Nonlinear Control of Rigid Manipulators Based on Approximate Solution of HJB Equation

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Abstract: The paper presents the design of nonlinear state feedback controller for a rigid manipulator. In order to obtain such controllers, a partial differential equation, HJB equation, should be solved, which is difficult to find a closed solution of that. In this paper, an efficient method using Taylor series expansion of nonlinear terms is used to tackle this problem. The tracking performance of the robotic system, using linear and nonlinear control actions is investigated. Simulation results show that the nonlinear control action has better response than that of linearized counterpart.

Key-Words: - Rigid robot manipulator, nonlinear state feedback control, optimal control, HJB equation, Taylor series expansion.

1 Introduction

Over the past decade, the problem of trajectory tracking for robot manipulators has attracted the attention of many researches, and various algorithms have been proposed for the control of robot manipulators, which consists of a wide range of strategies, such as adaptive controllers and sliding mode control (Spong, 1992; Spong, and Ortega, 1992; Cai, and Dai, 2001; Zhihong, and Yu, 1997; Keleher, and Stonier, 2002; Battoliti, and Lanari, 1996). One of these approaches is the application of *LQR* controllers. In this technique a state feedback is utilized such that a defined cost function for the system is minimized. This cost function is defined as a quadratic function of the system state variables and inputs that results in a first order controller. In spite of its simplicity, the use of this controller has some disadvantages, such as sensitivity of the controller to the variation of system parameters or the limited range of controllable disturbances. In other words, the domain of validity of the *LQR* controllers in contrast to the actual systems, that are nonlinear, has considerable limitations. These limitations encouraged control engineers to introduce nonlinear controllers for the design of controllers. In spite of their complexity, nonlinear controllers have the advantage of increasing the region of stability. In (Yazdanpanah, Khorasani, Patel, 1999) it has been proved that nonlinear feedback controllers always have a larger estimation of domain of validity than controllers with linear feedback. In (Yazdanpanah, Khorasani, Patel, 1998) this claim has been shown for a flexible link manipulator. In this paper the above issues will be considered by approximate solving of the *HJB* equation using Taylor Series expansion. Using the obtained controller the response of the closed-loop system for tracking of fixed point and oscillating reference signals are obtained, which the nonlinear controller results better performance than the linear one.

The paper is organized as follows: In Section 2 a nonlinear model of a two-link robot manipulator is given. Then in Section 3 nonlinear optimal control laws are obtained by solving the *HJB* equation using Taylor Series expansion of that. Some simulation results are provided in Section 4. Finally, the paper is concluded in Section 5.

2 System model

We begin with a general analysis of an n-joint rigid robotic manipulator system whose dynamics may be described by the second-order nonlinear vector differential equation:

$$M(q)\dot{q}+h(q,\dot{q})=u(t) \quad (1)$$

where $\dot{q}(t)$ is the $n\times 1$ vector of joint angular positions, $M(q)$ is the $n\times n$ symmetric positive definite inertia matrix, $h(q,\dot{q})$ is the $n\times 1$ vector containing Coriolis, centrifugal forces and gravity torques, $u(t)$ is the $n\times 1$ vector of applied joint torques (control inputs).

The dynamic equations of the two-link robotic manipulator are expressed in state variable form as $x_1=q_1, x_2=\dot{q}_1,\ x_3=q_2, x_4=\dot{q}_2$ with $x=[x_1\ \ x_2\ \ x_3\ \ x_4]^T$. The dynamics of this specific system is given by the equations

$$\dot{x}_1=x_2 \quad (2\text{-a})$$

$$\dot{x}_2=\frac{1}{a_1}[bx_2(x_2+x_4)\left(1+\frac{a_2^2}{a_1a_2-a_2^2}\right)+\gamma_1 g+u_1 - \frac{a_2}{(a_1a_2-a_2^2)^2}\left(a_1(\gamma_2 g-bx_4+u_2)-a_1(\gamma_1 g+u_2)\right)] \quad (2\text{-b})$$

$$\dot{x}_3=x_4 \quad (2\text{-c})$$

$$\dot{x}_4=\frac{1}{a_1a_2-a_2^2}[a_1(\gamma_2 g-bx_4^2+u_2)-a_2(bx_2(x_2+x_4)+\gamma_1 g+u_2)] \quad (2\text{-d})$$

where

$$a_1=(m_1+m_2)r_1^2+m_2r_2^2+2m_2r_1r_2\cos(x_3)+J_1 \quad (3\text{-a})$$

$$a_2=m_2r_2^2+2m_2r_1r_2\cos(x_3) \quad (3\text{-b})$$

$$b=m_2r_1r_2\sin(x_3) \quad (3\text{-c})$$

$$\gamma_1=-((m_1+m_2)r_1\cos(x_3)+m_2r_2\cos(x_1+x_3)) \quad (3\text{-d})$$

$$\gamma_1=-(m_2r_2\cos(x_1+x_3)) \quad (3\text{-e})$$

3 Nonlinear optimal control design

Optimal control is the determination of control signals due to optimize a predefined cost function while fulfilling some constraints. Using dynamic programming and optimality principle results in a nonlinear partial differential equation known as ***HJB*** equation. This equation has the following form (Uinter, 2000; Yang and Zhou, 1999):

Assume a system with the following differential equation

$$\underline{\Im}=\underline{a}(\underline{x}(t),\underline{u}(t),t) \quad (4)$$

where $\underline{x}$ is the state variables and $\underline{u}$ is the system input vector. The problem of optimal control design is to control the above system such that the following cost function is minimized

$$J=h(\underline{x}(t_f),t_f)+\int_{t_o}^{t_f} g(\underline{x}(\tau),\underline{u},\tau)d\tau \quad (5)$$

where g, h are definite functions and t_o, t_f are constants. So, according the *HJB* equation

$$J_t^*(\underline{x}(t),t)+H(\underline{x}(t),\underline{u}^*(\underline{x}(t),J_{\underline{x}},t),t)=0 \quad (6)$$

where J^* is the minimum of the cost function, u^* is the input vector that minimizes H, and H is Hamilton function that is defined as follows

$$H(\underline{x}(t),\underline{u}^*(\underline{x}(t),J_{\underline{x}},t),t)=g(\underline{x}(t),\underline{u}(t),t) + J_{\underline{x}}^{*T}(\underline{x}(t),t).[\underline{a}(\underline{x}(t),\underline{u}(t),t)] \quad (7)$$

As it can be seen the HJB equation is a partial differential equation, and finding exact analytical solution for J^* is so difficult. However, there are methods to find approximate solution for J^* that one of them is the use of *Taylor Series Expansion* of desired order. According to (7) u^* is a function of $J_{\underline{x}}^*$, so expressing $J_{\underline{x}}^*$ in the form of Taylor Series expansion of order n leads to a controller of order *(n-1)*. In order to obtain J_x^* in the form of Taylor Series expansion of order n, the following method has been proposed (Jalili-Kharaajoo and Moezzi-Madani, 2003; Jalili-Kharaajoo and Yazdanpanah, 2003).

1. Using a substitution we define new state variables as the deviation of the state variables from their steady state initial values.

2. If the system differential equations consist of nonlinear terms, they will be replaced by their Taylor Series expansion of order *(n-1).*

3. J^* is written as n ordered polynomial of state variables $(x_1,x_2,...,x_m)$. In this form, expressing J^* is equivalent to a Taylor Series expansion of the state variables $(x_1,x_2,...,x_m)$, the coefficients of all terms are considered unknown. Due to express J^* as the Taylor Series expansion completely, all possible terms up to order n should be included. All possible terms up to order i for the variables $(x_1,x_2,...,x_m)$ can be obtained by expansion of $(x_1+x_2+...+x_m)^i$ regardless of their coefficients.

4. The Taylor Series expansion of J^* is given to the HJB equation (7) and the coefficients of different terms are sorted. Then the coefficients of all terms in the form of $x_1^{i1}x_2^{i2}...x_m^{im}$ are set equivalent to zero. Using this method, some nonlinear equations of unknown coefficients in Taylor Series expansion of J^* are obtained.

5. The nonlinear equations obtained this way are solved by numerical methods like Newton-Rafson. So, the value of each coefficient can be calculated.

In order to design an optimal controller a cost function should be firstly considered. For this system, the following cost function is considered

$$J = \int_0^\infty (\frac{1}{2} p_1 x_1^2 + \frac{1}{2} p_2 x_2^2 + \frac{1}{2} p_3 u_1^2 + \frac{1}{2} p_4 u_2^2) \quad (8)$$

where p_1, p_2, p_3, p_4 and are positive constants.

Using the above procedure some nonlinear equations are obtained that can be solved using proper mathematics software. Here we have used MAPLE

4 Simulation results

For simulation the following parameters are considered

$$r_1 = 1.0m, r_2 = 0.8m, J_1 = 5Kgm, J_2 = 5Kgm$$
$$m_1 = 0.5Kg, m_2 = 1.5kg, g = 9.8Kgm/s^2$$
$$p_1 = 50, p_2 = 50, p_3 = 1, p_4 = 1$$

Using the proposed method the nonlinear optimal control law for u_1 and u_2 can be obtained. Here are the first and third order controllers.

The first order optimal control laws:

$$u_1 = -2.311x_1 + 10.435x_2 + 34.788x_3 + 5.699x_4$$
$$u_2 = -10.421x_1 - 8.374x_2 + 12.545x_3 + 4.112x_4$$

The third order optimal control laws:

$$\begin{aligned} u_1 = & -2.312x_1 + 10.421x_2 + 34.784x_3 + 5.711x_4 + .231x_1x_2 \\ & + 1.256x_1x_3 - 3.406x_1x_4 + .799x_2x_3 - .023x_2x_4 - .952x_3x_4 \\ & - 1.002x_1^2x_2x_3 + .041x_1^2x_2x_4 - 2.239x_1^2x_3x_4 + .007x_2^2x_1x_3 \\ & - .442x_2^2x_1x_4 - .891x_2^2x_3x_4 + .998x_3^2x_1x_2 - .556x_3^2x_1x_4 \\ & + 3.221x_3^2x_2x_4 + .945x_4^2x_1x_2 + 1.003x_4^2x_1x_3 - .439x_4^2x_2x_3 \\ & + 1.203x_1^2 + .777x_2^2 - .048x_3^2 + 3.405x_4^2 - .287x_1^3 + .798x_2^3 \\ & + 1.965x_3^2 + .038x_4^3 \end{aligned}$$

$$\begin{aligned} u_2 = & -10.421x_1 - 8.374x_2 + 12.545x_3 + 4.112x_4 + .456x_1x_2 \\ & + 2.304x_1x_3 + .341x_1x_4 - 1.203x_2x_3 - .343x_2x_4 - 1.002x_3x_4 \\ & + .095x_1^2x_2x_3 + .213x_1^2x_2x_4 + .006x_1^2x_3x_4 + .032x_2^2x_1x_3 \\ & - .442x_2^2x_1x_4 + .211x_2^2x_3x_4 - .0234x_3^2x_1x_2 + .998x_3^2x_1x_4 \\ & + 1.201x_3^2x_2x_4 + 3.001x_4^2x_1x_2 - .058x_4^2x_1x_3 - .159x_4^2x_2x_3 \\ & - 1.359x_1^2 + .034x_2^2 + .491x_3^2 + 1.045x_4^2 + .108x_1^3 - .426x_2^3 \\ & - .887x_3^2 + .281x_4^3 \end{aligned}$$

In this section the MATLAB simulation highlights the operation of the manipulator when tracking to a steady state value; q_1 and q_2 converge to 0.85 and 1.25 respectively. The reference signals are

$$q_{r1} = 0.85 - \frac{7}{5}e^{-t} + \frac{7}{20}e^{-4t}$$
$$q_{r2} = 1.25 + e^{-t} - \frac{1}{4}e^{-4t}$$

The initial state values of the system are selected as:

$$[x_1 \;\; x_2 \;\; x_3 \;\; x_4]^T = [0.8 \quad 0 \quad 0.8 \quad 0]^T$$

In Fig. 1 the closed-loop system responses using the first order control law (dashed line) and the third order controller (solid line) are depicted. As it is seen, the performance of the system with nonlinear controller is better than the linear one.

In order to investigate further, the effect of nonlinear control action on the performance of the system, the tracking problem of an oscillatory reference signal is considered. Here the desired trajectory reference signals are defined as

$$q_{r1} = 0.175(1 - \cos(2\pi t)) + 0.175$$
$$q_{r2} = 0.22(1 - \cos(2\pi t)) + 0.22$$

The initial state values of the system are selected as:

$$[x_1 \;\; x_2 \;\; x_3 \;\; x_4]^T = [0.1 \quad 0.1 \quad 1.3 \quad 0]^T$$

As it can be seen from Fig. 2, the performance of the closed-loop system using the third order controller (solid line) is better than that of the first order one (dashed line) and using the former the system needs less energy to track the reference signal sooner.

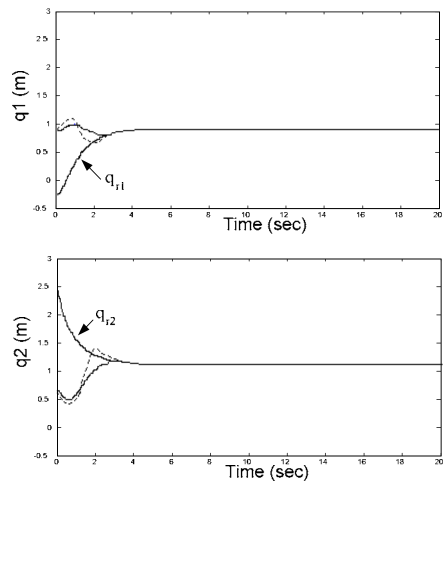
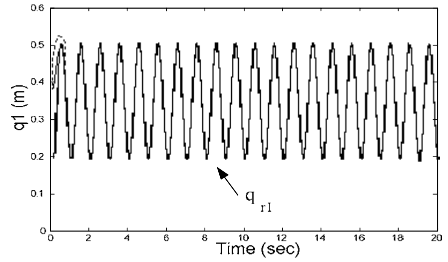


Fig. 1: Closed-loop system responses for tracking a fixed point reference signal using the first order controller (dashed line) and the third order controller (solid line).



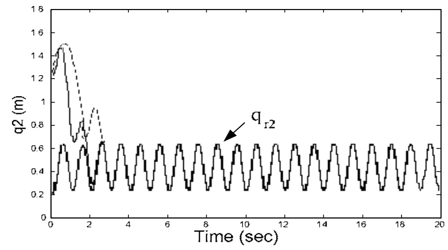


Fig. 2: Closed-loop system responses for tracking a oscillatory reference signal using the first order controller (dashed line) and the third order controller (solid line).

6 Conclusion

In this paper, an optimal nonlinear state feedback controller was applied to a two-link robot manipulator. For this, the approximate solution of HJB equation was obtained using Taylor series expansion of nonlinear terms. The fixed point and oscillatory reference signal tracking performance of the closed-loop system with linear and nonlinear controllers was compared. Simulation results showed that the performance of the nonlinear controller, which was a third order controller, was better than that of the linear one and the energy of output signal until reaching to the reference signal using the nonlinear controller was less than that of linear control action.

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