On the essential spectrum of the operators generated by PDE systems of stratified fluids

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Abstract: - We establish the localization and the structure of the spectrum of normal vibrations described by systems of partial differential equations modelling small displacements of stratified fluid in the homogeneous gravity field. We also compare the spectral properties of gravitational and rotational operators. The similarity of the essential spectrum for stratified and rotational flows corresponds to the analogy in the propagation of gravitational and Coriolis waves in viscous fluids, whose consideration includes the study of qualitative properties of the solutions, such as existence, uniqueness, smoothness, asymptotics, etc.

Key-Words: - Partial differential equations, essential spectrum, Sobolev spaces, stratified fluid, internal waves.

1 Introduction

Let us consider a PDE system which describes small displacements of an exponentially stratified viscous fluid in the gravity field

$$\begin{cases} \rho_* \dfrac{\partial \vec{u}}{\partial t} + \vec{e}_3 g\rho - \mu\Delta\vec{u} + \nabla p = 0 \\ \dfrac{\partial \rho}{\partial t} - \dfrac{N^2\rho_*}{g}u_3 = 0 \\ div\,\vec{u} = 0 \end{cases} \qquad (1)$$

together with the PDE system describing the rotational movement of a viscous fluid

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} + \vec{\omega}\times\vec{u} - \mu\Delta\vec{u} + \nabla p = 0 \\ div\,\vec{u} = 0 \end{cases} . \qquad (2)$$

Here $x \in \Omega \subset R^3$, $t \geq 0$, $\vec{u}(x,t) = (u_1, u_2, u_3)$ is the velocity field, $p(x,t)$ is the scalar field of the dynamic pressure , $\rho(x,t)$ is the dynamic density, $\vec{\omega} = (0,0,\omega)$, $\vec{e}_3 = (0,0,1)$, and ρ_* , μ , g , N , ω are positive constants. The equations (1) are deduced under the assumption that the function of stationary distribution of density is performed by $\rho_* e^{-Nx_3}$. The system (2) describes the rotation over the vertical axis.

The systems (1) and (2) were studied from different angles , some of the results may be found in [2]- [5].

In [2] we prove that the essential spectrum of normal vibrations for the operators generated by (2) with $\mu = 0$, is the interval of the real axis $[-\omega, \omega]$, and we also construct an explicit example of non-uniqueness for the spectral parameter belonging to the essential spectrum.

In [3] the following result is stated:

Theorem 1 .

The solution of a Cauchy problem for (2) has the following asymptotic property : the velocity field decreases as $\left(1/t\right)^{5/2}$, $t \to \infty$, where the decay of order $\left(1/t\right)^{3/2}$ is due to the viscosity and the influence of the Coriolis term is $1/t$.

In [4], [5], [7] we prove that for the system (1) the distribution of energy is the same. Namely, from the point of view of *t*-asymptotics, the effects of gravitation and rotation are analogous in viscous fluids :

Theorem 2.

Let us consider the system (1) in the semi-space

$$R^3_+ = \{(x_1, x_2, x_3) : (x_1, x_2) \in R^2, x_3 \geq 0\},$$

together with the boundary conditions

$$\left.\frac{\partial u_1}{\partial x_3}\right|_{x_3=0} = \left.\frac{\partial u_2}{\partial x_3}\right|_{x_3=0} = \left.u_3\right|_{x_3=0} = 0 . \qquad (3)$$

Then, for certain initial conditions, the solution of (1),(3) has the following asymptotic representation :

$$\vec{u}(x,t)=\frac{\vec{C}(x)}{(\mu\, t)^{\frac{3}{2}}(Nt)}+o\left(t^{-\frac{5}{2}}\right)\ ,\quad t\to\infty\quad .$$

Let us observe that the mentioned analogy between gravitational and rotational waves in the dissipation of energy, leads to the corresponding analogy in spectral properties.

Indeed, for the systems (1) and (2) with $\mu = 0$, the singular solutions have the following forms, respectively :

$$E(x,t)=\frac{1}{4\pi|x_3|}\int_0^{\frac{N t |x_3|}{|x|}} J_0(\alpha)d\alpha \quad , \text{ and}$$

$$E(x,t)=\frac{1}{4\pi|\bar{x}|}\int_0^{\frac{\omega t |\bar{x}|}{|x|}} J_0(\alpha)d\alpha\ ,\ |\bar{x}|=\sqrt{x_1^2+x_2^2}\ .$$

Summing up all these results, it seems appropriate to express the conjecture that the operators generated by the system (1) should posess spectral properties, analogous to the system (2) , namely, the essential spectrum of such operators should be the interval [-*N*,*N*] . In this paper we prove that this conjecture is true.

2 Problem Formulation

Let us consider the system

$$\begin{cases} \rho_* \dfrac{\partial \vec{u}}{\partial t}+\vec{e}_3 g\rho+\nabla p=0 \\ \dfrac{\partial\rho}{\partial t}-\dfrac{N^2\rho_*}{g}u_3=0 \\ div\,\vec{u}=0 \end{cases} \quad . \ (4)$$

Differentiating the second equation of (4) with respect to t , we obtain

$$\begin{cases} \dfrac{\partial^2\vec{u}}{\partial t^2}+\vec{e}_3 N^2 u_3+\nabla P=0 \\ div\,\vec{u}=0 \end{cases} \quad , \ (5)$$

where $P=\dfrac{1}{\rho_*}\dfrac{\partial p}{\partial t}$.

For the system (4), let us consider the boundary value problem

$$\vec{u}\cdot\vec{n}\big|_{\partial\Omega}=0 \quad , \qquad (6)$$

where $\vec{n}$ is the vector of the external normal for the bounded domain $\Omega\subset R^3$.

Let $G(\Omega)$ be the space of potential fields in $L_2(\Omega)$:

$$G_2(\Omega)=\{\vec{u}\in L_2(\Omega): \vec{u}=\nabla\varphi;\varphi\in W_2^1(\Omega)\}.$$

Furthermore, let $\overset{0}{J}(\Omega)$ be the space of solenoidal fields :

$$\overset{0}{J}(\Omega)=\{\vec{u}\in C^1(\Omega): div\,\vec{u}=0\,,\vec{u}\cdot\vec{n}\big|_{\partial\Omega}=0\}.$$

Finally, let us introduce the space $J_2(\Omega)$ as a closure of $\overset{0}{J}(\Omega)$ in the norm of $L_2(\Omega)$.

It can be shown ([1]), that $L_2(\Omega)$ permits the following orthogonal decomposition :

$$L_2(\Omega)=J_2(\Omega)\oplus G_2(\Omega)\ .$$

Let *P* be the operator of the orthogonal projection of $L_2(\Omega)$ onto $J_2(\Omega)$. Now, let us define the operator *B* :

$$B\vec{u}=P\{u_3\vec{e}_3\}$$

with the domain

$$D(B)=J_2(\Omega)\ .$$

Thus, the system (5) transforms into

$$\begin{cases} \dfrac{\partial^2\vec{u}}{\partial t^2}+N^2B\vec{u}=0 \\ \vec{u}\in J_2(\Omega) \end{cases} \quad . \ (7)$$

For the system (7) we consider the problem of normal vibrations

$$\vec{u}(x,t)=\vec{v}(x)e^{i\lambda t} \quad . \ (8)$$

Therefore, we can finally write the system (7) as

$$\begin{cases} \lambda^2\vec{v}-N^2B\vec{v}=0 \\ \vec{v}\in J_2(\Omega) \end{cases} \quad . \ (9)$$

Our aim is to investigate the spectrum of the operator *B* . From the physical point of view, the separation of variables (8) serves as a tool to establish the possibility to represent every non-stationery process described by (4) as a linear superposition of the normal vibrations. The knowledge of the spectrum of the normal vibrations, its structure and localization, may be very useful for studying the stability of the flows. Finally, the spectrum of operator *B* is important in the investigation of weakly non-linear flows, since the bifurcation points where the small non-linear solutions arise, belong to the spectrum of linear normal vibrations, i.e., to the spectrum of operator *B* .

3 Problem Solution

Lemma 3.

B is a positive self-adjoint operator in $J_2(\Omega)$.

Proof. Evidently, $\|B\vec{u}\|_{L_2(\Omega)}\le\|\vec{u}\|_{L_2(\Omega)}$ and thus

$$\|B\| \le 1 \ .$$

Let $\vec{u}, \vec{v} \in J_2(\Omega)$. Then,

$$(\vec{u}, B\vec{v}) = (\vec{u}, P\{v_3\vec{e}_3\}) = (P\vec{u}, \{v_3\vec{e}_3\}) =$$
$$= \int_\Omega u_3 v_3 dx = (B\vec{u}, \vec{v}).$$

Since B is bounded, its self-adjointness follows from its symmetry.
Finally,

$$(\vec{u}, B\vec{u}) = \int_\Omega |u_3(x)|^2 dx \ge 0 \ ,$$

which concludes the proof.

Lemma 4.

The kernel of B is the subspace $H_J(\Omega)$ which consists of all elements of $J_2(\Omega)$ with trivial third component.

Proof. Obviously, $H_J(\Omega) \subset Ker(B)$. Suppose that $\vec{u} \in Ker(B)$ and $\vec{u} \notin H_J(\Omega)$. Then, we obtain that $(\vec{u}, B\vec{u}) = \int_\Omega |u_3(x)|^2 dx = 0$, which implies $u_3 = 0$ and thus $H_J(\Omega) = Ker(B)$.

Corollary.

$\lambda = 0$ is an eigenvalue of infinite multiplicity for B. Its corresponding eigenvectors compose all the subspace $H_J(\Omega)$.

Now, let us consider the same separation of variables for the function $P(x,t)$:

$$P(x,t) = q(x)e^{i\lambda t} \ , \ q \in W_2^1(\Omega) \ .$$

If $q(x)$ is a solution of the system

$$\begin{cases} -\lambda^2 v_1 + \dfrac{\partial q}{\partial x_1} = 0 \\ -\lambda^2 v_2 + \dfrac{\partial q}{\partial x_2} = 0 \\ \left(-\lambda^2 + N^2\right) v_3 + \dfrac{\partial q}{\partial x_3} = 0 \\ \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3} = 0 \end{cases} , \qquad (10)$$

then $q(x)$ satisfies the equation

$$\Delta q = -div\left(N^2 v_3 \vec{e}_3\right) \ ,$$

which implies

$$div\left(N^2 v_3 \vec{e}_3 + \nabla q\right) = 0 \ .$$

Thus, the projection operator B obtains its explicit form as

$$N^2 B\vec{v} = N^2 v_3 \vec{e}_3 + \nabla q \ .$$

We shall establish now the structure of the spectrum of the operator B.

Theorem 5.

The essential spectrum of the operator $N^2 B$ is the interval of the real axis $[-N, N]$. Moreover, the points $0, \pm N$ are eigenvalues of infinite multiplicity.

Proof. First we recall that the essential spectrum is composed of the points belonging to the continuous spectrum, limit points of the point spectrum and the eigenvalues of infinite multiplicity ([8]). We shall use the following criterion which is attributed to Weyl ([8]): A necessary and sufficient condition that a real finite value μ be a point of the essential spectrum of a self-adjoint operator B is that there exist a sequence of elements $x_n \in D(B)$ such that

$$\begin{array}{l} \|x_n\| = 1, \ x_n \to 0 \ \text{weakly and} \\ \|(B - \mu I)x_n\| \to 0 \end{array} . \qquad (11)$$

Let us denote $\lambda^2 = \mu$, $\mu \ne 0$. Then, the system (10) takes the matrix form

$$\begin{pmatrix} -\mu & 0 & 0 & \dfrac{\partial}{\partial x_1} \\ 0 & -\mu & 0 & \dfrac{\partial}{\partial x_2} \\ 0 & 0 & N^2 - \mu & \dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \qquad (12)$$

One can easily see that the main symbol of the differential operator in (12) is

$$L(\xi) = \begin{pmatrix} -\mu & 0 & 0 & \xi_1 \\ 0 & -\mu & 0 & \xi_2 \\ 0 & 0 & N^2 - \mu & \xi_2 \\ \xi_1 & \xi_2 & \xi_3 & 0 \end{pmatrix} .$$

As

$$\det L(\xi) = \mu\left(-\mu|\xi|^2 + N^2|\xi|^2\right) \ ,$$
$$|\xi|^2 = \xi_1^2 + \xi_2^2 \ , \ |\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \ ,$$

we may conclude that the operator $N^2 B$ is not elliptic in sense of Douglis-Nirenberg if and only if $\mu \in [0, N^2]$. ([6]) .

Now, let us consider $\mu_0 \in (0, N^2)$ and choose a vector ξ such that

$$-\mu_0|\xi|^2 + N^2|\xi|^2 = 0 \ , \ |\xi| \ne 0 \ .$$

Therefore, there exists

$$\eta = (\eta_1, \eta_2, \eta_3, \eta_4) \ , \ \eta_i \ne 0 \ , \ 1 \le i \le 4 ,$$

such that $L(\xi)\eta = 0$:

$$\begin{cases} -\mu_0\eta_1 + \xi_1\eta_4 = 0 \\ -\mu_0\eta_2 + \xi_2\eta_4 = 0 \\ \left(-\mu_0 + N^2\right)\eta_3 + \xi_{31}\eta_4 = 0 \\ \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 = 0 \end{cases} \qquad (13)$$

Solving (13) with recpect to η , we obtain

$$\begin{cases} \eta_1 = \dfrac{\xi_1}{\mu_0} \\ \eta_2 = \dfrac{\xi_2}{\mu_0} \\ \eta_3 = \dfrac{\xi_3}{\mu_0 - N^2} \\ \eta_4 = 1 \end{cases} \qquad (14)$$

We observe that $\eta_i \neq 0$, $1 \le i \le 4$.

Now, let us choose a function

$$\psi_0(x) \in C_0^\infty(\Omega) \ , \ \int_{|x|\le 1} \psi_0^2(x)dx = 1 \ .$$

We fix $x_0 \in \Omega$ and define

$$\psi_k(x) = k^{\frac{3}{2}}\psi_0\left(k(x - x_0)\right) \ , \ k = 1,2,...$$

One can easily see that

$$\left\|\psi_k\right\|_{L_2(\Omega)} = 1 \ , \ \left\|\frac{\partial\psi_k}{\partial x_j}\right\|_{L_2(\Omega)} = C_j^1 k \ ,$$

$$\left\|\frac{\partial^2\psi_k}{\partial x^2{}_j}\right\|_{L_2(\Omega)} = C_j^2 k^2 \ , \qquad (15)$$

where the constants $C_j^i \neq 0$ do not depend on k .

We define the Weyl sequence

$$\tilde{v}^k = \left(v_1^k, v_2^k, v_3^k, q^k\right)$$

as follows :

$$\begin{cases} v_j^k(x) = \eta_j e^{ik^3<x,\xi>}\left(\psi_k - \dfrac{1}{ik^3\xi_j}\dfrac{\partial\psi_k}{\partial x_j}\right), \ j = 1,2,3 \\ q^k(x) = -\dfrac{i}{k^3}\psi_k e^{ik^3<x,\xi>} \\ <x,\xi> = x_1\xi_1 + x_2\xi_2 + x_3\xi_3 \ , \ k = 1,2,.... \end{cases} \qquad (16)$$

Now we have to verify that the sequence $\tilde{v}^k$ defined above, satisfies the conditions (11). Note that a Weyl sequence is an explicit solution of a system of partial differential equations.

For the functions (16), the weak convergence to zero is evident. Let us introduce the matrix differential operator M :

$$M = \begin{pmatrix} 0 & 0 & 0 & \dfrac{\partial}{\partial x_1} \\ 0 & 0 & 0 & \dfrac{\partial}{\partial x_2} \\ 0 & 0 & N^2 & \dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} & 0 \end{pmatrix} .$$

Thus, the system (12) can be expressed as

$$\left(M - \mu I_3\right)\tilde{v} = 0 \ ,$$

where

$$I_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Let us prove that

$$\lim_{k\to\infty}\left\|\tilde{f}^k\right\|_{L_2(\Omega)} = 0 \ ,$$

where

$$\tilde{f}^k = \left(M - \mu_0 I_3\right)\tilde{v}^k \ .$$

We have

$$f_1^k = -\mu_0 v_1^k + \frac{\partial q^k}{\partial x_1} = \left(-\mu_0\eta_1 + \xi_1\eta_4\right)\times$$

$$\times\psi_k e^{ik^3<x,\xi>} - \frac{i}{k^3}e^{ik^3<x,\xi>}\frac{\partial\psi_k}{\partial x_1}\left[\frac{\mu_0\eta_1}{\xi_1} + 1\right]$$

By virtue of (13), (14) we have

$$-\mu_0\eta_1 + \xi_1\eta_4 = 0 \ .$$

Therefore,

$$\left\|f_1^2\right\|_{L_2(\Omega)} \le Const \cdot k^{-2} \to 0 \ \text{as} \ k \to \infty \ .$$

Analogously,

$$f_2^k = -\mu_0 v_2^k + \frac{\partial q^k}{\partial x_2} = \left(-\mu_0\eta_2 + \xi_2\eta_4\right)\times$$

$$\times\psi_k e^{ik^3<x,\xi>} - \frac{i}{k^3}e^{ik^3<x,\xi>}\frac{\partial\psi_k}{\partial x_2}\left[\frac{\mu_0\eta_2}{\xi_2} + 1\right]$$

From (13),(14) it follows that

$$-\mu_0\eta_2 + \xi_2\eta_4 = 0 \ .$$

Thus,

$$\left\|f_2^2\right\|_{L_2(\Omega)} \le Const \cdot k^{-2} \to 0 \ \text{as} \ k \to \infty \ .$$

In the similar way,

$\left(-\mu_0 + N^2\right)\eta_3 + \xi_3\eta_4 = 0$ implies

$$\left\| f_3^2 \right\|_{L_2(\Omega)} \le Const \cdot k^{-2} \to 0 \text{ as } k \to \infty \ .$$

Analogously, from $<\xi,\eta>=0$ we obtain

$$\lim_{k\to\infty} \left\| f_4^2 \right\|_{L_2(\Omega)} = 0 \quad .$$

To verify that the norms $\left\| \tilde{v}^k \right\|_{L_2(\Omega)}$ are separated from zero, it is sufficient to prove that at least the norms of one component of the field $\tilde{v}^k$ are separated from zero as $k \to \infty$.

Let us consider the two summands

$$v_1^k = v_{11}^k + v_{12}^k \quad ,$$

where

$$v_{11}^k = \eta_1 e^{ik<x,\xi>} \psi_k(x) \ ,$$

$$v_{12}^k = \eta_1 e^{ik<x,\xi>} \frac{i}{k^3 \xi_1} \frac{\partial \psi_k(x)}{\partial x_1} \ .$$

Evidently,

$$\lim_{k\to\infty} \left\| v_{12}^k \right\|_{L_2(\Omega)} = \lim_{k\to\infty} \frac{|\eta_1|}{k^3 |\xi_1|} \left\| \frac{\partial \psi_k}{\partial x_1} \right\|_{L_2(\Omega)} = 0 \ .$$

However,

$$\left\| v_{11}^k \right\|_{L_2(\Omega)} = \left\| \eta_1 \psi_k e^{ik^3<x,\xi>} \right\|_{L_2(\Omega)} = |\eta_1| \left\| \psi_k \right\|_{L_2(\Omega)} =$$

$$= |\eta_1| \neq 0 \ .$$

In this way, we have proved that the sequence (16) satisfies Weyl conditions (11). Since the essential spectrum is closed, the points $\mu = 0$, N^2 , belong to it. Returning to the initial spectral parameter λ , we obtain that the essential spectrum of the operator $N^2 B$ is the interval $[-N, N]$.

We have seen that $\lambda = 0$ is an eigenvalue of infinite multiplicity . The same statement holds for the points $\lambda = \pm N$.

Indeed, for $\lambda = \pm N$ the system (10) transforms into

$$\begin{cases} -N^2 v_1 + \partial q / \partial x_1 = 0 \\ -N^2 v_2 + \partial q / \partial x_2 = 0 \\ \partial q / \partial x_3 = 0 \\ \partial v_1 / \partial x_1 + \partial v_2 / \partial x_2 + \partial v_3 / \partial x_3 = 0 \end{cases} .$$

It can be easily seen that any function of the type $(0,0,\varphi(x_1,x_2),0)$, $\varphi \in C_0^\infty$, satisfies the last system.

Thus, theorem 5 is proved.

4 Conclusion

The remarkable analogy of gravitational and rotational waves discussed above could serve as an example of how mathematical description of physical forces of different origin may help us to understand the unity of the Nature’s manifestations.

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