A Linear Programming Approach for Calculation of All Stabilizing Parameters of Lead-Lag Compensator for Continuous-Time Plants

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Abstract: - In this paper, a new method to calculate all stabilizing parameters of Lead-Lag type controller for a given continuous-time plant is proposed. Lead-Lag controllers are used extensively in industry and there is no solution to the problem: can a plant be stabilized by a Lead-Lag controller. Using the earlier results on calculation of all stabilizing PID gains, we provide a computationally constructive characterization of all stabilizing Lead-Lag parameters. In this approach, a linear programming technique is used that is the main advantage of the method. The method is utilized for a numerical example.

Key-Words: - Lead-lag type compensator, all stabilizing parameters, continuous-time systems, linear programming.

1 Introduction

Lead-Lag type controllers have a wide-spread use in industry applications [1,2]. So, we need to have some efficient methods to determine the stability of the system with the controller. For this purpose, there are some classical approaches, the root locus technique, the Nyquist stability criterion, and the Routh-Hurwitz criterion [3], but in many situations especially in the case of high order plants, the use of these approaches is not straight forward. In recent years a linear programming approach is developed to calculate all stabilizing PID gains based on generalization of Hermite-Biehler Theorem [4,5]. We will use the same approach to determine the whole set of Lead-Lag parameters that can stabilize a given plant. The main characteristic of the proposed method is the use of linear programming that can be performed using a digital computer.

In the paper, Section 2 presents a generalization of Hermite-Biehler Theorem. In Section 2 feedback stabilization using a constant gain is considered. Section 4 proposed the procedure of determining of stabilizing Lead-Lag parameters for a given plant. A numerical example shows the method in details in section 5. Some concluding remarks are pointed out in Section 6.

2 A generalization of Hermit-Behler Theorem

In this section, we present the generalization of the Hermite-Biehler Theorem [6]. To do so, we first introduce the standard *signum* function $\mathrm{sgn}:\Re\rightarrow\{-1,0,1\}$ defined by

$$\mathrm{sgn}(x)=\begin{cases}-1 & if \quad x<0\\ 0 & if \quad x=0\\ 1 & if \quad x>0\end{cases}$$

Let $\delta(s)=\delta_o+\delta_1 s+...+\delta_n s^n$ be a given real polynomial of degree *n*. write $\delta(s)=\delta_e(s^2)+s\delta_o(s^2)$ where $\delta_e(s^2)$ and $s\delta_o(s^2)$ are the components of $\delta(s)$ made up the even and odd powers of *s* respectively. For every frequency $w\in\Re$, denote $\delta(jw)=p(w)+jq(w)$ where $p(w)=\delta_e(-w^2), q(w)=w\delta_o(-w^2)$.

Furthermore, define

$$\delta_f(jw)=p_f(w)+jq_f(jw)$$

where

$$p_f(w)=\frac{p(w)}{f(w)}, q_f(w)=\frac{q(w)}{f(w)};$$

and

$$f(w)=(1+w^2)^{\frac{n}{2}}$$

Define the signature of polynomial $\delta(s)$ by $\sigma(\delta(s))$, as

$\sigma(\delta(s))$:= number of open left half plan zeros of $\delta(s)$ - number of open right half plan zeros of $\delta(s)$

Then, we can state the following [6]:

Theorem 1 *(A Generalization of Hermite-Biehler Theorem) Let* $\delta(s)$ *be a given real polynomial of degree n with no jw axis roots except for possibility one at the origin. Let* $0=w_o<w_1<w_2<...<w_{m-1}$ *be the real, non-*

negative, distinct finite zeros of $q_f(w)$ *with odd multiplicities. Also define* $w_m = \infty$. *Then*

$$\sigma(\delta)=\begin{cases}\{\mathrm{sgn}[p_f(w_o)]-2\mathrm{sgn}[p_f(w_1)] \\ +2\mathrm{sgn}[p_f(w2)]+(-1)^{m-1}2\mathrm{sgn}[p_f(w_{m-1})] \\ +(-1)^m\mathrm{sgn}[p_f(w_m)]\}.(-1)^{m-1}\mathrm{sgn}[q(\infty)] \\ \qquad if \quad n \quad is\, even \\ \{\mathrm{sgn}[p_f(w_o)]-2\mathrm{sgn}[p_f(w_1)] \\ +2\mathrm{sgn}[p_f(w2)]+(-1)^{m-1}2\mathrm{sgn}[p_f(w_{m-1})]\} \\ .(-1)^{m-1}\mathrm{sgn}[q(\infty)] \\ \qquad if \quad n \quad is\, odd\end{cases} \tag{1}$$

Remark 1 *When the polynomial* $\delta(s)$ *is Hurwitz, Theorem 1 immediately implies the interlacing property [7]. That is why Theorem 1 generalizes the Hermite-Biehler Theorem to the case of not necessarily Hurwitz polynomials. Furthermore, the interlacing property of the Hermite-Biehler Theorem gives a graphical interpretation of the Hermite-Biehler Theorem while Theorem 1 gives an analytical characterization to the case of not necessarily Hurwitz polynomials.*

3 Feedback stabilization using a constant gain

In this section we summarize the results of [4], which provide a complete analytical solution to the feedback constant gain stabilization problem shown in Fig. 1, which is the lowest order compensator design problem possible. Here *r* is the command signal, *y* is the output, *N(s)* and *D(s)* are coprime polynomials, and *C(s)* is the controller. Here, *C(s)=k.*

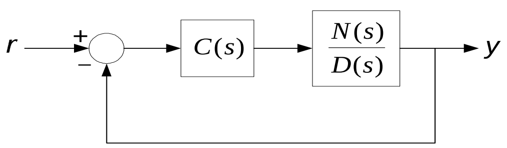

Fig. 1. Feedback control system.

The closed loop characteristics polynomial $\delta(s)$ is given by

$$\delta(s) = D(s) + kN(s) \tag{2}$$

where *k* is a scalar.

The objective is to determine those values of *k*, if any, for which the closed loop system is stable, i.e., $\delta(s)$ is Hurwitz.

There are several classical approaches for solving this problem: the root locus technique, the Nyquist stability criterion, and the Routh-Hurwitz criterion. Of these approaches, the root locus technique and the Nyquist stability criterion solve this problem by plotting the root loci of $\delta(s)$ and the Nyquist plot of $C(s) = N(s)/D(s)$, respectively. Hence, both of these methods are graphical in nature and fail to provide us with an analytical characterization of all stabilizing parameters. The Routh-Hurwitz criterion, on the other hand, dose provides us with an analytic solution. However, the stabilizing parameters must be determined by solving a set of polynomial inequalities, a task which is not straight forward especially for higher order plants. For this porpoise we will use the results of [4] as follows:

Consider (2) with the even-odd decomposition

$$\begin{aligned}N(s) &= N_e(s^2) + sN_o(s^2) \\ D(s) &= D_e(s^2) + sD_o(s^2)\end{aligned} \tag{3}$$

Suppose that the degree of *D(s)* is *n* while the degree of *N(s)* is *m* and $m \le n$. Let $e(s^2)$ be the greatest common divisor (gcd) of $N_e(s^2)$ and $N_o(s^2)$, we define

$$N_e'(s^2) = \frac{N_e(s^2)}{e(s^2)}, N_o'(s^2) = \frac{N_o(s^2)}{e(s^2)} \tag{4}$$

Let

$$N'(s) = N_e'(s^2) + sN_o'(s^2) \tag{5}$$

and define

$$N^*(s) = N'(-s) = N_e'(s^2) - sN_o'(s^2) \tag{5}$$

Clearly $N'(s)$ have no *jw* axis roots except possibly a single root at the origin. Let m' be the degree of $N'(s)$. Now, multiplying $\delta(s)$ by $N^*(s)$ and examining the resulting polynomial, we obtain

$$\begin{aligned}&\sigma(\delta(s)N^*(s)) = \sigma(\delta(s)) + \sigma(N^*(s)) \\ &= \sigma(\delta(s)) + \sigma(N'(-s)) = \sigma(\delta(s)) - \sigma(N'(s))\end{aligned} \tag{6}$$

Substituting *s=jw*, we obtain

$$\delta(jw)N^*(jw) = p(w,k) + jq(w) \tag{7}$$

where

$$\begin{aligned}p(w,k) &= p_1(w) + kp_2(w) \\ p_1(w) &= [D_e(-w^2)N_e'(-w^2) + w^2 D_o(-w^2)N_o'(-w^2)] \\ p_2(w) &= [N_e(-w^2)N_e'(-w^2) + w^2 N_o(-w^2)N_o'(-w^2)] \\ q(w) &= w[D_o(-w^2)N_e'(-w^2) - D_e(-w^2)N_o'(-w^2)]\end{aligned} \tag{8}$$

Also, define

$$\begin{aligned}p_f(w,k) &= \frac{p(w,k)}{(1+w^2)^{\frac{m'+n}{2}}}, \\ q_f(w) &= \frac{q(w)}{(1+w^2)^{\frac{m'+n}{2}}}\end{aligned} \tag{9}$$

Definition 1 *Let* $0 = w_o < w_1 < w_2 < \ldots < w_{l-1}$ *be the real, non-negative, distinct finite zeros of*

$q_f(w)$ with odd multiplicities. Then A_t is the set of strings defined as

$$A_t = \begin{cases} \{\{i_o, i_1, ..., i_l\} & for \quad n+m' \quad even\} \\ \{\{i_o, i_1, ..., i_{l-1}\} & for \quad n+m' \quad \text{odd}\} \end{cases} \quad (10)$$

where $i_o \in \{-1,0,1\}$ and $i_t \in \{-1,1\}$ for $t \neq 0$.

Remark 2 *For $n+m'$ even, the number of string in A_t is 3×2^l while for $n+m'$ odd, the number of strings in A_t is $3\times 2^{l-1}$.*

Theorem 2 *(Main Results on Constant Gain Stabilization) Consider the polynomial $\delta(s) = D(s) + kN(s)$ where D(s), N(s) are polynomials of degree n, m respectively, $n \geq m$ and k is a scalar gain. Let the integer m and the polynomials $N'(s), p(w,k), p_1(w), p_2(w), q(w), p_f(w,k), q_f(w)$ be as already defined. Let $0 = w_o < w_1 < w_2 < ... < w_{l-1}$ be the real, non-negative, distinct finite zeros of $q_f(w)$ with odd multiplicities. Also define $w_l = \infty$. Then there exist a k such that $\delta(s) = D(s) + kN(s)$ is Hurwitz if and only if the following conditions hold:*

There exist a string $\iota = \{i_o, i_1, ...\} \in A_t$ such that

(i) if $p_2(w_t) = 0$ for some t =1,2,…,l, then $i_t = \text{sgn}[p_1(w_t)]$

(ii)

$$\sigma(\delta) = \begin{cases} \{i_o - 2i_1 + 2i_2 + ... + (-1)^{l-1} 2i_{l-1} \\ + (-1)^m i_l\}.(-1)^{m-1}\text{sgn}[q(\infty)] \\ \qquad if \quad n \quad is\, even \\ \{i_o - 2i_1 + 2i_2 + ... + (-1)^{l-1} 2i_{l-1}\} \\ .(-1)^{l-1}\text{sgn}[q(\infty)] \\ \qquad if \quad n \quad is\, odd \end{cases} \quad (11)$$

(iii)

$$\max_{i_t \in \iota, i_t.\text{sgn}[p_2(w_t)]=1} \left[-\frac{1}{G(jw_t)} \right] < \min_{i_t \in \iota, i_t.\text{sgn}[p_2(w_t)]=-1} \left[-\frac{1}{G(jw_t)} \right] \quad (12)$$

Furthermore, If the above three conditions are satisfied by string $\iota_1, \iota_2, ..., \iota_s$, then the set of all k such that $\delta(s)$ is Hurwitz is given by $K = \bigcup_{r=1}^{s} K_r$ where

$$K_r = (\max_{i_t \in \iota, i_t.\text{sgn}[p_2(w_t)]=1} \left[-\frac{1}{G(jw_t)} \right], \min_{i_t \in \iota, i_t.\text{sgn}[p_2(w_t)]=-1} \left[-\frac{1}{G(jw_t)} \right])_{r=1,2,...,s} \quad (13)$$

4 Calculation of all Stabilizing Lead-Lag compensator parameters

In this section, we make use of results mentioned above to provide a complete analytical solution to calculate the whole set of stabilizing Lead-Lag type controller parameters for the feedback control system shown in Fig. 1. The general form of Lead-Lag compensators is as the following

$$C(s) = \frac{ks+a}{s+b} \quad (14)$$

Then, the closed-loop characteristic polynomial $\delta(s)$ is given by

$$\delta(s) = (ks+a)N(s) + (s+b)D(s) \quad (15)$$

Our objective is to determine those values of *a*,*b*, and *k*, if any, for which the closed-loop system is stable, i.e. $\delta(s)$ is Hurwitz. We have

$$\begin{aligned} &\delta^*(s) = \delta(s^2)N^*(s^2) = \\ &a[N_e(s^2)N'_e(s^2) - s^2N_o(s^2)N'_o(s^2)] \\ &+ b[D_e(s^2)N'_e(s^2) - s^2D_o(s^2)N'_o(s^2)] \\ &- s^2(D_o(s^2)N'_e(s^2) - D_e(s^2)N'_o(s^2)] \\ &+ ks[N_e(s^2)N'_e(s^2) - s^2N_o(s^2)N'_o(s^2)] \\ &+ bs[D_o(s^2)N'_e(s^2) - D_e(s^2)N'_o(s^2)] \\ &s[D_e(s^2)N'_e(s^2) - s^2D_o(s^2)N'_o(s^2)] \end{aligned} \quad (16)$$

Substituting *s*=*jw*, we obtain

$$\begin{aligned} &\delta^*(jw) = \delta(jw)N^*(jw) \\ &= a\delta_{ea} + b\delta_{eb} + \delta_{ec} + jw[k\delta_{ok} + b\delta_{ob} + \delta_{oc}] \end{aligned} \quad (17)$$

where

$$\begin{aligned} &\delta_{ea} = N_eN'_e + w^2N_oN'_o, \delta_{eb} \\ &= D_eN'_e + w^2D_oN'_o, \delta_{ec} = w^2(D_oN'_e - D_eN'_o) \\ &\delta_{ok} = N_eN'_e + w^2N_oN'_o, \delta_{ob} \\ &= D_oN'_e - D_eN'_o, \delta_{oc} = D_eN'_e + w^2D_oN'_o \end{aligned} \quad (18)$$

Now for the fix values of $b = b_o$, we have

$$\delta^*(jw) = a\delta_{ea} + \delta'_{ec} + jw[k\delta_{ok} + \delta'_{oc}] \quad (19)$$

where

$$\begin{aligned} &\delta_{ea} = N_eN'_e + w^2N_oN'_o, \delta'_{ec} \\ &= w^2(D_oN'_e - D_eN'_o) + b_o(D_eN'_e + w^2D_oN'_o) \\ &\delta_{ok} = N_eN'_e + w^2N_oN'_o, \delta'_{oc} \\ &= D_eN'_e + w^2D_oN'_o + b_o(D_oN'_e - D_eN'_o) \end{aligned} \quad (20)$$

Now, we can use the results of [4]. It is remarkable that we are no longer able to obtain a closed form solution. Instead, a linear programming problem has to be solved for each fixed b as the following algorithm

Algorithm:

Step 1 For a fixed value of *b* at b=b_o:

Step2 for a fixed value of *k*, determine $\{0 = w_o < w_1 < w_2 < ... < w_{l-1}\}$, the real, distinct finite zeros of $(k\delta_{ok} + \delta'_{oc})$ with odd multiplicities and let $w_l = \infty$. Also define *n* := degree of $\delta(s)$ and m' := degree of $N'(s)$.

Step 3

(a) if $\delta_{ea}(w_t) = 0$ for some $t = 0,1,2,...,l$ then set $i_t = \text{sgn}[\delta'_{ec}(w_t)]$, where

$$\text{sgn}(x) = \begin{cases} 1 & if \quad x > 0 \\ 0 & if \quad x = 0 \\ -1 & if \quad x < 0 \end{cases} \tag{21}$$

(b) Now choose strings $\imath \in A_t$ (A_t is defined as (10)) satisfying *(a)* above such that

$$n - \sigma(N'(s)) = \begin{cases} \{i_o - 2i_1 + 2i_2 + ... \\ + (-1)^{l-1} 2i_{l-1} \\ + (-1)^l 2i_l\}(-1)^{l-1} \text{sgn}[q(\infty, k_p)] \\ \qquad for \ m' + n \ even \\ \{i_o - 2i_1 + 2i_2 + ... \\ + (-1)^{l-1} 2i_{l-1}\}(-1)^{l-1} \text{sgn}[q(\infty, k_p)] \\ \qquad for \ m' + n \ \text{odd} \end{cases} \tag{22}$$

where σ := number of open left half plan of the system – number of open right half plan of the system.

For each such string, go to Step 3.

If no such string exists, then there is no solution.

Step 4 Determine the admissible values of *a* by solving the following linear inequalities: for $t = 0,1,2,...,\ l$ and $\delta_{ea}(w_t) \neq 0$

$$\begin{cases} \delta'_{ec}(w_t) + a\delta_{ea}(w_t) > 0 & if \quad i_t = 1 \\ \delta'_{ec}(w_t) + a\delta_{ea}(w_t) < 0 & if \quad i_t = -1 \end{cases} \tag{23}$$

Using a linear programming technique:

If the admissible set of a for (23) = Ø, then there is no solution.

Else, for fixed *k* and fixed *b*, the admissible values of a satisfying (23) is the region for which $\delta(s)$ is Hurwitz.

Step 5 Sweep $k \in [k_{\min}, k_{\max}]$ and go to Step 2. So, the stabilizing set of (a,k) for fixed *b* can be obtained.

Step 6 Sweep $b_o \in [b_{\min}, b_{\max}]$ and go to Step 1.

Using the above algorithm the whole set of (a,b,k) for which $\delta(s)$ is Hurwitz, cab be obtained.

5 Illustrative example

To clearly understand the algorithm, we now present a simple example.

Example Consider the problem of choosing stabilizing Lead-Lag gains for the plant $G(s) = \dfrac{N_1(s)}{D_1(s)}$ where

$$\begin{aligned} D_1(s) &= s^6 + 4s^5 + 8s^4 + 32s^3 + 46s^2 + 46s + 17, \\ N_1(s) &= s^4 + 2s^3 - 4s^2 + s + 2 \end{aligned} \tag{24}$$

The closed loop characteristic polynomial is

$$\delta(s) = (ks + a)N_1(s) + (s + b)D_1(s) \tag{25}$$

We consider the even-odd decompositions for the polynomials $N_1(s)$ and $D_1(s)$, i.e.

$$\begin{aligned} D_1(s) &= D_{1e}(s^2) + sD_{1o}(s^2),\ N_1(s) \\ &= N_{1e}(s^2) + sN_{1o}(s^2) \end{aligned} \tag{26}$$

where

$$\begin{aligned} & D_{1e}(s^2) = s^6 + 8s^4 + 46s^2 + 17, D_{1o}(s^2) \\ & = 4s^4 + 32s^2 + 46 \\ & N_{1e}(s^2) = s^4 - 4s^2 + 2, N_{1o}(s^2) = 2s^2 + 1 \end{aligned} \tag{27}$$

since $\gcd(N_{1e}, N_{1o}) = 1$, it follows that

$$\begin{aligned} N^*(s) &= N(-s) = N_{1e}(s^2) - sN_{1o}(s^2) \\ &= (s^4 - 4s^2 + 2) - s(2s^2 + 1) \end{aligned} \tag{28}$$

Therefore, from (6) we obtain

$$\begin{aligned} & \delta_{ea} = (w^4 + 4w^2 + 2)^2 + w^2(-2w^2 + 1)^2 \\ & \delta_{eb} = [(-w^6 + 8w^4 - 46w^2 + 17) \\ & + w^2(4w^4 - 32w^2 + 46)] = (w^4 + 4w^2 + 2) \\ & \delta_{ec} = w^2(-2w^2 + 1)[(4w^4 - 32w^2 + 46) \\ & - (-w^6 + 8w^4 - 46w^2 + 17)] \\ & \delta_{ok} = (w^4 + 4w^2 + 2)^2 + w^2(-2w^2 + 1)^2 \\ & \delta_{ob} = (4w^4 - 32w^2 + 46)(w^4 + 4w^2 + 2) \\ & - (-w^6 + 8w^4 - 46w^2 + 17)(-2w^2 + 1) \\ & \delta_{oc} = (-w^6 + 8w^4 - 46w^2 + 17)(w^4 + 4w^2 + 2) \\ & + w^2(4w^4 - 32w^2 + 46)(-2w^2 + 1) \end{aligned} \tag{29}$$

Now, we can use the proposed algorithm, which the stabilizing set of (a,b,k) are shown in Fig. 2.

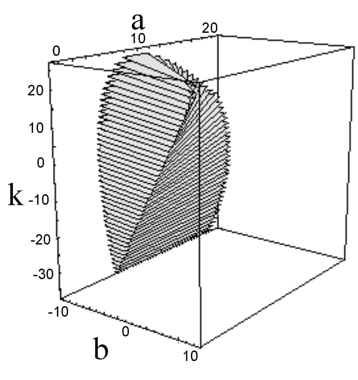

Fig. 2. The stabilizing set of (a,b,k).

6 Conclusion

A linear programming approach was proposed for calculation of all stabilizing parameters of continuous-tome Lead-Lag controllers. The procedure was obtained by using the earlier results on calculation of all stabilizing PID gains. The results obtained in this paper are expected to have a significant impact on industrial control applications where Lead-Lag controllers find extensive use.

References:

[1] Minhoto Teixeiva M.C., Direct Expressions for Ogata's Lead-Lag Design Method Using Root Lucas. *IEEE Trans. On Education*, vol. 37, No. 1, pp. 63-64, 1997.

[2] Akgul M., O, Morgul, *Fuzzy Controller Design for Parametric Controllers*. *Proc. 12th IEEE Int. Sym. On Intelligent Control, Istanbul,* Turkey, pp. 67-72, 1997.

[3] Ogata K., *Modern Control Engineering*. Englewood Cliffs, NJ: Prentice- Hall, Third Edition, 1997.

[4] Ho M.T., A. Datta and S.P. Bhattacharyya, A Linear Programming Characterization of All Stabilizing PID Controllers. *Proc. American Control Conference (ACC)*, pp.3922-3928, 1997.

[5] Ho M.T., A. Datta and S.P. Bhattacharyya, A New Approach To Feedback Stabilization. *Proc. 35th IEEE Conf. On Decision & Control*,pp.4643-4648,1996.

[6] Ho M.T., A. Datta and S.P. Bhattacharyya, A generalization of Hermite-Biehler Theorem. *Proc. 34th Conf. Decision and Control (CDC)*, pp130-131, 1995.

[7] Bhattacharyya, S.P., H. Chapellat, Keel, L.H., *Robust Control: The Parametric Approach,* Prentice Hall, 1995.