A Derivation of Interactor for Non-Square Plants and Pseudo Inverted Interactorizing by State Feedback

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Abstract:— An interactor matrix plays some important roles in control system analysis and synthesis. Recently, a simple derivation of the interactor matrix using pseudoinverse. Unfortunately, this method is limited to square systems. In this paper, it will be presented a simple derivation of an interactor matrix for non-square transfer function matrices. A pseudo inverted interactorizing will be proposed and achieved by using state feedback.

Key Words:–interactor matrix, non-square transfer matrix, pseudoinverse, null-space.

1 Introduction

A spectral factorization of transfer function matrices are used to solve the many control problems such as filtering [1], [2], singular LQ problem [3] and H_∞ problem [4]. A primary solution was given in [5], which was based on the direct operations for the transfer function matrix and was not adequate for computer calculations. Later, a method using state space representation was shown [6]. Unfortunately, the method can not be used for strictly proper transfer function matrices.

In this paper, it will be presented a state space formula of spectral factorization for strictly proper transfer function matrices. The basic idea is as follows: first, calculate a unitary interactor matrix [7], [3] for a given transfer matrix in order to make the compensated transfer matrix be proper, and then use the classical method in [6]. A derivation of interactor was given in [7], which is hard to carry out by computers. The authors presented a simple method using state space representation of given transfer function matrix [8]. But this method is limited to non-singular transfer functions. Therefore, it will be presented a method to derive an interactor for non-square transfer function matrices. It will be shown that the problem can be solved by calculating the null space of certain matrix.

As an application of spectral factorization, it will discussed a pseudo inverted interactorizing using by state feedback. The notion of inverted interactorizing was presented in [9]. But the method is useful for the square and minimum phase plants. The method is closely related to the singular LQ regulation problem [3]. In fact, the singular LQ regulation can be solved by the inverted interactorizing of the minimum phase image of given plant. But only square plants are discussed in [3]. It will be introduced a pseudo inverted interactorizing for non-square plants, instead. Then, a method to achieve the pseudo inverted interactorizing will be shown by using state feedback.

2 Simple Derivation of Interactor Matrix

For a given $m \times p$ strictly proper and full rank transfer function matrix, $G(z)$, there exists an $m \times m$ polynomial matrix, $L(z)$, which satisfies the following equation.

$$\lim_{z\to\infty} L(z)G(z) = K \quad (full\ rank). \tag{1}$$

Such an $L(z)$ is called an interactor matrix of $G(z)$ [1]. If $K = I_m$, $L(z)$ is called an identity interactor [11]. At first, a derivation of an identity interactor $\xi(z) := K^{-1}L(z)$ is considered.

Let $(A,\ B,\ C)$ denote a minimal realization of $G(z)$ and

[1] Although the definition [7] is restricted the structure of $L(z)$ (lower triangular), we do not consider such a restriction since it is not essential in this paper.

define $\boldsymbol{T}_{k-1}$ and $\boldsymbol{J}_{k-1}$ by

$$\boldsymbol{T}_{k-1} = \begin{bmatrix} CB & 0 & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{k-1}B & CA^{k-2}B & \cdots & CB \end{bmatrix}, \quad (2)$$
$$\boldsymbol{J}_{k-1} = [I_m \ 0_{m\times m(k-1)}].$$

Define w as the least integer k which satisfies the following equation.

$$\text{rank} \begin{bmatrix} \boldsymbol{T}_{k-1} \\ \boldsymbol{J}_{k-1} \end{bmatrix} = \text{rank } \boldsymbol{T}_{k-1}. \quad (3)$$

Let an identity interactor, $\xi(z)$, be described by

$$\begin{aligned} &\xi(z) = z\xi_1 + z^2\xi_2 + \cdots + z^w\xi_w = \boldsymbol{\xi} z S_{I_m}^{w-1}(z), \\ &\boldsymbol{\xi} := [\xi_1 \ \xi_2 \ \cdots \ \xi_w], \quad \xi_i \in R^{m\times m} \\ &S_{I_m}^{w-1}(z) := [I_m \ zI_m \ \cdots \ z^{w-1}I_m]^T, \end{aligned} \quad (4)$$

then, from Mutoh and Ortega (1993) (or Huang *et al* 1997), the following equation holds:

$$\boldsymbol{\xi}\boldsymbol{T}_{w-1} = \boldsymbol{J}_{w-1}. \quad (5)$$

Conversely, the identity interactor $\xi(z)$ can be obtained by solving this equation and the solvability of this is asserted from eqn.(3). Thus, using Moore-Penrose pseudoinverse $\boldsymbol{T}_{w-1}^{\dagger}$ of $\boldsymbol{T}_{w-1}$, $\boldsymbol{\xi}$ can be calculated by

$$\boldsymbol{\xi} = \boldsymbol{J}_{w-1}\boldsymbol{T}_{w-1}^{\dagger}. \quad (6)$$

Example 1 Consider the following transfer function matrix [7].

$$G(z) = \begin{bmatrix} \frac{1}{z+1} & \frac{1}{z+2} \\ \frac{1}{z+3} & \frac{1}{z+4} \end{bmatrix}$$

In this case, $(A,\ B,\ C)$ can be given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -8 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 3 & 1 & 4 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}.$$

In this case, $w = 3$ and using the pseudoinverse of

$$\boldsymbol{T}_2 = \begin{bmatrix} CB & 0 & 0 \\ CAB & CB & 0 \\ CA^2B & CAB & CB \end{bmatrix},$$

we have

$$\boldsymbol{\xi} = \begin{bmatrix} 0.75 & 0.75 & 0.25 & -1.25 & 0.5 & -0.5 \\ -0.5 & -0.5 & 0 & 1 & -0.5 & 0.5 \end{bmatrix},$$
$$\xi(z) = \frac{z}{2} \begin{bmatrix} z^2 + 0.5z + 1.5 & -z^2 - 2.5z + 1.5 \\ -z^2 - 1 & z^2 + 2z - 1 \end{bmatrix}.$$

3 Properties of the Interactor Matrix

In this section, we consider the properties of the interactor by the proposed method. For the pseudoinverse of $\boldsymbol{T}_{w-1}$, we have the following Lemma.

Lemma 1 There exists a matrix $\boldsymbol{P}$ such that

$$\begin{bmatrix} CB \\ CAB \\ \vdots \\ CA^{w-1}B \end{bmatrix} \boldsymbol{\xi}\boldsymbol{\xi}^T + \begin{bmatrix} 0_{m\times mw} \\ \boldsymbol{T}_{w-2} \end{bmatrix} \boldsymbol{P}\boldsymbol{\xi}^T = \boldsymbol{\xi}^T. \quad (7)$$

(Proof). From eqns.(2) and (6), the first m rows of $\boldsymbol{T}_{w-1}^{\dagger}$ must be $\boldsymbol{\xi}$. So using some matrix $\boldsymbol{P}$, we can write

$$\boldsymbol{T}_{w-1}^{\dagger} = \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{P} \end{bmatrix}. \quad (8)$$

Since $\boldsymbol{T}_{w-1}^{\dagger}$ is the Moore-Penrose pseudoinverse of $\boldsymbol{T}_{w-1}$,

$$(\boldsymbol{T}_{w-1}\boldsymbol{T}_{w-1}^{\dagger})^T = \boldsymbol{T}_{w-1}\boldsymbol{T}_{w-1}^{\dagger}. \quad (9)$$

Substituting eqn.(8) into the above equation, we have

$$\begin{bmatrix} CB \\ CAB \\ \vdots \\ CA^{w-1}B \end{bmatrix} \boldsymbol{\xi} + \begin{bmatrix} 0_{m\times mw} \\ \boldsymbol{T}_{w-2} \end{bmatrix} \boldsymbol{P} = \begin{bmatrix} \boldsymbol{\xi}^T & \boldsymbol{P}^T \end{bmatrix} \boldsymbol{T}_{w-1}^T. \quad (10)$$

By postmultiplying the above equation by $\boldsymbol{\xi}^T$ and then using eqn.(5), eqn.(7) can be obtained. □

The explicit form of $\boldsymbol{P}$ will be given in Appendix A.

Theorem 1 Let

$$\begin{aligned} &\xi^{\sim}(z) = \xi^T(z^{-1}) = z^{-1}\xi_1^T + z^{-2}\xi_2^T + \cdots + z^{-w}\xi_w^T, \\ &F = \boldsymbol{\xi} \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^w \end{bmatrix}, \quad \mathcal{O}_{w-1}(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{w-1} \end{bmatrix}, \\ &A_F := A - BF. \end{aligned} \quad (11)$$

If $\boldsymbol{\xi}$ is given by

$$\boldsymbol{\xi} = \boldsymbol{J}_{w-1}\boldsymbol{T}_{w-1}^{\dagger}, \quad (12)$$

then the following properties hold:

$$\textbf{P1} \ \ \xi(z)\xi^{\sim}(z) = \boldsymbol{\xi}\boldsymbol{\xi}^T, \quad (13)$$
$$\textbf{P2} \ \ \mathcal{O}_{w-1}(C, A_F)B = \boldsymbol{\xi}^{\dagger}, \quad (14)$$
$$\textbf{P3} \ \ CA_F^w = 0. \quad (15)$$

(Proof of **P1**). From eqn.(5),

$$\begin{bmatrix} \xi_w & 0 & \cdots & 0 \\ \xi_{w-1} & \xi_w & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_2 & \xi_3 & \cdots & \xi_w \end{bmatrix} \boldsymbol{T}_{w-2} = 0. \tag{16}$$

Using eqn.(7),

$$\begin{aligned} & \begin{bmatrix} \xi_w & 0 & \cdots & 0 \\ \xi_{w-1} & \xi_w & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1 & \xi_2 & \cdots & \xi_w \end{bmatrix} \boldsymbol{\xi}^T \\ &= \begin{bmatrix} \xi_w & 0 & \cdots & 0 \\ \xi_{w-1} & \xi_w & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1 & \xi_2 & \cdots & \xi_w \end{bmatrix} \begin{bmatrix} CB \\ CAB \\ \vdots \\ CA^{w-1}B \end{bmatrix} \boldsymbol{\xi}\boldsymbol{\xi}^T \\ &\quad + \begin{bmatrix} \xi_w & 0 & \cdots & 0 \\ \xi_{w-1} & \xi_w & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1 & \xi_2 & \cdots & \xi_w \end{bmatrix} \begin{bmatrix} 0_{m\times mw} \\ \boldsymbol{T}_{w-2} \end{bmatrix} \boldsymbol{P}\boldsymbol{\xi}^T \\ &= \begin{bmatrix} \xi_w & 0 & \cdots & 0 \\ \xi_{w-1} & \xi_w & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1 & \xi_2 & \cdots & \xi_w \end{bmatrix} \begin{bmatrix} CB \\ CAB \\ \vdots \\ CA^{w-1}B \end{bmatrix} \boldsymbol{\xi}\boldsymbol{\xi}^T \\ &= \begin{bmatrix} 0_{m(w-1)\times m} \\ I_m \end{bmatrix} \boldsymbol{\xi}\boldsymbol{\xi}^T, \end{aligned} \tag{17}$$

which implies that $\xi(z)$ has the all-pass property, i.e., **P1** holds.

(Proof of **P2**). Since F is the state feedback gain of the inverted interactorizing, the closed-loop transfer function matrix $G_F(z) = \xi^{-1}(z)$ is given by

$$G_F(z) = \sum_{i=1}^{\infty} z^{-i} CA_F^{i-1}B. \tag{18}$$

From the all-pass property of $\xi(z)$, there exists an integer k such that

$$CA_F^j B = 0, \quad \forall j \geq k. \tag{19}$$

Since

$$G_F^{\sim}(z)G_F(z) = (\xi^{\sim}(z))^{-1}\xi^{-1}(z) \tag{20}$$

holds,

$$\begin{aligned} & B^T\mathcal{O}_{k-1}^T(C, A_F) \begin{bmatrix} CB & 0 & \cdots & 0 \\ CA_FB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA_F^{k-1}B & CA_F^{k-2}B & \cdots & CB \end{bmatrix} \\ &= [(\boldsymbol{\xi}\boldsymbol{\xi}^T)^{-1} \ 0_{m\times m(k-1)}], \\ & B^T\mathcal{O}_{k-1}^T(C, A_F) \begin{bmatrix} CA_F^iB \\ \vdots \\ CA_F^{k+i}B \end{bmatrix} = 0_{p\times p} \quad (i \geq 1). \end{aligned} \tag{21}$$

On the other hand, since $G_F(z)\xi(z) = I_m$, we have

$$\begin{aligned} & \begin{bmatrix} CB & 0 & \cdots & 0 \\ CA_FB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA_F^{w-1}B & CA_F^{w-2}B & \cdots & CB \\ \vdots & \vdots & & \vdots \\ CA_F^{k-1}B & CA_F^{k-2}B & \cdots & CA_F^{k-w}B \end{bmatrix} \begin{bmatrix} \xi_w \\ \xi_{w-1} \\ \vdots \\ \xi_1 \end{bmatrix} \\ &= \begin{bmatrix} 0_{m(w-1)\times m} \\ I_m \\ 0_{m(k-w)\times m} \end{bmatrix} \end{aligned} \tag{22}$$

for $k \geq w$. Shifting the above relation, the following equation holds.

$$\begin{aligned} & \begin{bmatrix} CB & 0 & \cdots & 0 \\ CA_FB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA_F^{w-2}B & CA_F^{w-3}B & \cdots & CB \\ \vdots & \vdots & & \vdots \\ CA_F^{k-1}B & CA_F^{k-2}B & \cdots & CA_F^{k-w}B \end{bmatrix} \begin{bmatrix} \xi_1 & \cdots & \xi_w \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \xi_1 \end{bmatrix} \\ &= \begin{bmatrix} I_{mw} \\ 0_{m(k-w)\times mw} \end{bmatrix} \\ &\quad - \begin{bmatrix} CA_FB & \cdots & CA_F^{w-1}B \\ \vdots & & \vdots \\ CA_F^wB & \cdots & CA_F^{2w-1}B \\ \vdots & & \vdots \\ CA_F^kB & \cdots & CA_F^{k+w-1}B \end{bmatrix} \begin{bmatrix} \xi_2 & \cdots & \xi_w & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \xi_w & \cdots & 0 & 0 \end{bmatrix} \end{aligned} \tag{23}$$

Premultiplying the above equation by $B^T\mathcal{O}_{k-1}^T(C, A_F)$ and then using eqn.(21), it follows that

$$\begin{aligned} & B^T\mathcal{O}_{w-1}^T(C, A_F) \\ &= B^T\mathcal{O}_{w-1}^T(C, A_F) \begin{bmatrix} CB & 0 & \cdots & 0 \\ CA_FB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA_F^{w-2}B & CA_F^{w-3}B & \cdots & CB \\ \vdots & \vdots & & \vdots \\ CA_F^{k-1}B & CA_F^{k-2}B & \cdots & CA_F^{k-w}B \end{bmatrix} \\ &\quad \times \begin{bmatrix} \xi_1 & \cdots & \xi_w \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \xi_1 \end{bmatrix} \\ &= [(\boldsymbol{\xi}\boldsymbol{\xi}^T)^{-1} \ 0_{m\times m(k-1)}] \begin{bmatrix} \xi_1 & \cdots & \xi_w \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \xi_1 \end{bmatrix} \\ &= (\boldsymbol{\xi}\boldsymbol{\xi}^T)^{-1}\boldsymbol{\xi} = (\boldsymbol{\xi}^{\dagger})^T. \end{aligned} \tag{24}$$

Thus, **P2** holds.

(Proof of **P3**). From eqn.(14),

$$B^T \mathcal{O}_{w-1}^T(C, A_F)\mathcal{O}_{w-1}(C, A_F)B = (\boldsymbol{\xi}\boldsymbol{\xi}^T)^{-1} \tag{25}$$

holds. On the other hand, from eqn.(21),

$$B^T \mathcal{O}_{k-1}^T(C, A_F)\mathcal{O}_{k-1}(C, A_F)B = (\boldsymbol{\xi}\boldsymbol{\xi}^T)^{-1}. \tag{26}$$

If $k > w$, then $CA_F^k B = 0$ and it contradicts to the definition of k. Thus, $k \leq w$ in eqn.(19). Therefore,

$$CA_F^w [B \ A_F B \ \cdots \ A_F^i B \ \cdots] = 0 \tag{27}$$

can be obtained. From the reachability of $(A_F,\ B)$, **P3** holds. □

P1 implies that $\xi(z)$ has the all-pass property in the discrete-time. Therefore, all zeros of the interactor lie at the origin. Since F is the state feedback gain for the inverted interactorizing [9], **P2** and **P3** imply that the Markov parameters of the closed-loop system are given by the pseudoinverse of the coefficient matrix of interactor.

The following Corollary is obvious consequence if the general solution of eqn.(5) is considered.

Corollary Using the free parameter $\boldsymbol{\Lambda} \in \boldsymbol{R}^{m\times mw}$, define

$$\boldsymbol{L}_0 = \boldsymbol{\xi} + \boldsymbol{\Lambda}(I_{mw} - \boldsymbol{T}_{w-1}\boldsymbol{T}_{w-1}^{\dagger}). \tag{28}$$

Then, the identity interactor can be parametrized as follows:

$$L(z) = z\boldsymbol{L}_0 S_{I_m}^{w-1}(z) + \xi_0, \tag{29}$$

where $\xi_0 \in \boldsymbol{R}^{m\times m}$ is an arbitrary.

4 Non-Square Transfer Matrices

4.1 Tall Transfer Matrices

In this case, $m > p$. Without loss of generality, assume that K has the following form:

$$K = \begin{bmatrix} I_p \\ 0_{(m-p)\times p} \end{bmatrix} \tag{30}$$

According to the above portion, $\boldsymbol{L}$ is also divided as follow:

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{L}_1 \\ \boldsymbol{L}_2 \end{bmatrix} \begin{matrix} \} \, p\text{-rows} \\ \} \, (m-p)\text{-rows} \end{matrix}. \tag{31}$$

Thus, algebraic equation corresponding to eqn.(5) can be written by

$$\boldsymbol{L}_1\boldsymbol{T}_{w-1} = \begin{bmatrix} I_p & 0_{p\times p(w-1)} \end{bmatrix} \tag{32}$$

$$\boldsymbol{L}_2\boldsymbol{T}_{w-1} = 0_{(m-p)\times pw}. \tag{33}$$

In order to guarantee the unitarily of interactor, eqn.(32) should be solved by using pseudoinverse as in Theorem 1, and then the orthogonal unit bases in the left null space of

$$\begin{bmatrix} & L_w^T & L_{w-1}^T & \cdots & L_1^T & 0 & \cdots & 0 \\ & 0 & L_w^T & \cdots & L_2^T & L_1^T & \cdots & 0 \\ \boldsymbol{T}_{w-1} & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & \cdots & L_w^T & L_{w-1}^T & \cdots & L_1^T \end{bmatrix} \tag{34}$$

should be calculated.

Example 2 Consider the following transfer function matrix.

$$G(z) = \begin{bmatrix} \frac{1}{z+1} & \frac{1}{z+2} \\ \frac{1}{z+3} & \frac{1}{z+4} \\ \frac{1}{z+5} & \frac{1}{z+6} \end{bmatrix}$$

In this case, $(A,\ B,\ C)$ can be given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -15 & -23 & -9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -48 & -44 & -12 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

$$C = \begin{bmatrix} 15 & 8 & 1 & 24 & 10 & 1 \\ 5 & 6 & 1 & 12 & 8 & 1 \\ 3 & 4 & 1 & 8 & 6 & 1 \end{bmatrix}.$$

Then, using the pseudoinverse of

$$\boldsymbol{T}_2 = \begin{bmatrix} CB & 0 & 0 \\ CAB & CB & 0 \\ CA^2B & CAB & CB \end{bmatrix},$$

the solution $\boldsymbol{L}_1$ of eqn(**??**) is given by

$$\begin{aligned} \boldsymbol{L}_1 &= \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} \\ &= \begin{bmatrix} .1846 & .1846 & .1846 & .0359 & -.3333 & -.7026 \\ -.1385 & -.1385 & -.1385 & .0564 & .3333 & .6103 \end{bmatrix. \\ &\qquad \begin{matrix} .4962 & -.4962 & -.0038 \\ -.4346 & .3692 & .0654 \end{matrix}\Bigg]. \end{aligned}$$

Then, a soution $\boldsymbol{L}_2$ of eqn.(33) is given by calculating the orthogonal unit bases of the left null space of

$$\begin{bmatrix} & L_3^T & L_2^T & L_1^T & 0 & 0 \\ \boldsymbol{T}_2 & 0 & L_3^T & L_2^T & L_1^T & 0 \\ & 0 & 0 & L_3^T & L_2^T & L_1^T \end{bmatrix}$$

to guarantee the unitarily of interactor, which is given by as follow:

$$\begin{aligned} \boldsymbol{L}_2 = \big[\, & .2116 \ .2116 \ .2116 \ .4231 \ 0 \ -.4231 \\ & -.2909 \ .5818 \ -.2909 \,\big]. \end{aligned}$$

Thus, the interactor is given by

$$L(z) = z\begin{bmatrix} .4962z^2+.0359z+.1846 & -.4962z^2-.3333z+.1846 & -.0038z^2-.7026z+.1846 \\ -.4346z^2+.0564z-.1385 & .3692z^2+.3333z-.1385 & .0654z^2+.6103z-.1385 \\ -.2909z^2+.4231z+.2116 & .5818z^2+.2116 & -.2909z^2-.4231z+.2116 \end{bmatrix}.$$

4.2 Fat Transfer Matrices

In this case, a special form of K cannot be assumed. However, $\boldsymbol{T}_{w-1}^{\dagger}$ can be calculated. Therefore, $\boldsymbol{L}$ can be written by

$$\boldsymbol{L} = \boldsymbol{J}_{w-1}\boldsymbol{T}_{w-1}^{\dagger} = K(\boldsymbol{T}_{w-1}^{\dagger})(1:p) \tag{35}$$

where $(\boldsymbol{T}_{w-1}^{\dagger})(1:p)$ denote the matrix consisting of the first p-th rows of $\boldsymbol{T}_{w-1}^{\dagger}$. Then, substituting eqn.(35) to eqn.(5),

$$K\left\{(\boldsymbol{T}_{w-1}^{\dagger})(1:p)\boldsymbol{T}_{w-1} - \begin{bmatrix} I_p & 0_{p\times p(w-1)} \end{bmatrix}\right\} = 0. \tag{36}$$

By calculating the left null space of the above matrix, K and thus $\boldsymbol{L}$ can be determined.

Example 3 Consider the following transfer function matrix.

$$G(z) = \begin{bmatrix} \dfrac{1}{z+1} & \dfrac{1}{z+2} & \dfrac{1}{z+3} \\ \dfrac{1}{z+4} & \dfrac{1}{z+5} & \dfrac{1}{z+6} \end{bmatrix}$$

In this case, $(A,\ B,\ C)$ can be given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -4 & -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -10 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -18 & -9 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

$$C = \begin{bmatrix} 4 & 1 & 5 & 1 & 6 & 1 \\ 1 & 1 & 2 & 1 & 3 & 1 \end{bmatrix}.$$

Then,

$$(\boldsymbol{T}_2^{\dagger})(1:3)\boldsymbol{T}_2 - \begin{bmatrix} I_3 & 0_{3\times 6} \end{bmatrix}$$
$$= \begin{bmatrix} -.1471 & .2941 & -.1471 & .0588 & 0 & -.0588 & .0588 & .0588 & .0588 \\ .2941 & -.5882 & .2941 & -.1176 & 0 & .1176 & -.1176 & -.1176 & -.1176 \\ -.1471 & .2941 & -.1471 & .0588 & 0 & -.0588 & .0588 & .0588 & .0588 \end{bmatrix}$$

and K, $\boldsymbol{L}$ and $L(z)$ can be obtained as follows:

$$K = \begin{bmatrix} -.6802 & .0262 & .7326 \\ .6088 & .5768 & .5447 \end{bmatrix},$$

$$\boldsymbol{L} = \begin{bmatrix} -.3763 & -.3763 & -.2113 & .9177 & -.2355 & .2335 \\ .1060 & .1060 & .1430 & -.1751 & .0107 & -.0107 \end{bmatrix},$$

$$L(z) = z\begin{bmatrix} -.2355z^2-.2113z-.3763 & .2335z^2+.9177z-.3763 \\ .0107z^2+.1430z+.1060 & -.107z^2-.1751z+.1060 \end{bmatrix}$$

5 Pseudo Inverted Interactorizing by State Feedback

Throughout this section, assume that $G(z)$ has no invariant zeros on the unit circle, and $m \geq p$, which means $G(z)$ has more outputs channel than inputs. Let (A,B,C) denote a minimal realization of a given transfer matrix $G(z)$ and $L(z)$ denote a unitary interactor of $G(z)$. Then, a realization of $L(z)G(z)$ can be written by $(A,B,\hat{C}.\hat{D})$ defining

$$\hat{C} = L_1CA + L_2CA^2 + \cdots + L_wCA^w \tag{37}$$
$$\hat{D} = L_1CB + L_2CAB + \cdots + L_wCA^{w-1}B \tag{38}$$

where L_i is defined in eqn.(4). It is well known that the spectral factor $\tilde{G}_1(z)$ of $L(z)G(z)$ which satisfies

$$G^{\sim}(z)L^{\sim}(z)L(z)G(z) = \tilde{G}_1^{\sim}(z)\tilde{G}_1(z) \tag{39}$$

is given by

$$\tilde{G}_1(z) = \left[\begin{array}{c|c} A & B \\ \hline \tilde{C} & (I_p + (B^TXB)^{1/2}) \end{array}\right] \tag{40}$$

where

$$\tilde{C} = (I_p + B^TXB)^{-1/2}(B^TXA + \hat{D}^T\hat{C}) \tag{41}$$

and X is the positive definite solution of the following discrete-time Riccati equation [6]:

$$\begin{aligned} X &= A^TXA + \hat{C}^T\hat{C} \\ &\quad -(A^TXB + \hat{C}^T\hat{D})(I_p + B^TXB)^{-1}(B^TXA + \hat{D}\hat{C}). \end{aligned} \tag{42}$$

Note that a spectral factor $\tilde{G}_1(z)$ of $L(z)G(z)$ is also a spectral factor of $G(z)$ since $L(z)$ is a unitary interactor and

$$G^{\sim}(z)L^{\sim}(z)L(z)G(z) = G^{\sim}(z)G(z).$$

Therefore, the above spectral factor $\tilde{G}_1(z)$ can be considered as a minimum phase image of $G(z)$.

Since a unitary interactor of spectral factor $\tilde{G}_1(z)$ is I_m, the feedback gain F_t of the inverted interactorizing for $\tilde{G}_1(z)$ is given by [9]

$$F_t = I_m \cdot \tilde{C} = \tilde{C}. \tag{43}$$

Then, the control input $u(t)$ is given by

$$u(t) = F_t x(t). \tag{44}$$

Applying the above feedback gain F_t, the poles of closed-loop system divide the following three parts:

(a) the poles corresponding to the zeros of the interactor.

(b) the poles corresponding to the stable zeros of $\tilde{G}_1(z)$.

(c) the poles corresponding to the minimum phase image of anti-stable zeros of $\tilde{G}_1(z)$.

Therefore, by canceling the poles of (b), the closed-loop system is given by the inverted interactorizing part plus inner matrix (all-pass transfer matrix) part.

6 Conclusions

In this paper, we presented an easy and direct method to compute an interactor matrix. For this derivation, we only calculate the pseudoinverse and null space of some Toeplitz matrix. The interactor can be applicable for spectral factorization and inner-outer factorization for strictly proper transfer function matrices. We also presented the notion of pseudo inverted interactorizing. This is a natural extension of the inverted interactorizing for square transfer matrix.

A Appendix A

From the existence of $\boldsymbol{P}$,

$$(I_{m(w-1)} - \boldsymbol{T}_{w-2}\boldsymbol{T}_{w-2}^{\dagger})(\begin{bmatrix} \xi_2^T \\ \xi_3^T \\ \vdots \\ \xi_w^T \end{bmatrix} - \boldsymbol{M}\boldsymbol{\xi}\boldsymbol{\xi}^T) = 0 \tag{45}$$

holds. Using the singular value decomposition, $\boldsymbol{T}_{w-2}$ can be written as

$$\boldsymbol{T}_{w-2} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T. \tag{46}$$

Since eq.(5) implies

$$[\xi_2 \ \xi_3 \ \cdots \ \xi_w]\boldsymbol{T}_{w-2} = 0, \tag{47}$$

postmultiplying eq.(47) by $V \begin{bmatrix} \Sigma^{-2} & 0 \\ 0 & 0 \end{bmatrix} V^T$ gives

$$[\xi_2 \ \xi_3 \ \cdots \ \xi_w](\boldsymbol{T}_{w-2}^{\dagger})^T = 0. \tag{48}$$

Using a free parameter matrix $\boldsymbol{Z} \in \boldsymbol{R}^{m(w-1)\times mw}$, the general solution of eq.(7) is given by

$$\boldsymbol{P} = (\boldsymbol{Z} - \boldsymbol{T}_{w-2}^{\dagger}\boldsymbol{T}_{w-2}\boldsymbol{Z}\boldsymbol{\xi}^T(\boldsymbol{\xi}^{\dagger})^T) - \boldsymbol{T}_{w-2}^{\dagger}\boldsymbol{M}\boldsymbol{\xi}. \tag{49}$$

Then define a free parameter matrix $\boldsymbol{Z}$ by

$$\boldsymbol{Z} = \left[0_{m(w-1)\times m} \ \boldsymbol{T}_{w-2}^{\dagger} \right]. \tag{50}$$

It is easy to verify that the above $\boldsymbol{Z}$ (in $\boldsymbol{T}_{w-1}^{\dagger}$) satisfies the rest of conditions for the pseudoinverse, i.e.,

$$\begin{aligned} (\boldsymbol{T}_{w-1}^{\dagger}\boldsymbol{T}_{w-1})^T &= \boldsymbol{T}_{w-1}^{\dagger}\boldsymbol{T}_{w-1}, \\ \boldsymbol{T}_{w-1}\boldsymbol{T}_{w-1}^{\dagger}\boldsymbol{T}_{w-1} &= \boldsymbol{T}_{w-1}, \\ \boldsymbol{T}_{w-1}^{\dagger}\boldsymbol{T}_{w-1}\boldsymbol{T}_{w-1}^{\dagger} &= \boldsymbol{T}_{w-1}^{\dagger}. \end{aligned} \tag{51}$$

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