x_1\wedge\dots\wedge x_n}}=\frac{n-(n+1)(\overline{x_1\Diamond\dots\Diamond x_n})}{n-1}=1-\omega$$

These definitions show that now *α* and *ω* depend on *n*. For example, for the geometric mean the andness is

$$\alpha(n)=\frac{n}{n-1}\left[1-\left(\frac{n}{n+1}\right)^{n-1}\right]$$

$$\alpha(2)=2/3=0.667 \quad \lim_{n\to\infty}\alpha(n)=1-\frac{1}{e}=0.632$$

In the case of power means and exponential means α and ω also depend on the parameter r: $\alpha = \mathrm{A}(n,r)$, $\omega=\Omega(n,r)$. Similarly, r can be computed from the desired value of α or ω:

$$r=\rho_n(\alpha)=\rho_n(1-\omega)$$

$$\mathrm{A}(n,\rho_n(\alpha))=\alpha\,, \quad \Omega(n,\rho_n(1-\omega))=\omega$$

In [12] Larsen proposed the following approximation:

$$r=\rho_n(\alpha)=\frac{\omega}{\alpha}=\frac{1}{\alpha}-1\,, \quad \alpha\le 1/2 \quad (orand)$$

Exact numeric values of parameter r for n=2 and nine characteristic levels of andness and orness are shown in Table 1. Larsen's approximation yields a satisfactory accuracy: $\rho_2(0.5)=1$, $\rho_2(0.375)=1.67$, $\rho_2(0.25)=3$, $\rho_2(0.125)=7$, and $\rho_2(0)=+\infty$. A more precise numerical approximation is:

$$r=\frac{-0.742+3.363\omega-4.729\omega^2+3.937\omega^3}{(1-\omega)\omega}\,, \quad 0\le\omega\le 1$$

Table 1. The values $r=\rho_2(\alpha)$ for power means and exponential means

Symbol	Andness		Orness		r for power mean	r for expo. mean
	Level	α	Level	ω		
D	Min	0	Max	1	$+\infty$	$+\infty$
D+	Very Low	0.125	Very High	0.875	9.53	14.0
DA	Low	0.25	High	0.75	3.93	5.40
D-	Medium Low	0.375	Medium High	0.625	2.02	2.14
A	Medium	0.5	Medium	0.5	1	0
C-	Medium High	0.625	Med Low	0.375	0.26	-2.14
CA	High	0.75	Low	0.25	-0.72	-5.40
C+	Very High	0.875	Very Low	0.125	-3.51	-14.0
C	Max	1	Min	0	$-\infty$	$-\infty$

The exponential mean satisfies $\rho_2(\alpha)=-\rho_2(1-\alpha)$. It provides an alternative to the power means for the cases where the mandatory property (Section 6) is not desirable.

5 Weights and Relative Importance

In the MCDM area, there is no consensus on the meaning of weights. Choo et al. [2] identify 13 different interpretations of weights in MCDM. These interpretations include weights as degrees of relative importance of component criteria. Weights can also express the level of confidence, the level of evaluator's expertise, etc.

In PCD models, weights are used to express relative importance of input preferences, and this interpretation is used in the paper. Traditionally, relative importance is considered constant. In a general case, however, weights can depend on preferences and Losonczi means [13] provide a convenient mechanism for realizing this property.

In PCD models, we assume the independence of weights and andness/orness. Indeed, in system evaluation practice, evaluators independently think about the relative importance of individual inputs, and about the desired level of their simultaneity (andness). In this area, we sometimes encounter the "low weight – high andness paradox". The low weight is interpreted as low importance. However, the high andness means the requirement for high simultaneity, which indirectly means that all components are necessary and consequently very important. So, a low weight (e.g. less than 5%) and high andness (e.g. more than 75%) in a general case can be considered a contradiction, and should be avoided.

There are two approaches to modeling weights:

- multiplicative approach
- implicative approach

The multiplicative approach is used in power means and in exponential means, where weights multiply satisfaction degrees as in the case of arithmetic mean:

$$y=W_1x_1+W_2x_2+\ldots+W_nx_n$$

Weights mean importance because they determine both the level of penalty for a low satisfaction and the level of reward for a high satisfaction. High level of penalty/reward for an input can obviously mean only one thing: the input is important.

The implicative approach is based on the concept that it is not acceptable that something is important and it has low satisfaction:

$$W_i\to x_i=\overline{W_i\wedge\overline{x_i}}=\overline{W_i}\vee x_i\ \left(=\max\left(1-W_i,x_i\right)\right)$$

where $\overline{x}$ denotes the standard negation, $\overline{x}=1-x$. This as same as requiring, for all i, x_i to be high if it is important. Hence, the implicative importance weighted aggregate is, for the pure *and*, (cf. [11]):

$$y_\wedge=\left(\overline{W_1}\vee x_1\right)\wedge\cdots\wedge\left(\overline{W_n}\vee x_n\right)$$

and, through duality [12], for the pure *or*:

$$y_\vee=\overline{\left(\overline{W_1}\vee\overline{x_1}\right)\wedge\cdots\wedge\left(\overline{W_n}\vee\overline{x_n}\right)}=\left(W_1\wedge x_1\right)\vee\cdots\vee\left(W_n\wedge x_n\right)$$

By the IAWA operators [12], $y_\wedge$ and $y_\vee$ are obtained for andness 1 and 0 respectively, with $\vee$ and $\wedge$ in the importance weighting functions,

$\overline{W_i} \vee x_i$ and $W_i \wedge x_i$, chosen as the dual t-conorms and t-norms algebraic sum and product. The IAWA operators provide implicative importance weighting (in [12] just called "importance weighting") for all degrees of andness in *I*, through, essentially, an andness-orness weighted sum of the importance weighting functions at andness 1 and 0. At andness ½, the AIWA operator represents the weighted arithmetic mean.

6 Mandatory Requirements

In the area of system evaluation, we regularly have the situation where one or more of inputs represent mandatory requirements. Suppose that in the case of computer evaluation the final stage of aggregating preferences includes two components: hardware (x_1) and software (x_2). The global preference of the evaluated computer is $y = x_1 \Diamond x_2$. If x_1=0 (inappropriate hardware) we must reject such a computer (the andor function must generate the result *y*=0). Similarly, if x_2=0 (e.g. no software), then again *y*=0. Obviously, both good hardware *and* good software are mandatory requirements that all computers must satisfy. Therefore, we need a PCD function that satisfies the condition $x_1 \Diamond 0 = 0 \Diamond x_2 = 0$ (rejection of system that does not satisfy mandatory requirements).

Unfortunately, in this case, the pure conjunction $y = x_1 \Diamond x_2 = x_1 \wedge x_2$ cannot be used, because such a rigid criterion would not be acceptable in regular cases where $x_1 > 0$ and $x_2 > 0$. Indeed, the majority of evaluators would not accept the equality $0.5 \wedge 0.5 = 1 \wedge 0.5 = 0.5$ that claims that a system with an average hardware and software is equivalent to the system having perfect hardware and an average software. In other words, instead of pure conjunction we need a *partial conjunction* that satisfies the following *mandatory requirements conditions*:

$$0 \Diamond x_2 = x_1 \Diamond 0 = 0$$

$$x_1 \Diamond (x_1 + a) > x_1 , \quad x_1 > 0 , \quad a > 0$$

For example, the geometric mean $\sqrt{x_1 x_2}$ obviously satisfies these conditions, and so do the weighted power means $\lambda_p(x_1,...,x_n;r)$ for $r \le 0$.

In addition to the use of mandatory requirements in PCD, this property is indispensable for generating the partial absorption function [6], and other more complex logic function [7].

Exponential means $\lambda_e(x_1,...,x_n;r)$ do not satisfy the mandatory requirements conditions. However, this is a desirable feature in other applications, where the missing satisfaction of one criterion should not eliminate the evaluated object, such as in object recognition and information retrieval.

7 Sufficient Requirements

If we take a function that is dual to a partial conjunction that satisfies the mandatory requirements conditions, we get a *partial disjunction* (*orand*) that satisfies the following *sufficient requirements conditions*:

$$1 \Diamond x_2 = x_1 \Diamond 1 = 1$$

$$x_1 \Diamond (x_1 - a) < x_1 , \quad x_1 < 1 , \quad a > 0$$

For example, since the geometric mean satisfies the mandatory requirements conditions, it follows that $x_1 \nabla ... \nabla x_n = 1 - [(1-x_1)(1-x_2)...(1-x_n)]^{1/n}$ satisfies the sufficient requirements conditions. However, such requirements occur seldom in practical problems.

8 De Morgan's PCD Functions

De Morgan's law is a convenient mechanism for creating PCD functions that have various specific properties. Let Δ denote partial conjunction (*andor*) and let ∇ denote partial disjunction (*orand*). These are special cases of the PCD $\Diamond$: if α>1/2 then $\Diamond$ becomes Δ, and if ω>1/2 then $\Diamond$ becomes ∇. De Morgan's laws can be written as follows:

$$x_1 \nabla ... \nabla x_n = 1 - (1-x_1) \Delta ... \Delta (1-x_n)$$

$$x_1 \Delta ... \Delta x_n = 1 - (1-x_1) \nabla ... \nabla (1-x_n)$$

These formulas show how to make a conjunctive partial absorption if we have a model of disjunctive partial absorption and vice versa. In addition, if $x_1 \Delta ... \Delta x_n$ is a partial conjunction with andness *α*=*c*, then $1-(1-x_1)\Delta ... \Delta(1-x_n)$ is the partial disjunction with orness *ω*=*c*. For example, if the partial conjunction is modeled using the geometric mean then the corresponding (dual) partial disjunction can be modeled as follows:

$$x_1 \Delta ... \Delta x_n = (x_1 x_2 ... x_n)^{1/n}$$

$$x_1 \nabla ... \nabla x_n = 1 - [(1-x_1)(1-x_2)...(1-x_n)]^{1/n}$$

The first function can be used to model the mandatory requirements ($x_i = 0$ yields *y*=0, and it is necessary to have $x_i > 0,\ i = 1,...,n$ to produce *y*>0). Similarly, the second function can be used to model the sufficient requirements (it is sufficient to have $x_i = 1,\ \ i \in \{1,..,n\}$, to produce *y*=1). The same effects, at a higher level of andness/orness, can be achieved using the harmonic mean:

$$x_1 \Delta \ldots \Delta x_n = n/(1/x_1 + 1/x_2 \ldots 1/x_n)$$
$$x_1 \nabla \ldots \nabla x_n = 1 - n/[1/(1-x_1) + 1/(1-x_2) + \ldots + 1/(1-x_n)]$$

If we want to avoid mandatory and sufficient requirements we could use the quadratic mean to model the orand and andor, as follows:

$$x_1 \nabla \ldots \nabla x_n = [(x_1^2 + x_2^2 + \ldots + x_n^2)/n]^{1/2}$$
$$x_1 \Delta \ldots \Delta x_n = 1 - \{[(1-x_1)^2 + (1-x_2)^2 + \ldots + (1-x_n)^2]/n\}^{1/2}$$

In this case neither orand can model sufficient requirements, nor can andor model mandatory requirements.

Generally, De Morgan's PCD functions are defined as follows:

$$y = \begin{cases} \lambda(x_1,\ldots,x_n;\omega) \\ 1-\lambda(1-x_1,\ldots,1-x_n;1-\omega) \end{cases} \quad \text{either } \omega \ge \frac{1}{2} \text{ or } \omega \le \frac{1}{2}$$

Of course, such functions always satisfy De Morgan's laws.

Some PCD functions do not satisfy De Morgan's laws. One such example is the PCD based on weighted power means [4]; however, the corresponding errors are sufficiently small so that the function is suitable for practical applications.

De Morgan's approach can be used to define functions that satisfy De Morgan's laws, e.g.:

$$\lambda_p(x_1,\ldots,x_n;\rho_n(\alpha)) = \begin{cases} \left(\sum_{i=1}^{n} W_i x_i^{\rho_n(\alpha)}\right)^{1/\rho_n(\alpha)} \\ 1-\left(\sum_{i=1}^{n} W_i (1-x_i)^{\rho_n(1-\alpha)}\right)^{1/\rho_n(1-\alpha)} \end{cases}$$
$$\text{either} \quad \alpha \ge 1/2 \quad \text{or} \quad \alpha \le 1/2$$

This function has the property: if $\alpha \ge 2/3$ then it satisfies mandatory requirements, and if $\alpha < 2/3$, it does not satisfy the mandatory requirements. For example, the andness $\alpha = 2/3$ can be achieved using the geometric mean $y = \sqrt{x_1 x_2}$; in this case, the mandatory requirements are satisfied. The same level of andness (but without mandatory requirements) can be achieved using the function $y = 1 - [0.5(1-x_1)^r + 0.5(1-x_2)^r]^{1/r}, \quad r = 2.41$.

9 Functions with Constant Andness

Means that don't have adjustable parameter can be interpreted as logic functions with constant andness. For example, the Losonczi mean in the case where $\Phi_i(x_i) = F_i(x_i) = x_i$ generates the antiharmonic mean [7] where the weight of each variable equals its value:

$$\lambda(x_1,x_2) = (x_1^2 + x_2^2)/(x_1 + x_2), \quad x_1 + x_2 \ne 0 .$$

Such weights emphasize the importance of large values. This is a typical disjunctive property, and the corresponding orness is ω=0.77. By using De Morgan's law we can easily create a dual function that has andness α=0.77:

$$\lambda(x_1,x_2) = 1 - \frac{(1-x_1)^2 + (1-x_2)^2}{2 - x_1 - x_2}, \quad x_1 + x_2 \ne 2$$

10 Associativity and Distributivity

Errors in weight assessment increase with increasing number of aggregated inputs [8]. Associativity ($(x_1 \Diamond x_2 \Diamond x_3) = (x_1 \Diamond x_2) \Diamond x_3 = x_1 \Diamond (x_2 \Diamond x_3)$) is a desirable algebraic property that helps reducing the errors in weight assessment by grouping input preferences in small groups of up to 5 inputs. Similarly, transfigurations of preference aggregation structures are possible if the partial conjunction and the partial disjunction are mutually distributive, e.g.

$$x_1 \Delta (x_2 \nabla x_3) = (x_1 \Delta x_2) \nabla (x_1 \Delta x_3)$$
$$x_1 \nabla (x_2 \Delta x_3) = (x_1 \nabla x_2) \Delta (x_1 \nabla x_3)$$

Andor functions based on weighted power means do not satisfy algebraic properties of distributivity and associativity. According to [5], the errors are not significant: (1) if the andness/orness are 0, ½, or 1, these errors are 0, and (2) the average distributivity error is 1.4%, and the average associativity error is regularly less than 1%. This level of errors is not significant because errors in estimating preferences, andness, and weights are regularly significantly larger [8].

11 Pseudo averaging operators

This paper is focused on applications of PCD operators in logic decision models, with emphasis on system evaluation. Such models use the "quality range relation" (the basic property of means) $x_1 \wedge \ldots \wedge x_n \le \lambda(x_1,\ldots,x_n;\omega) \le x_1 \vee \ldots \vee x_n$ that reflects the concept that the quality of system cannot be better than the quality of its best component, or worse than the quality of its worst component. This directly yields the idempotency $\lambda(x,\ldots,x;\omega) = x$ and insensitivity $\partial\lambda(x,\ldots,x;\omega)/\partial\omega = 0, \quad \omega \in I$.

It is well-known these properties are only satisfied by mean (over averaging) operators. Thus the intersection operators (t-norms) and the union operators (t-conorms) do not have these properties, but have other properties, such as associativity and the existence a neutral (identity) element (namely 1 and 0, respectively), see, for instance, [17].

Some authors propose to combine intersection and union operators to obtain averaging like operators with adjustable levels of andness/orness. The idea of such models is either a linear (additive) combination $x_1 \Diamond x_2 = q(x_1 \nabla x_2) + (1-q)(x_1 \Delta x_2)$, or a multiplicative form $x_1 \Diamond x_2 = (x_1 \nabla x_2)^q (x_1 \Delta x_2)^{1-q}$, $0 \le q \le 1$ where q plays the role of orness and 1-q the role of andness.

Such operators, proposed by Zimmermann and Zysno [19], are $x_1 \Diamond x_2 = (1-\bar{x}_1\bar{x}_2)^q (x_1 x_2)^{1-q}$ (based on geometric mean) or $x_1 \Diamond x_2 = q(1-\bar{x}_1\bar{x}_2) + (1-q)(x_1 x_2)$ (based on the arithmetic mean). Pseudo averaging models of the form $x_1 \Diamond x_2 = c_0 + c_1 x_1 + c_2 x_2 + c_{12} x_1 x_2$ are frequently used in the utility theory [1,10].

12 Fuzzy Logic Averaging Operators

In fuzzy logic, the term "averaging operators" for mappings $H:[0,1]^n \to [0,1], n>1$ that are monotonic increasing in all its arguments, continuous, symmetric in all its arguments, and idempotent. We notice that PCD functions satisfy these requirements and therefore also are averaging operators.

A particular interesting family of averaging operators are the OWA (ordered weighted averaging) operators [11,12,15,16,18]. An OWA operator is characterized by a vector of positional weights $(v_1,\ldots,v_n) \in I^n$ that satisfies $v_1 + \cdots + v_n = 1$. The aggregate y of an argument vector $(x_1,\ldots,x_n) \in I^n$ is defined by:

$$y = v_1 x_{(1)} + \cdots + v_n x_{(n)}$$

where $(\cdot)$ is an index permutation such that $x_{(1)} \ge \cdots \ge x_{(n)}$. The orness of such an operator is a function of the OWA weighting tuple, namely:

$$orness = \frac{v_1(n-1) + v_2(n-2) + \ldots + v_{n-1}}{n-1}$$

The andness is defined, as usual, by andness = 1 – orness, that is,

$$andness = \frac{v_2 + \ldots + v_{n-1}(n-2) + v_n(n-1)}{n-1}$$

From this definition, if $v_i = 1/n,\ \ i=1,\ldots n$, the OWA operator represents the arithmetic mean, with *orness*=*andness*=1/2; if $v_1 = 1\ \ (v_i = 0 \text{ for } i>1)$,it represents the pure *or* (the max operator), with orness = 1; if $v_n = 1\ \ (v_i = 0 \text{ for } i<n)$,it represents the pure *and* (the min operator), with orness = 0. If n=2, then $orness = v_1$ and $andness = v_2$ (the weight of the larger argument represents the orness and the weight of the smaller argument represents the andness). The measures of orness and andness can be shown to comply with the measures defined in Section 4.

A certain advantage of the OWA family is that it for all degrees or orness in (0, 1) allows us to construct multiple averaging operators with different properties. For instance, the following OWA weighing tuples (illustrated with n = 5) all have orness = 0.5:

$(1/n, 1/n, 1/n, 1/n, 1/n)$ (arithmetic mean)

$(0,0,1,0,0)$ (median)

$(0,1/3,1/3,1/3,0)$ (olympic mean)

OWA operators are further characterized by the dispersion of there OWA weights. The dispersion is, in its normalized form, defined by:

$$disp(v_1,\ldots,v_n) = -(v_1 \ln v_1 + \cdots + v_n \ln v_n)/\ln n$$

For instance, the dispersion of the above shown three OWA weighting vectors are, respectively, 1, 0, and 0.683.

An extension to (implicative) importance weighted OWA operators is analyzed in [11] and presented in [12] in a form that represents the weighted arithmetic mean for orness 0.5. There are several other extensions and properties of OWA operators described in the literature, but the above brief introduction will suffice for the scope of this paper.

An iterative approach to order weighted like PCD operators was proposed in [7]. Let the input preferences be sorted so that $x_i \le x_{i+1}$, $i>0$. If n=2 the iterative model is

$$y = x_1 \Diamond x_2 = \alpha(x_1 \wedge x_2) + \omega(x_1 \vee x_2) = \alpha x_1 + \omega x_2 .$$

In the case of 3 variables, the function $y = x_1 \Diamond x_2 \Diamond x_3$ is defined by the following iterative procedure:

```
while( x3 - x1 > ε )            //  ε = a small error
{
   x12 = αx1 + ωx2;   x13 = αx1 + ωx3;   x23 = αx2 + ωx3;
   x1 = x12;   y = x2 = x13;   x3 = x23;
}
```

This procedure can be expressed in a matrix form as follows:

$$\begin{bmatrix} y \\ y \\ y \end{bmatrix} = \lim_{k\to+\infty} \begin{bmatrix} \alpha & \omega & 0 \\ \alpha & 0 & \omega \\ 0 & \alpha & \omega \end{bmatrix}^k \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The resulting PCD for 3 variables is:

$$y = x_1 \Diamond x_2 \Diamond x_3 = \frac{\alpha^2 x_1 + \alpha\omega x_2 + \omega^2 x_3}{\alpha^2 + \alpha\omega + \omega^2}$$

$$y = \begin{cases} x_1, & \alpha = 1 \\ (x_1 + x_2 + x_3)/3, & \alpha = 1/2 \\ x_3, & \alpha = 0 \end{cases}$$

By expanding this procedure for 4 variables we have

$$y = x_1 \Diamond x_2 \Diamond x_3 \Diamond x_4 = [\alpha^3(1+2\alpha+2\alpha^2)x_1 + \\ + \alpha^2(1+\alpha-2\alpha^3)x_2 + \alpha(1-\alpha^2-2\alpha^3+2\alpha^4)x_3 + \\ +(1-\alpha-\alpha^2-\alpha^3+4\alpha^4-2\alpha^5)x_4]/(1+4\alpha^4)$$

$$y = \begin{cases} x_1, & \alpha = 1 \\ (x_1+x_2+x_3+x_4)/4, & \alpha = 1/2 \\ x_4, & \alpha = 0 \end{cases}$$

This process can be continued in a similar way to generate the PCD for more than 4 variables.

The iterative order weighted PCD is consistent with OWA averaging operators for $\alpha = 0, 0.5, 1$; however it uses the "a priori andness/orness" that is different from the OWA approach and the mean value approach.

13 Conclusions and future work

The PCD operators can be organized, interpreted, and used in a variety of ways. We focused on interpretations in continuous preference logic, and briefly introduced the fuzzy logic averaging operator interpretation. The organization of the PCD operators based on weighted power means is shown to be the most attractive for interpretations in preference logic and applications in system evaluation. We proposed a new PCD organization based on exponential means. We introduced a distinction between two kinds of importance weighting, namely multiplicative and implicative. These new concepts deserve future research. Comparison of various approaches to definition of andness and orness, as well as new forms of PCD based on Losonczi means, also deserve future research.

We have presented and discussed several properties of PCD operators. As means all the functions satisfy the quality range relation, and are commutative, monotonic, continuous, and idempotent. For the usability of such operators, the level of andness/orness should be easily adjustable. The importance of criteria (or, rather, of satisfying criteria) should be easily adjustable through weights.

The weighted operators should provide a generalization of the unweighted operator, such that the latter is retained when the criteria are evenly weighted. The two kinds of importance weighting generalizations, multiplicative and implicative, should be supported; the choice between these kinds depends on the kind of the decision problem. The multiplicative form is primarily applied for estimating the level of satisfaction of requirements, as applied in, for instance, system evaluation. The implicative form primarily is applied for ranking of options according to their satisfaction of joint criteria (constraints), as applied in selection, classification, and recognition problem solving.

An often required property in system evaluation is the mandatory requirements property. This says that, regardless of the andness and the (positive) importance weights, the aggregate must evaluate to zero, if at least one of the criteria is not satisfied at all. For other kinds of problems, absence of the mandatory property may be required; for instance, in recognition, where the failure to satisfy a single criterion should not invalidate the option, but just "punish", depending on the importance of satisfying the criterion.

Associativity and distributivity are not properties of PCD operators. However, similar properties may be computationally useful and is indeed possible, with an ignorable small error, with weighted power means. The De Morgan duality applying to PCD operators (around the andness 0.5) may also be computationally useful.

Finally, for multiplicative importance weighted PCD operators, we need the sensitivity property $\partial y / \partial x_i > 0$. In addition, assuming that all inputs are not equal, the condition $\partial y / \partial \omega > 0$, where ω is the orness, must be satisfied. The property $\partial y / \partial \omega > 0$ does not hold in general for implicative importance weighting.

We have seen that the (multiplicative) weighted power means provide a useful set of properties; the mandatory property is obtained for non-positive values of the parameter *r*, yielding andness degree above about 2/3. The lowest andness for which the mandatory property is obtained is the geometric mean ($r = 0$) with andness between 0.667 (at $n = 2$) and 0.632 ($n \gg 1$). If absence of the mandatory property is required with multiplicative weighting for all degree of andness, then a possible choice is the exponential mean. A nice feature of the exponential mean is it symmetry: for $r=0$ it generates the arithmetic mean, and the andness for *r* equals the orness for $-r$.

The implicative importance weighting is provided by the IAWA operators [12] that are based on the power means. These operators do not have the mandatory requirements property.

The PCD function emerges in various forms in system evaluation, classification, recognition, and in other areas. In the continuous logic it is interpreted as a model of adjustable simultaneity and replaceability. In the fuzzy logic it is interpreted as the averaging operator. Various application areas generate a rich spectrum of desired and achieved

PCD forms and properties. Our goal was to present various forms and interpretations of PDF in a unifying and comparative way that might cause more convergence in the future research.

All components of decision models (input preferences, weights, and andness) are determined by experts who frequently make substantial errors [8]. Training of experts and the use of appropriate software tools can reduce such errors, but will never eliminate them. Consequently, the necessary precision of decision models is limited by the precision of decision makers who specify inputs and parameters. Therefore, in the area of system evaluation (and possibly in other related areas), the perfection of logic decision models is desirable but not necessary.

Acknowledgments:
Henrik Legind Larsen was supported by the Danish Natural Science ResearchCouncil (Project No. 51-00-0376) and the Center for Teleinfrastructure (CTIF).

References:
[1] Belton, V. and Stewart, T.J.: *Multiple Criteria Decision Analysis: an Integrated Approach*. Kluwer Academic Publishers, 2002.
[2] Choo, E.U, B. Schoner, W.C. Wedley: Interpretation of Criteria Weights in Multicriteria Decision Making. Computers and Industrial Engineering 37 (1999) pp. 527-541.
[3] Dujmović, J.J., A Generalization of Some Functions in Continuous Mathematical Logic, (In Serbo-Croatian). Proceedings of the Informatica Conference, Bled, Yugoslavia, 1973.
[4] Dujmović, J.: Weighted Conjunctive and Disjunctive Means and their Application in System Evaluation. Journal of the University of Belgrade, EE Dept., Series Mathematics and Physics, No. 483, 1974, pp. 147-158, 1974.
[5] Dujmović, J.J., Extended Continuous Logic and the Theory of Complex Criteria. Journal of the University of Belgrade, EE Dept., Series Mathematics and Physics, No. 537, 1975, pp. 197-216.
[6] Dujmović, J.J., Partial Absorption Function. Journal of the University of Belgrade, EE Dept., Series Mathematics and Physics, No. 659, 1980, pp. 156-163.
[7] Dujmović, J.J., Preferential Neural Networks. Chapter 7 in *Neural Networks - Concepts, Applications, and Implementations,* Vol. II. Edited by P. Antognetti and V. Milutinović, Prentice-Hall Advanced Reference Series, Prentice-Hall, 1991, pp. 155-206.
[8] Dujmović, J.J. and W.Y. Fang: An Empirical Analysis of Assessment Errors for Weights and Andness in LSP Criteria. Proceedings of MDAI'04, Springer Verlag, 2004.
[9] Gini, C. et al., *Means*. Italian original, *Le Medie*, Milano 1958. Russian translation, *Средние Величины* published by Statistika, Moscow, 1970.
[10] Keeney, R.L. and H. Raiffa, *Decisions with Multiple Objectives: Preferences and Value Tradeoffs.* J. Wiley, 1976
[11] Larsen, H.L.: Importance weighted OWA aggregation of multicriteria queries. *Proceeding of the North American Fuzzy Information Processing Society conference, New York, 10–12 June 1999 (NAFIPS'99).* Pp. 740-744.
[12] Larsen, H.L.: Efficient Andness-directed Importance Weighted Averaging Operators, International Journal of Uncertainty, *Fuzziness and Knowledge-Based Systems* **12** (Suppl.): 67–82 (2003).
[13] Losonczi, L., A New Class of Means. (In German) *Acta Sci. Math (Acta Univ. Szeged)* Vol. 32, 1971, pp.71-78.
[14] Mitrinović, D., P.S. Bullen and P.M. Vasić, *Means and their Inequalities* (in Serbo-Croatian). Publications of the University of Belgrade EE Department, Series: Mathematics and Physics, No. 600, 1977.
[15] Torra, V. The Weighted OWA Operator. Int. J. of Intel. Systems, 12 , 1997, pp. 153-166.
[16] Yager, R.R., On Ordered Weighted Averaging Aggregation Operators in Multi-Criteria Decision Making. IEEE Trans. SMC 18 (1988) pp. 183-190.
[17] Yager, R.R., and Larsen, H.L.: *Retrieving Information by Fuzzification of Queries. International Journal of Intelligent Information Systems* **2**(4), 1993, pp. 421–441.
[18] Yager, R.R., Generalized OWA Aggregation Operators. Fuzzy Optimization and Decision Making, 3 (2004) pp. 93-107.
[19] Zimmermann, H.J., and P. Zysno, Latent Connectives in Human Decision Making. Fuzzy Sets and Systems 4, 1980, pp. 37-51.