Regularization Algorithms for Electric Tomography Images Reconstruction

ALEXANDRE GREBENNIKOV
Facultad de Ciencias Fisico Matematicas
Benemerita Universidad Autonoma de Puebla,
Av. San Claudio y Rio verde, Ciudad Universitaria,
72570 Puebla, Pue.
MEXICO
e-mail: agrebe@fcfm.buap.mx

Abstract: - Electrical Tomography consists in reconstruction the image of the body interior on the measuraments made on the it's surface. Mathematically it can be described as a coefficient inverse problem for the Laplace equation, written in the divergent form. The coefficient is the functions of the space variables and characterize the electrical properties of a media. Well know and the most developed now approach is Electrical Impedance Tomography, that includes the resistence and capacitance tomographies. It use the measuraments of the voltages on the surface produced under the the known injected currents. This method has some advantiges, but it's algorithmic realization is sufficiently hard, because of nonlinear structure of the mathematical model. We propose here another approach for the plane case, based on the original use of the Radon transformation. We use regularization by spline-approximation method for the explicit realization of the inverse Radon transformation, that leads to the fast algorithm of the image reconstruction. These approach and algotithms are justified with the numerical experiments on the simulated model problems.

Key-words: - images reconstruction, electrical tomography, spline, regularization.

1. Introduction

Electrical Tomography consists in reconstruction the image of the body interior on the electrical characteristics measurements made on its surface [1], [2]. Well know and the most developed now approach is Electrical Impedance Tomography (EIT), that includes the resistance and capacitance tomographies. In a plane case EIT can be mathematically described as a coefficient inverse problem for the Laplace equation, written in the divergent form

$$\frac{\partial}{\partial x}\left(T\ u_x'(x,y)\right)+\frac{\partial}{\partial y}\left(T\ u_y'(x,y)\right)=0, \qquad (1)$$

where $x, y \in \Omega$ - some region on a plane, $u(x,y)$ is potential, the function $T = T(x,y)$ characterize the admittivity of a media. The goal of EIT is to reconstruct the admittvity distribution of the interior of the object on the knowledge of measurements of the voltages on the surface. It use the so-called complete electrode model, when the voltages on the surface are produced under the known injected currents. This approach leads to nonlinear and ill-posed problem. As the rule, the Tikhonov regularization scheme is used to solve it. This method has some advantages, but its algorithmic and software realization for desired image reconstruction are sufficiently complicated because of nonlinear structure of the corresponding mathematical model.

We propose here another more simple approach for the measurement of the external data that is explicated here more detailed for the plane case. Proposed measurement scheme leads in the radial symmetric case to the classic Radon transformation [10]. To solve the corresponding inverse problem we use in the radial symmetric case the explicit form of the inverse Radon transformation and regularization by spline-approximation method for it calculation. It leads to the fast algorithm of the image reconstruction and a simple computer programs. In the general case we propose a new $G-$transformation, related with the Radon transformation. We obtain the basic integral equation and propose the simplified algorithm for its numerical solution. These approach and algorithms are justified with the numerical experiments on the simulated model problems.

2. Coefficient inverse problem for the elliptic equation

The inverse problem for the equation (1) consists in reconstruction of the coefficient T. This scheme can appear in the component quantification problem for the fluids of complex mixture (for example, mixture of gas, oil and water). If the solution u is known and initial conditions for T are given, then in principle it is possible to solve the partial differential equation of the first order concerning T by the characteristics method [7]. The main system of the differential equations is the next

$$-\frac{dT}{T} = \frac{\Delta u dx}{u_x^{'}} = \frac{\Delta u dy}{u_y^{'}}, \tag{2}$$

where $u_x^{'}, u_y^{'}$ are nonzero derivatives of the function $u(x,y)$ on x and y accordingly. If the function u is given in a discreet form on some red, the approach of the construction numerical algorithms for this type of equations on the base of the local spline approximation formulas was proposed in [8], [9]. If the input data are noised values of the function it is possible to realize the regularization with the Full Spline Approximation Method (F.S.A.M.) [14] -[17]. We use here the adopted for considered case F.S.A.M., that includes four recursion steps: 1) pre-smoothing the input data by explicit approximation cubic splines S_k; 2) pre-reconstruction of required functions by calculations with given formulas or by numerical solution of the equation, describing the process; 3) post-processing, including the post-smoothing and projecting of the pre-reconstruction on the set, that characterizes special properties of the exact solution of the problem; 4) stop rule in the form of the residual principle. The theoretical justification of the regularization properties of this algorithm for sufficiently general cases can be chosen in [3] - [6], [14] -[17].

3. The General Ray principle and its application to the Electric tomography

3.1.The General Ray principle and the basic integral equation

Let us consider the problem of the image reconstruction of the structure, consisting of the component with different characteristics, under the influence of the known external physical field or rays. The image of the distribution of this characteristics are described inside the plane domain Ω by some function $g(x,y)$, that must be reconstructed based on indirect boundary observations. To construct the algorithms of the desired image reconstruction we propose here the General Ray (GR) principle. It consists in the next assumptions:

1) the considered influence of the external physical field or rays can be simulated mathematically by the plane vector field $\overrightarrow{V}(l)$ parallel to the direction of the ray along the straight line l ;.

2) this field is homogeneous on the direction orthogonal l;

3) the field $\overrightarrow{V}(l)$ is characterized with some function $u(x,y)$;

4) we can measure the values of the difference $v = u(x^1,y^1) - u(x^0,y^0)$ in any boundary points. $P_1 = (x^1,y^1)$ and $P_0 = (x^0,y^0)$. of the domain;

5) the values of derivatives $\partial u(x,y)/\partial l$ on all directions l give us the possibility to reconstruct the function $g(x,y)$.

It is easy to observe that the tomography such as electrical, ultrasonic and radioisotope imaging [1], can be considered under this GR-principle due to the choice of the measurement scheme.

The line l has the parametric presentation $p = x\cos\varphi + y\sin\varphi$, where $|p|$ is a length of the perpendicular, passed from the center of coordinates to the line l, φ is the angle between the axis x and this perpendicular. Hence, using this parametrization, we present the function $v = v(p,\varphi)$. If $v(p,\varphi)$ is given for all p,φ,then, using the Radon transformation [10]

$$\begin{aligned} R[u] &= \int_l \frac{\partial u(x,y)}{\partial l} dt, \\ x &= p\cos\varphi - t\sin\varphi, \\ y &= p\sin\varphi + i\cos\varphi, \end{aligned} \tag{3}$$

we can obtain as the mathematical model of the GR-principle the basic linear integral equation

$$G[u] \equiv R[u_x]\cos\varphi + R[u_y]\sin\varphi = v(p,\varphi) \tag{4}$$

of the first kind with respect to the function $u(x,y)$. Investigation of the $G-$transformation shows that G is unbounded operator from the L_2 into L_2, bounded from the Sobolev space $W_2^{(1)}$ into L_2, that defines the character of the instability at the solving the equation (4). It is possible to prove the uniqueness of the solution of the basic integral equation on the set of the functions u that equal to zero in one fixed point $(\overline{x},\overline{y})$ on the boundary.

We will consider the Electrical Tomography scheme, when the external field $\overrightarrow{V}(l)$ is the electromagnetic field. It initiates some distribution of the electric potential $u(x,y)$ inside the domain Ω, which we define for simplicity as a unit circle. The function $g(x,y)$ is the admittivity function $T(x,y)$. We suppose that the measurement scheme is a "parallel", i.e., we have the $2(2n-2)$ electrodes uniformly and symmetrically distributed on the unit circle at the points $\{t_i,p_i\}$ such as to the every pare $P_i=\{t_i,p_i\}$ corresponds $\overline{P}_i=\{-t_i,p_i\}$. These electrodes serve as sources of the electric field and also as measurement units. The realization of the General Ray principle consists in the measurement of the difference of the potential for the angle $\varphi=0$ between points P_i and $\overline{P}_i, i=1,...,(2n-2)$. Then for values $\varphi_i=\pi(i-1)/(2n-2)$ we rotate the scheme of measurements on this angle, that corresponds to the scanning by the rotating field $\overrightarrow{V}(x)$.

3.2. Electrical tomography in the case of the radial symmetry.

In the case of the radial symmetry the potential $u=u(r)$ does not depend on the angle φ and it is sufficient to use in the measurement scheme only $\varphi=0$. Equation (1) transforms into the equation

$$\frac{1}{r}(Tru_r^{'}(r))_r^{'}=0, \tag{5}$$

the mentioned basic integral equation (4) transforms into well known Abel´s equation:

$$\int_p^1 \frac{w(t)tdt}{\sqrt{t^2-p^2}}=v(p),\ p\in[0,1], \tag{6}$$

with respect to the function $w(r)=u_r^{'}(r)$ that has the explicit relation with $T(p)$ by the formula $T(p)=c/pw(p)$, $c=const$. Hence, the reconstruction of the function $u(p)$ give us possibility to reconstruct the desired electric admittvity distribution $g=T(p)$. The instability of the solution of the equation (7) is equivalent to the instability of the problem of the numerical differentiation. We use as he input data the measured values of the difference of potentials $\widetilde{v}_i=v(p_i)+\xi_i$, in the boundary points of the n parallel lines, corresponding to $p_i=ih$, $h=1/n$, $|\xi_i|\le\delta$, $i=1,...,n$. The formula for calculation of the function w is based on the approximate calculation of the explicit inverse Radon transformation:

$$\begin{aligned}\widetilde{w}(p) \;=\;& \widetilde{v}_j/(\pi\sqrt{t_{j+1}^2-p^2})+ \\ & \sum_{i=k+1}^{n}\frac{\widetilde{v}_i}{\pi}\left[1/\sqrt{t_{i+1}^2-p^2}-1/\sqrt{t_i^2-p^2}\right].\end{aligned} \tag{7}$$

We applied the described above F.S.A.M. algorithm. Let us present outcomes of some model numerical experiments. We underline, that in this scheme we use as the input data the values of the simulated potential on the boundary only, not inside the circle. For the exact $T(r)=T_1=0.5$, $r\in[0,\ 0.3]$; $T(r)=T_2=2$, $r\in[0.3,\ 0.7]$; $T(r)=T_3=1$, $r\in[0.7,\ 1]$; we calculated the exact $u(r)=2\ln r\ -1.5\ln(0.3)\ +0.5\ln(0.7)\ +1$, $r\in[0,\ 0.3]$; $u(r)=0.5\ln r\ +0.5\ln(0.7)+1$, $r\in[0.3,\ 0.7]$; $u(r)=\ln r+1$, $r\in[0.7,\ 1]$. The simulation consists in the construction of the model potential distribution in the domain Ω for the known $T(p)$ under the influence of the known external electric field. We considered the plane vector field $\overrightarrow{V}(x)$ parallel to the direction of r independent on φ. The simulated relative exact values of the function $v(p)$ can be calculated by formulas: $v(p)=2[(1/T_2-1/T_3)\overline{y}\ +(1/T_1-1/T_2)\overline{z}\ +\overline{x}/T_3]$, $p\in[0,\ 0.3]$; $v(p)=2[(1/T_2-1/T_3)\overline{y}\ +\overline{x}/T_3]$, $p\in[0.3,\ 0.7]$; $v(p)=2\overline{x}/T_3$, $p\in[0.7,\ 1]$, where $\overline{x}=(1-p^2)^{1/2}$, $\overline{y}=(0.7^2-p^2)^{1/2}$, $\overline{z}=(0.3^2-p^2)^{1/2}$. We used the values $v(p_i)$ with the additional random errors as the input data.

We note that in considered case T presents some piece-wise constant function, corresponding the electric properties of the mixture components. Although the theoretical foundation of F.S.A.M. is given for the smooth functions, however, the proposed algorithm gives good results of the reconstruction of the coefficient T in considered case too. Moreover, if the values of this constants $\{T_i\}$ are known a priori, we include this information into the algorithm as the last post-processing step, that consists in the projection the result of the post-smoothing on the set $\{T_i\}$. This projection can be realized with respect to the absolute or the relative criteria. If the values $\{T_i\}$ have not very different scale, the absolute criterion gives good results, otherwise we need to use the relative criterion.

In Fig. 1 the results of the coefficient T reconstruction for $n = 21$, $\delta = 0.05$ are presented as maps of isolines. Graph (a): the exact $T(x, y)$; graph (b): reconstruction on noised simulated data without regularization; graph (c): reconstruction on noised simulated data by spline-approximation method without post-processing; graph (d): F.S.A.M. reconstruction on noised simulated data with the post-processing (absolute criterion).

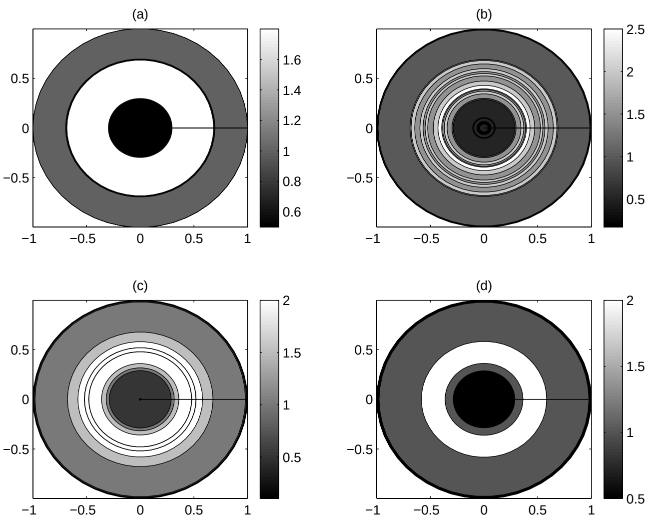

Figure 1: Regularization effect at the coefficient T reconstruction in the radial symmetric case.

3.3. The general case.

It is possible to look for the approximate solution of the integral equation (4) in the general case by spline-approximation method with realization the pre-reconstruction using the collocation scheme [17]. The most simple variant corresponds to the linear splines in the collocation scheme and to the cubic splines for recursive smoothing. We will locate in Cartesian coordinates the approximation of the potential of electric field $u(x, y)$ as a bipolinomial spline $s(x, y) = \sum_{i,j=1}^{2n-1} c_{i,j} \; s_i \; (x)$ $s_j(y)$, $[x, y] \in \Omega \subset [-1, 1] \times [-1, 1]$. B-splines s_i are constructed on the corresponding uniform grids on x and y. Substituting $s(x, y)$ into equation (4) and using collocation conditions on the corresponding grids $\{p_l\}, \{\varphi_k\}, l, k = 1, ..., 2n - 2$, we obtain the system of linear algebraic equations concerning $\{c_{i,j}\}$. Together with the condition $s(\overline{x}, \overline{y}) = 0$ and with periodicity and symmetry it gives us sufficient number of equations to determine desired coefficients. Then it is possible to construct approximation for $u(x, y)$ and recuperate the function $T(x, y)$ from equations (2). In principle, proposed scheme, based on the $G-$transformation for electric tomography images reconstruction, is more simple then traditional schemes of electrical tomography. But it is also unstable as the problem of twice numerical differentiation and we need regularize it, for example with the spline-approximation algorithms. Proposed scheme has also sufficiently grand number $(2n - 1)^2$ of equations. That is why we developed the simplified approximate algorithm for the solution of the equation (4). It consists in the following approximate presentation of the function $\gamma(x, y) = \partial u(x, y)/\partial l$ for $\varphi = 0$ and x, y belonging to the line l:

$$\gamma(p, t) = \sum_{i,j=1}^{2n-2} a_{i,j} s_i \; (p) s_j(t), \tag{8}$$

using $B-$splines, constructed on the grids $\{p_i\}, \{t_j\}$. Substituting (8) it into equation (4) we have for every p_i relations

$$\sum_{i=1}^{2n-2} s_i \; (p_i) b_i \quad = \quad v(p_i, 0), \tag{9}$$

$$b_i \quad = \quad \sum_{j=1}^{2n-2} a_{i,j} \int_{t_1}^{t_{2n-1}} s_j(t) dt. \tag{10}$$

From the explicit formulas of spline-approximation theory [3], [4] we obtain approximate àveraged values $b_i \approx v(p_i, 0)$. To reconstruct the tomography image function $T(x, y)$ we use the described scheme for all $\{\varphi_k\}$ with rotating of the investigated object and inverse rotation after the approximation. We extend the approximate averaged values along the line l for all p and use the coincidence condition for calculate recuperation of the $T(x, y)$. Proposed "averaged-rotating" algorithm appears effective for the piece-wise constant function $T(x, y) = T_i$, $x, y \in \Omega_i$,if the values of this constants $\{T_i\}$ are known a priori and it is necessary to reconstruct the distribution of subdomains Ω_i inside the circle. We include the special projection procedure into our algorithm as the last post-processing step. We obtain the good quality of the reconstruction of the tomography image if the values $\{T_i\}$ have not very different scale.

Some results of numerical experiments with exact simulated data are presented on the Fig. 2, 3. In the little circle $T = T_1$, in the middle circle $T = T_2$, in the unit circle $T = T_3$. Graphs (a) - exact image; (b), (c), (d) - reconstructed by "averaged-rotating" algorithm for $n = 11$, 21, 31 correspondingly.

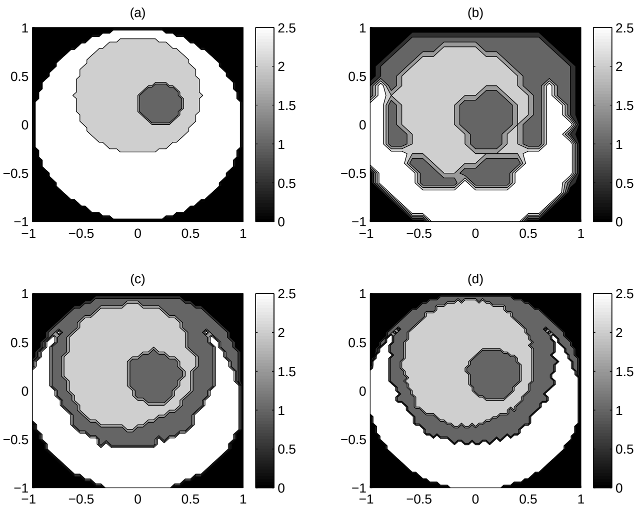

Figure 2: Reconstruction the electrical tomography image for T1=1, T2=2, T3=3 with n=11, 21, 31.

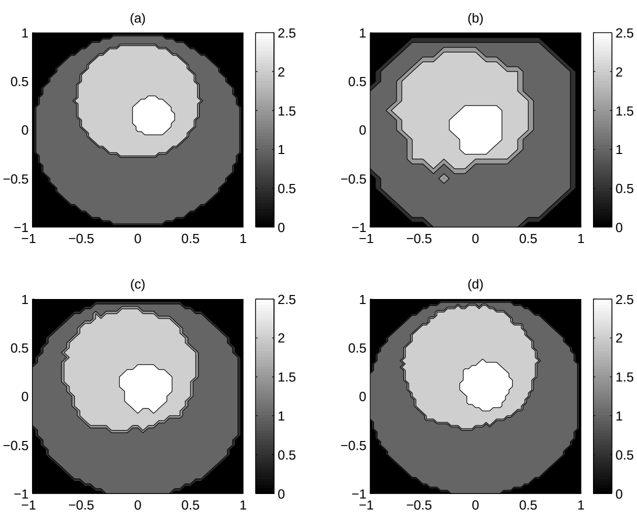

Figure 3: Reconstruction the electrical tomography image for T1=3, T2=2, T3=1 with n=11, 21, 31.

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