CONSTRAINED CONTROLLABILITY OF DISTRIBUTED PARAMETER SYSTEM

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Abstract: In the paper constrained controllability of distributed parameter dynamical system defined in infinite-dimensional domain is considered. Using spectral theory of unbounded differential operators, necessary and sufficient conditions constrained controllability are formulated and proved. Remarks and comments on the relationships between different kinds of controllability are also given. Simple numerical examples of controllable systems are presented.

Key Words: Distributed parameter systems. Linear systems. Controllability. Linear operators.

1 Introduction*

Controllability is one of the fundamental concept in mathematical control theory [1], [3], [6]. Roughly speaking, controllability generally means, that it is possible to steer dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. In literature there are many different definitions of controllability which depend on class of system [1], [3], [6], [9], [11], [13], [15].

Problems of controllability for linear control systems defined in infinite-dimensional Banach spaces, have attracted a good deal of interest over the past 20 years. For infinite dimensional dynamical systems it is necessary to distinguish between the notions of approximate and exact controllability [1], [3], [6], [11], [12], [13], [14] and [15]. It follows directly from the fact, that in infinite-dimensional spaces there exist linear subspaces which are not closed.

Most of the literature in this direction so far has been concerned, however, with unconstrained controllability, and little is known for the case when the control is restricted to take on values in a preassigned subset of the control space. Until now, scare attention has been paid to the important case where the control of a system are nonnegative. In this case controllability is possible only if the system is oscillating in some sense.

The present paper is devoted to a study of controllability for positivity-preserving dynamical systems [9] if the controls are taken to be positive. In analogy to the usual definition of controllability it is possible to introduce the concept of positive controllability for positive systems [9].

The present paper is devoted to a study of constrained approximate controllability for linear infinite-dimensional distributed parameter dynamical systems. For such dynamical systems direct verification of constrained approximate controllability is rather difficult and complicated [8]. Therefore, we shall concentrate on special case, when the values of controls are taken from a given closed convex cone [10].

2 System Description

Let us consider distributed parameter dynamical system described by the following partial differential equation defined on infinite domain [2], [7]

$$v_t(z,t) = A_k v(z,t) + b_1(z)u_1(t) + b_2(z)u_2(t) \tag{1}$$

with initial condition

$$v(z,0) \in L_2(R) \tag{2}$$

where $z \in R$ and $t \geq 0$, $b_1(z) \in L_2(R)$, $b_2(z) \in L_2(R)$, and k is an integer number. In the next section we shall also consider dynamical system of the form (2.1) but with only one scalar control (i.e. $b_2(z) = 0$).

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In the sequel it is generally assumed, that the admissible controls $u_1(t) \in L_2([0,t_1],R^+)$, and $u_2(t) \in L_2([0,t_1],R^+)$.

$A_k : D(A_k) \rightarrow L_2(R)$ is a linear unbounded differential operator defined as follows

$$D(A_k) = \{v(z) \in L_2(R) : A_k v(z) \in L_2(R)\} \tag{3}$$

$$A_k v(z) = v_{zz}(z) + (k - z^2)v(z) \tag{4}$$

Now, for the convenience, let us collect some well known facts about the operator A_k [2], [6], [7], . Operator A_k is selfadjoint with compact resolvent and is an infinitesimal generator of an analytic semigroup of linear bounded operators $S_k(t) : L_2(R) \rightarrow L_2(R)$, for $t \geq 0$. Moreover, operator A_k has only pure discrete point spectrum $\sigma(A_k) = \{s_{kn}\}$, where the eigenvalues :

$s_{kn} = -2n + k - 1$, for $n = 0,1,2,...$

are all of multiplicity one.

The corresponding eigenfunctions

$g_n(z)=(2^n n!)^{-0,5}(\pi)^{-0,25}\exp(-0,5z^2)H_n(z) \quad n=0,1,2,...$

where

$H_n(z) = (-1)^n \exp(z^2) d^n/dz^n(\exp(-z^2))$

are Hermite's polynomials, and form a complete ortonormal system in a separable Hibert space $L_2(R)=V$.

It is well known [2], [3], [7], that abstract ordinary differential equation (1) with initial conditions $v(z,0) \in D(A_k)$ has for each $t_1 > 0$ an unique solution $v(t;v(z,0),u)$ such that $v(t) \in D(A_k)$ for $t \in (0,t_1]$.

Definition 2.1 . [1], [3], [6]. Dynamical system (1) is said to be approximately controllable with nonnegative controls if for any initial condition $v(z,0) \in V$, any given final condition $v_f \in V$ and each positive real number ε there exist a finite time $t_1 < \infty$ (depending generally on $v(z,0)$ and v_f) and admissible controls $u_1(t) \in L^2([0,t_1],R^+)$, and $u_2(t) \in L^2([0,t_1],R^+)$ such that

$$\left\| v(t_1; v(z,0), u_1, u_2) - v_f \right\|_V \leq \varepsilon$$

The above notion of approximate controllability is defined in the sense that we want to reach a dense subspace of the entire state space. However, in many instances for systems with restrictions on the controls, it is known that all states are contained in a closed positive cone V^+ of the state space. In this case approximate controllability in the sense of the above definition is impossible but it is interesting to know conditions under which the reachable states are dense in V^+. This observation leads to the concept of so-called positive approximate controllability.

Definition 2.2. [9] Dynamical system (1) is said to be positively approximately controllable if for any initial condition $v(z,0) \in V^+$, any given final condition $v_f \in V^+$ and each positive real number ε there exist a finite time $t_1 < \infty$ (depending generally on $v(z,0)$ and v_f) and admissible controls $u_1(t) \in L^2([0,t_1],R^+)$, and $u_2(t) \in L^2([0,t_1],R^+)$ such that

$$\left\| v(t_1; v(z,0), u_1, u_2) - v_f \right\|_V \leq \varepsilon$$

From the above definitions directly follows, that approximate controllability with nonnegative controls always implies positive approximate controllability.

3 Constrained Controllability

Now, let us formulate several results concerning constrained approximate controllability of dynamical system (1).

Theorem 3.1. Dynamical system (1) is approximately controllable with nonnegative controls if and only if

$$b_{1n} b_{2n} < 0 \quad \text{for every } n=0,1,2,... \tag{5}$$

where

$$b_{jn} = \left\langle b_j(z), g_n(z) \right\rangle_V = \int_{-\infty}^{+\infty} b_j(z) g_n(z) dz \neq 0$$

for j=1,2 and every n=0,1,2,...

Proof. Proof of theorem is based on the results given in the papers [5], [6] and [10] concerning constrained approximate controllability. First of all, let us observe that dynamical system (1) satisfies all the assumptions stated in the paper [10]. Therefore, by theorem in [10] the following statement is valid: dynamical system (1) approximately controllable with nonnegative controls if and only if Fourier coefficients of the functions $b_1(z)$ and $b_2(z)$ corresponding to each eigenfunctions $g_n(z)$ have different signs. Therefore our theorem follows..

Corollary 3.1. [2] Dynamical system (1) is approximately controllable (with unconstrained controls) if and only if

$b^2_{1n} + b^2_{2n} \neq 0$ for every $n=0,1,2,...$ (6)

In other words dynamical system (1) is approximately controllable (with unconstrained controls) if and only if $b_{1n} \neq 0$ or $b_{2n} \neq 0$ for every $n = 0,1,2,...$ Therefore, approximate controllability with unconstrained controls may occur even for one scalar controls, which is impossible for approximate controllability with nonnegative controls.

Corollary 3.2. [2] Let $b_2(z) = 0$. Then dynamical system (1) is approximately controllable (with unconstrained controls) if and only if

$b_{1n} \neq 0$ for every $n=0,1,2,...$ (7)

In the next part of this section it is assumed that $b_2(z) = 0$, i.e. there is only one positive scalar control $u(t)\in R^+$.

Theorem 3.2. Dynamical system (1) is not positively approximately controllable.

Proof. In order to prove this theorem it is sufficient to point the final state $v_f \in V^+$ which cannot be reached approximately from a given initial state $v_0 \in V^+$. Let us take $v_0 = 0$. Without loss of generality we may assume that $b_n \neq 0$ for all $n = 0,1.2.,,,$ If it is not this case, then dynamical system (2.1) is not approximately controllable in any sense. Let us choose the final state v_f as follows

$v_f (z) = z\exp(-0,5z^2)$ for $z >0$ and

$v_f = 0$ for $z < 0$ if $b_2 < 0$

$v_f (z) = -z\exp(-0,5z^2)$ for $z < 0$ and

$v_f = 0$ for $z > 0$ if $b_2 > 0$

Hence, taking into account the form of eigenfunction $g_2(z)$ we conclude, that Fourier coefficient v_{f2} has different sign than b_2. Therefore, the Fourier coefficient of the solution $v_2(t)$ has different sign than v_{f2} for all $t>0$. Hence the final state v_f cannot be reached from zero initial state.

In practice, it is often not so important to reach approximately the entire positive cone V^+ of the state space V. Sometimes it suffices to reach approximately by nonnegative controls only particular positive states in the positive cone V^+. This observation leads directly to the concept of so-called positive stationary pairs.

Definition 3.3. [9] A pair $\{v_s,u_s\}\in V^+\times R^+$ is said to a positive stationary pair for dynamical system (2.1) if $A_k v_s + b_1 u_s = 0$.

Let us observe, that if $\{v_s,u_s\}$ is a positive stationary pair, then $v(z,t) = v_s$ is a nonzero constant solution of (2.1) for $u(t) = u_s$, and $v(z,0) = v_s$. Moreover, there exists strong connection between existence of positive stationary pairs and stability of dynamical system (2.1).

Theorem 3.3. Let $-2n+k-1 \neq 0$. Then to each $u_s \in R^+$ there exists exactly one $v_s\in V^+$ such that $\{v_s,u_s\}$is a positive stationary pair.

Proof. If $-2n+k-1 \neq 0$ then zero is not an eigenvalue of the operator A_k and therefore belongs to the resolvent set, i.e. $0\in\rho(A_k)$. Hence, $-(A_k)^{-1}$ is a positive operator [9]. For all $u_s\in R^+$ we therefore obtain that $\{-(A_k)^{-1}bu_s , u_s\}$ is a positive stationary pair. On the other hand, $0\in\rho(A_k)$ implies that for each $u_s\in R^+$, there exists at most one v_s such that $\{v_s , u_s\}$ is a positive stationary pair.

4. EXAMPLE.

Let us consider dynamical system (2.1) with two nonnegative controls and the following functions

$b_1(z) = - \exp(2z - 0,5z^2 -1) \in L_2(R)$

$b_2(z) = \exp(z - 0,5z^2) \in L_2(R)$

Using the complete ortonormal system $g_n(z)$ we can express functions $b_1(z)$ and $b_2(z)$ as follows

$$b_1(z) = -\exp(2z - 0,5z^2 - 1) =$$

$$= -\sum_{n=0}^{n=\infty}(n!)^{-1}\exp(-0,5z^2)H_n(z) =$$

$$= -\sum_{n=0}^{n=\infty}(\pi)^{0,25}2^{0,5n}(n!)^{-0,5}(2^n n!)^{-0,5}(\pi)^{-0,25}\exp(-0,5z^2)H_n(z) =$$

$$= \sum_{n=0}^{n=\infty}b_{in}(z)g_n(z)$$

where

$b_{1n} = -(\pi)^{0,25}2^{0,5n}(n!)^{-0,5} < 0$, for every $n=0,1,2,...$

Moreover,

$$b(z) = \exp(z - 0{,}5z^2) =$$

$$e^{0,25}\sum_{n=0}^{n=\infty}(2^n n!)^{-1}\exp(-0{,}5z^2)H_n(z) =$$

$$= \sum_{n=0}^{n=\infty} e^{0,25}(2^n n!)^{-0,5}(\pi)^{-0,25}(2^n n!)^{-0,5}\exp(-0{,}5z^2)H_n(z) =$$

$$\sum_{n=0}^{n=\infty} b_n g_n(z)$$

where

$b_{2n} = (2^n n!)^{-0,5}(\pi e)^{0,25} > 0$, for every n=0,1,2,...

Since inequality (3.1) is satisfied, than by Theorem 3.1 dynamical system is approximately controllable with nonnegative control.

5 Conclusion

In the paper approximate constrained controllability for linear infinite dimensional dynamial system has been considered. Using methods of functional analysis, specially theory of linear unbounded differential operators, necessary and sufficient conditions for approximate controllability have been formulated and proved. The obtained results can be extended to other types of infinite dimensional distributed parameter dynamical systems described by linear partial differential equations.

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