# Equations in Involution with a Scalar Second-Order Hyperbolic Equation in the Plane

MARTIN JURÁŠ
Department of Mathematics
North Dakota State University
Fargo, ND 58105–5075
United States
Martin.Juras@ndsu.nodak.edu http://www.math.ndsu.nodak.edu

*Abstract:* We find necessary and sufficient conditions for a scalar second-order hyperbolic equation in the plane

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

to admit an equation in involution of order $k$.

*Key-Words:* Hyperbolic PDE, equations in involution, generalized Laplace invariants, characteristic systems.

## 1 Introduction

Loosely speaking, a system of differential equations is in involution if there are no integrability conditions. This is trivially satisfied if one of the equations is a differential consequence of the other. A nontrivial example of equations in involution is a system

$$u_{xy} = \frac{2u}{(x+y)^2} \qquad \text{and} \qquad u_{xx} = -\frac{2u_x}{x+y} + \phi(x),$$

where $\phi$ is an arbitrary function. This can be easily verified by taking the total derivative of the first equation with respect to $x$ and the total derivative of the second equation with respect to $y$ to conclude that these are the same modulo the original equations. On the other hand, the equations $u_{xy} = u$ and $u_{xx} = 0$ are not in involution as the differentiation yields $u_y = 0$ and furthermore $u = 0$. If an equation admits sufficiently many equations in involution one may find the general solution by integrating a system of ordinary differential equations. Such equations are called *Darboux integrable*. A well known example of a Darboux integrable equation is Goursat's equation [3]

$$u_{xy} + \frac{2\sqrt{u_x u_y}}{x+y} = 0. \tag{1}$$

It admits two equations in involution

$$\frac{u_{xx}}{2\sqrt{u_x}} + \frac{\sqrt{u_x}}{x+y} = f(x) \quad \text{and} \quad \frac{u_{yy}}{2\sqrt{u_y}} + \frac{\sqrt{u_y}}{x+y} = g(y),$$

where $f(x)$ and $g(y)$ are arbitrary functions. The three equations give rise to the Frobenius system

$$\frac{\partial p}{\partial x} = 2f(x)\sqrt{p} - \frac{2p}{x+y}, \qquad \frac{\partial p}{\partial y} = -\frac{2\sqrt{pq}}{x+y},$$

$$\frac{\partial q}{\partial x} = -\frac{2\sqrt{pq}}{x+y}, \qquad \frac{\partial q}{\partial y} = 2g(y)\sqrt{q} - \frac{2q}{x+y},$$

where $p = u_x$ and $q = u_y$. Integrating this system we obtain the general solution of (1)

$$u = -\frac{(F(x)+G(y))^2}{x+y} + \int F'(x)^2 dx + \int G'(y)^2 dy,$$

where $F''(x) = f(x)$ and $G''(y) = g(y)$.

The objective of this paper is to find necessary and sufficient conditions for a second-order hyperbolic equation in the plane

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \tag{2}$$

to admit at least one equation in involution. We approach the problem by means of a series of differential forms intrinsically adapted to hyperbolic equation (2) called the Laplace adapted coframe [1].

## 2 The Laplace adapted coframe

Consider a non-degenerate, scalar equation in the plane (2), i.e. $(\partial F/\partial u_{xx}, \partial F/\partial u_{xy}, \partial F/\partial u_{yy}) \neq 0$. We assume that this equation is hyperbolic

$$\frac{1}{4}(\frac{\partial F}{\partial u_{xy}})^2 - \frac{\partial F}{\partial u_{xx}}\frac{\partial F}{\partial u_{yy}} > 0. \tag{3}$$

Moreover, we assume that all geometric objects (i.e. functions, forms, vector fields, etc.) in this paper are smooth.

Anderson and Kamran [1] extended the classical Laplace transformation for hyperbolic linear equations

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0,$$

and applied it to a formal linearization of a nonlinear hyperbolic equation (2) to derive a coframe on the infinitely prolonged equation manifold called the *Laplace adapted coframe*. We will briefly recall the construction of this coframe and its structure equations derived in [1] and [5].

Let $\pi\colon E \to M$ denote the trivial bundle with local coordinates $\pi\colon (x,y,u) \to (x,y)$ and let $\pi^\infty_M\colon\ J^\infty(E) \to M$ be the infinite jet bundle of local sections of $\pi$. The second-order equation (2) together with all its differential consequences, defines the *infinitely prolonged equation manifold* $\mathcal{R}^\infty \hookrightarrow J^\infty(E)$. There are two non-proportional real roots $(\mu, \lambda) = (m_x, m_y)$ and $(\lambda, \mu) = (n_x, n_y)$ of the *characteristic equation*

$$\frac{\partial F}{\partial u_{xx}}\lambda^2 - \frac{\partial F}{\partial u_{xy}}\lambda\mu + \frac{\partial F}{\partial u_{yy}}\mu^2 = 0$$

of (2) because the discriminant is positive (see (3). We define *characteristic total vector fields* on $\mathcal{R}^\infty$ (more precisely *characteristic directions*) by

$$X = m_x D_x + m_y D_y \quad \text{and} \quad Y = n_x D_x + n_y D_y,$$

where $D_x$ and $D_y$ denote the total derivative operators with respect to $x$ and $y$ restricted to $\mathcal{R}^\infty$. We have now developed enough terminology to give a precise definition of an equation in involution.

**Definition 2.1.** We say that the $k^{\text{th}}$-order equation

$$G(x, y, u, u_x, u_y, \dots u_{x^k}, u_{x^{k-1}y}, \dots, u_{xy^{k-1}}, u_{y^k}) = 0$$

is *in involution with* a second-order hyperbolic equation (2) if, restricted to the infinitely prolonged equation manifold $\mathcal{R}^\infty$, $X(G) = 0$ or $Y(G) = 0$.

We have

$$[\,X, Y\,] = PX + QY, \tag{4}$$

for some functions $P$ and $Q$ on $\mathcal{R}^\infty$. We denote the horizontal one-forms dual to $X$ and $Y$ by $\sigma$ and $\tau$ (*horizontal forms* are forms spanned by $dx$ and $dy$), i.e.

$$\sigma(X) = 1, \quad \sigma(Y) = 0, \quad \tau(X) = 0, \quad \tau(Y) = 1.$$

Restrictions of the *contact one-forms*

$$\theta = du - u_x\,dx - u_y\,dy, \quad \theta_x = du_x - u_{xx}\,dx - u_{xy}\,dy,$$
$$\theta_y = du_y - u_{xy}\,dx - u_{yy}\,dy,\ \dots,$$

to $\mathcal{R}^\infty$ generate the *contact ideal* $\mathcal{C}^\infty$, and are related by

$$\frac{\partial F}{\partial u_{xx}}\,\theta_{xx} + \frac{\partial F}{\partial u_{xy}}\,\theta_{xy} + \frac{\partial F}{\partial u_{yy}}\,\theta_{yy}$$
$$+ \frac{\partial F}{\partial u_x}\,\theta_x + \frac{\partial F}{\partial u_y}\,\theta_y + \frac{\partial F}{\partial u}\,\theta = 0.$$

We rewrite this equation, using the characteristic vector fields $X$ and $Y$, as

$$XY(\Theta) + A_0\,X(\Theta) + B_0\,Y(\Theta) + C_0\,\Theta = 0, \tag{5}$$

or as

$$YX(\Theta) + D_0\,X(\Theta) + E_0\,Y(\Theta) + G_0\,\Theta = 0, \tag{6}$$

where $V(\omega)$ denotes the Lie derivative of $\omega$ with respect to the vector field $V$, $\Theta = \rho\theta$ for some first-order function $\rho = \rho(x, y, u, u_x, u_y)$ on $\mathcal{R}^\infty$. Explicit formulas for the coefficients $A_0$, $B_0$, … , $G_0$ are found in [1]. Note that in [1] $A_0$ is denoted by $A$, $B_0$ is denoted by $B$, etc. For convenience we will denote $\Theta$ also by $\eta_0$ and $\xi_0$.

The first elements of the Laplace adapted coframe are $\sigma$, $\tau$, and $\Theta = \eta_0 = \xi_0$. The next two elements are defined by

$$\eta_1 = Y(\eta_0) + A_0\,\eta_0 \qquad \text{and} \qquad \xi_1 = X(\xi_0) + E_0\,\xi_0.$$

We have

$$X(\eta_1) = -B_0\,\eta_1 + H_0\,\eta_0, \quad Y(\xi_1) = -D_0\,\xi_1 + K_0\,\xi_0.$$

It is interesting to note that in terms of the coefficients of the forms (5) and (6) we have

$$H_0 = X(A_0) + A_0B_0 - C_0, \ \ K_0 = Y(E_0) + E_0D_0 - G_0.$$

We now proceed by induction. Assume that $\eta_1, \dots, \eta_i$ has been constructed. Then, provided $H_0 \neq 0$, $H_1 \neq 0$, $\dots$, $H_{i-1} \neq 0$, We set

$$A_i = A_{i-1} - Y(\ln H_{i-1}) - P, \quad \eta_{i+1} = Y(\eta_i) + A_i\,\eta_i.$$

The equation

$$X(\eta_{i+1}) = -B_i\,\eta_{i+1} + H_i\,\eta_i$$

defines $H_i$. This process continues until $H_p = 0$ for some $p \geq 0$, ($A_{i+1}$ is not defined), in which case we define

$$\eta_{p+i+1} = Y(\eta_{p+i}) \qquad \text{for all } i \geq 1.$$

Similarly, assuming that $\xi_1, \dots, \xi_i$ has been defined and provided $K_0 \neq 0$, $K_1 \neq 0, \dots, K_{i-1} \neq 0$, we set

$$E_i = E_{i-1} - X(\ln K_{i-1}) + Q, \quad \xi_{i+1} = X(\xi_i) + E_i\xi_i.$$

We define $K_i$ by the equation

$$Y(\xi_{i+1}) = -D_i\,\xi_{i+1} + K_i\,\xi_i.$$

This process continues until $K_q = 0$ for some $q \geq 0$, and we set

$$\xi_{q+i+1} = X(\xi_{q+i}) \qquad \text{for all } i \geq 1.$$

The forms $\sigma, \tau, \Theta, \eta_1, \xi_1, \eta_2, \xi_2, \dots$, define a coframe on $\mathcal{R}^\infty$ called the *Laplace adapted coframe*. The functions $H_0, H_1, \dots$, and $K_0, K_1, \dots$, are relative contact invariants called the *generalized Laplace invariants* [1]. These invariants play a key role in Darboux integrability. In fact a hyperbolic equation (2) is Darboux integrable if and only if $H_p = 0$ and $K_q = 0$ for some $p, q \geq 0$ [5].

It is useful to split the exterior derivative $d$ on $\mathcal{R}^\infty$ into two components

$$d = d_H + d_V,$$

where

$$d_H\omega = \sigma \wedge X(\omega) + \tau \wedge Y(\omega). \tag{7}$$

$d_H$ is called the *horizontal differential* and $d_V$ is referred to as *vertical differential*. Notice that this splitting does not occur on any *finitely* prolonged equation manifold. These two differentials define a double complex called the variational bicomplex associated with $\mathcal{R}^\infty$.

From (4) we have

$$d_H\sigma = -P\,\sigma \wedge \tau, \qquad d_H\tau = -Q\,\sigma \wedge \tau,$$

and the rest of the $d_H$ structure equations follow readily from the defining equations of the Laplace adapted coframe and (7). In [5] we derived the $d_V$ structure equations

$$\begin{aligned} &d_V\Theta \equiv 0 && \mathrm{mod}\{\,\Theta\,\}, \\ &d_V\eta_i \equiv 0 && \mathrm{mod}\{\,\xi_1, \Theta, \eta_1, \dots, \eta_i\,\} && i \geq 1, \\ &d_V\xi_i \equiv 0 && \mathrm{mod}\{\,\eta_1, \Theta, \xi_1, \dots, \xi_i\,\} && i \geq 1. \end{aligned}$$

Moreover, we proved that the horizontal forms $\sigma$ and $\tau$ satisfy

$$d_V\sigma = \sigma\wedge\mu_1 + \tau\wedge\alpha \quad \text{and} \quad d_V\tau = \sigma\wedge\beta + \tau\wedge\mu_2, \tag{8}$$

where $\alpha$, $\beta$, $\mu_1$, $\mu_2$ are contact one forms; $\alpha$ and $\beta$ are in the span of $\Theta$, $\xi_1$,$\eta_1$, $\xi_2$, and $\eta_2$ and the following relations hold:

$$\begin{aligned} d_VP &= X(\alpha) - Y(\mu_1) + P\mu_2 - Q\alpha, \\ d_VQ &= X(\mu_2) - Y(\beta) + Q\mu_1 - P\beta, \end{aligned}$$

$$d_V\beta = \beta \wedge (\mu_2 - \mu_1), \quad d_V\mu_2 = \alpha \wedge \beta = -d_V\mu_1,$$
$$d_V\alpha = \alpha \wedge (\mu_1 - \mu_2).$$

We shall also need commutation rules for $X$, $Y$ and $d_V$. By $\Omega^n(\alpha^1, \dots, \alpha^s)$ denote the module generated by the set of all $n$-fold exterior products $\alpha^{i_1} \wedge \ldots \wedge \alpha^{i_n}$, $1 \leq i_1, \leq \ldots \leq i_n \leq s$, over the ring of smooth functions on $\mathcal{R}^\infty$. Moreover, we denote $\Omega^*(\alpha^1, \dots, \alpha^s) = \bigcup_{n=0}^\infty \Omega^n(\alpha^1, \dots, \alpha^s)$. For $\omega \in \Omega^*(\Theta, \eta_1, \xi_1, \dots)$ we have

$$\begin{aligned} d_V[X(\omega)] - X(d_V\omega) &= \mu_1 \wedge X(\omega) + \beta \wedge Y(\omega), \\ d_V[Y(\omega)] - Y(d_V\omega) &= \alpha \wedge X(\omega) + \mu_2 \wedge Y(\omega). \end{aligned}$$

To hyperbolic equation (2) we associate two *characteristic Pfaffian systems* of order $k$, $\mathcal{C}_k(X)$ and $\mathcal{C}_k(Y)$ defined by

$$\begin{aligned} &\mathcal{C}_k(X) = \Omega^1(\tau, \Theta, \xi_1, \eta_1 \dots, \xi_k, \eta_k) \quad \text{and} \\ &\mathcal{C}_k(Y) = \Omega^1(\sigma, \Theta, \xi_1, \eta_1, \dots, \xi_k, \eta_k). \end{aligned}$$

Since

$$dG = d_HG + d_VG = X(G)\sigma + Y(G)\tau \mod \mathcal{C}^\infty,$$

then $X(G) = 0$ if and only if $dG \in \mathcal{C}_k(X)$ and $Y(G) = 0$ if and only if $dG \in \mathcal{C}_k(Y)$, provided $G$ is of order $k$. Thus we have proved

**Theorem 2.2.** *A $k^{\text{th}}$-order equation $G = 0$ is in involution with a second-order hyperbolic equation (2) if $dG$ restricted to the infinitely prolonged equation manifold $\mathcal{R}^\infty$ belongs to one of the two characteristic Pfaffian systems $\mathcal{C}^k(X)$ or $\mathcal{C}^k(Y)$.*

# 3 Integrable subsystems of characteristic systems

The question of finding equations in involution transforms into a question of finding integrable subsystems of characteristic systems. Let $\mathcal{C}_k^{(\infty)}(X)$ and $\mathcal{C}_k^{(\infty)}(Y)$ denote the *maximal integrable subsystems* of $\mathcal{C}_k(X)$ and $\mathcal{C}_k(Y)$, respectively. In [5], Theorem 5.1, page 367 we completely characterized $\mathcal{C}_k^{(\infty)}(X)$, $k \geq 1$ for hyperbolic equations with $H_p = 0$, for some $p \geq 0$.

**Theorem 3.1.** ([5]) *Let $H_p = 0$, $\widehat{\tau} = \tau - \Upsilon$ and let $\Upsilon \in \Omega^1(\Theta, \eta_1, \xi_1, \ldots, \xi_l, \eta_l)$ for some integer $l$ ($l \leq p$). Let $k \geq 1$.*

*If $p \geq 2$, then $\mathcal{C}_k^{(\infty)}(X) = \Omega^1(\widehat{\tau}, \eta_{p+1}, \ldots, \eta_k)$ for $k \geq p$, $\mathcal{C}_k^{(\infty)}(X) = \Omega^1(\widehat{\tau})$ if $l \leq k \leq p$, and $\mathcal{C}_k^{\infty}(X) = \{0\}$ if $k < l$.*

*If $p = 1$, then $\mathcal{C}_k^{(\infty)}(X) = \Omega^1(\widehat{\tau}, \eta_2, \ldots, \eta_k)$ for $k \geq 2$, $\mathcal{C}_1^{(\infty)}(X) = \Omega^1(\widehat{\tau})$ if $M_\tau = 0$, and $\mathcal{C}_1^{(\infty)}(X) = \{0\}$ if $M_\tau \neq 0$.*

*If $p = 0$, then $\mathcal{C}_k^{(\infty)}(X) = \Omega^1(\widehat{\tau}, \eta_1, \ldots, \eta_k)$ for $k \geq 2$, $\mathcal{C}_1^{(\infty)}(X) = \Omega^1(\widehat{\tau}, \eta_1)$ if $M_\tau = 0$, and $\mathcal{C}_1^{(\infty)}(X) = \{0\}$ if $M_\tau \neq 0$.*

In the above the one-form $\Upsilon = \Upsilon_p$ and the quantity $M_\tau$ will be defined later in the text. So, in fact, we only need to be concern with the case when $H_i \neq 0$ for all $i \geq 0$. But this requires the same arguments as assuming that

$$H_0 \neq 0,\ H_1 \neq 0,\ \ldots,\ H_{k-1} \neq 0.$$

From now on we assume that the above conditions are satisfied. This way we will also prove part of Theorem 3.1 using different arguments then in [5].

The $d_H$ structure equations immediately imply that $\dim \mathcal{C}_k^{(\infty)}(X)$, is 0 or 1. If $\dim \mathcal{C}_k^{(\infty)}(X) = 1$, then again using the $d_H$ structure equations we deduce

$$\mathcal{C}_k^{\infty}(X) = \Omega^1(\tau - \Sigma), \qquad d(\tau - \Sigma) \equiv 0 \bmod \{\, \tau - \Sigma \,\},$$

for some contact one-form $\Sigma$ that lies in the span of $\{\, \Theta, \eta_1 \xi_1, \ldots, \eta_{k-1}, \xi_{k-1}, \eta_k \,\}$. Splitting the differential into the horizontal and vertical components and using equation (8) we easily conclude that the necessary and sufficient conditions for

$$d(\tau - \Sigma) \equiv 0 \mod \{\, \tau - \Sigma \,\}$$

are

$$X(\Sigma) + Q\Sigma = \beta \qquad \text{and} \qquad d_V \Sigma = \Sigma \wedge (\mu_2 - Y(\Sigma)).$$

Let $V$ be a vector field on $\mathcal{R}^\infty$. A form $\omega$ is called a *relative $V$ invariant form* if $V(\omega) = \lambda\omega$ for some function $\lambda$. The following simple lemma plays a key role in our arguments.

**Lemma 3.2.** *Let $\Sigma$ be a contact form on $\mathcal{R}^\infty$ such that*

$$X(\Sigma) + Q\Sigma - \beta = 0. \tag{9}$$

*Then*

$$d_V \Sigma - \Sigma \wedge (\mu_2 - Y(\Sigma)) \tag{10}$$

*is a relative $X$ invariant contact form.*

*Proof:* Using the $d_V$ structure equations together with the commutation rules for $X$, $Y$, and $d_V$, we deduce that

$$\begin{aligned} 0 = d_V(X(\Sigma) + Q\Sigma - \beta) &= X(d_V \Sigma) + Q d_V \Sigma \\ &+ \beta \wedge [Y(\Sigma) - \mu_2] + [X(\mu_2) - Y(\beta) - P\beta] \wedge \Sigma \\ &+ \mu_1 \wedge [X(\Sigma) + Q\Sigma - \beta]. \end{aligned}$$

From Equation (9) we substitute for $\beta$ into the last equation to conclude

$$\begin{aligned} &X(d_V \Sigma - \Sigma \wedge (\mu_2 - Y(\Sigma))) \\ &+ Q\,[d_V \Sigma - \Sigma \wedge (\mu_2 - Y(\Sigma))] = 0. \end{aligned}$$

*q.e.d.*

At this point recall Proposition 4.5, page 286 [1] describing the relative $X$ invariant contact forms.

**Proposition 3.3.** *If $H_i \neq 0$ for all $i \geq 0$ then there are no nonzero relative $X$ invariant forms.*

*If $H_p = 0$ for some $p \geq 0$ and if $\omega \in \Omega^*(\Theta, \xi_1, \eta_1, \xi_2, \eta_2, \dots)$ is a relative $X$ invariant form, then*

$$\omega \in \Omega^*(\eta_{p+1}, \eta_{p+2}, \eta_{p+3}, \dots).$$

Let $\Sigma$ be a contact form of adapted order $\leq k$ satisfying (9). If $H_p = 0$ for some $p \geq k$, then

$$d_V\Sigma - \Sigma \wedge (\mu_2 - Y(\Sigma)) \in \Omega^2(\eta_{p+1}, \eta_{p+2}, \dots).$$

From the $d_V$ structure equations it easily follows that

$$d_V\Sigma - \Sigma \wedge (\mu_2 - Y(\Sigma)) \equiv 0 \mod \{\Theta, \xi_1, \eta_1, \dots, \xi_k, \eta_k\}.$$

Comparing the last two statements we deduce that

$$d_V\Sigma - \Sigma \wedge (\mu_2 - Y(\Sigma)) = 0.$$

If $H_i \neq 0$ for all $i \geq 0$ then again by Lemma 3.2 and Proposition 3.3 we conclude that the one-form (10) vanishes. Summarizing, we have proved

**Lemma 3.4.** *If $H_0 \neq 0$, $H_1 \neq 0$, $\dots$, $H_{k-1} \neq 0$, then $\dim \mathcal{C}_k^{(\infty)}(X) = 1$ if and only if*

$$X(\Sigma) + Q\Sigma - \beta = 0,$$

*for some one-form $\Sigma \in \Omega^1(\Theta, \xi_1\eta_1, \dots, \xi_k, \eta_k)$. Moreover, in this case $\mathcal{C}_k^{(\infty)}(X) = \Omega^1(\tau - \Sigma)$.*

We write $\beta$ defined by Equation (8) as

$$\beta = b_2\xi_2 + b_1\xi_1 + c_0\Theta + c_1\eta_1 + c_2\eta_2,$$

for some functions $b_1$, $b_2$, $c_0$, $c_1$, $c_2$. Note that $c_2 = M_\tau$ is a relative contact invariant that vanishes for a Monge–Ampère equation (see [4], sec. 8). The reader can find explicit formulas for $M_\tau$ in [4], sec. 9.

Let $H_0 \neq 0, H_1 \neq 0, \dots, H_{k-1} \neq 0$, $k \geq 1$. We write

$$\Sigma = \sum_{j=1}^{k-1} g_j\xi_j + f_0\Theta + \sum_{i=1}^{k} f_i\eta_i.$$

Using the defining equations for the Laplace-adapted coframe we compute

$$\begin{aligned} X(\Sigma) + Q\Sigma - \beta &= g_{k+1}\xi_{k+1} \\ &+ \sum_{j=1}^{k}(X(g_j) - (E_j - Q)g_j + g_{j-1} - b_j)\,\xi_j \\ &+ \sum_{i=0}^{k-1}(X(f_i) - B_if_i + H_if_{i+1} - c_i)\,\eta_i \\ &+ (X(f_k) - B_kf_k)\,\eta_k, \end{aligned}$$

where $g_0 = f_0$ and $b_i = c_i = 0$ for $i \geq 3$. Setting the above expression to zero and using Lemma 3.4 we obtain

**Theorem 3.5.** *Let $H_0 \neq 0, H_1 \neq 0, \dots, H_{k-1} \neq 0$, $k \geq 1$.*

*(i) The rank of $\mathcal{C}_1^{(\infty)}(X) \neq 0$ if and only if $I_1 = 0$ and $M_\tau = 0$. In this case $\mathcal{C}_1^{(\infty)}(X) = \Omega^1(\tau - \Upsilon_1)$.*

*(ii) If $k \geq 2$, then the rank of $\mathcal{C}_k^{(\infty)}(X) \neq 0$ if and only if $I_k = 0$. In this case $\mathcal{C}_k^{(\infty)}(X) = \Omega^1(\tau - \Upsilon_k)$. Here*

$$\Upsilon_k = b_2\xi_1 + \sum_{i=0}^{k} F_i\,\eta_i, \quad \text{and} \quad I_i = X(F_i) - B_iF_i - c_i$$

*where*

$$F_0 = -X(b_2) + (E_1 - Q)b_2 + b_1 \quad \text{and}$$
$$F_{i+1} = -\frac{1}{H_i}(X(F_i) - B_iF_i - c_i) \quad \text{for} \quad 0 \leq i < k,$$

*and as before we set $c_i = 0$ for $i \geq 3$.*

Theorems 3.1 and 3.5 combine to establish the following result.

**Corollary 3.6.** *The hyperbolic equation (2) admits at least one nontrivial equation in involution $G(x, y, u, u_x, u_y \dots, u_{x^k}, u_{x^{k-1}y}, \dots, u_{y^k}) = 0$ of order $k$, such that restricted to $\mathcal{R}^\infty$ $dG \in \mathcal{C}_k(X)$ if and only if*

*(i) $k = 1$: $M_\tau = 0$ and either $H_0 = 0$ or $I_1 = 0$;*

*(ii) $k \geq 2$: either $H_i = 0$ for some $i = 1, 2, \dots, k-1$ or $I_k = 0$.*

Similar results hold for the second characteristic system $\mathcal{C}_k(Y)$.

We now consider an equation of the form

$$u_{xy} = f(x, y, u, u_x, u_y). \tag{11}$$

It is a classical result that the hyperbolic Equation (2) is contact equivalent to Equation (11) if and only if there are two non-zero first-order functions $g_1$ and $g_2$, such that $dg_1 \in \mathcal{C}_1(X)$ and $dg_2 \in \mathcal{C}_1(Y)$; the contact transformation being determined by $\bar{x} = g_1$ and $\bar{y} = g_2$. Combining Corollary 3.6 together with a similar result for $\mathcal{C}_k(Y)$ (case $k = 1$) immediately yields a characterization of equations of type (11).

**Theorem 3.7.** *A hyperbolic equation (2) is contact equivalent to Equation (11) if and only if $M_\sigma = M_\tau = 0$ (by the results of [4] the equation is Monge-Ampère) and the following two conditions are satisfied:*

*(i) Either $H_0 = 0$ or $I_1 = 0$.*

*(ii) Either $K_0 = 0$ or $J_1 = 0$.*

*Here $J_1$ is an analogue of $I_1$ and $M_\sigma$ is an analogue of $M_\tau$. To define these quantities recall that the one-form $\alpha$ defined by Equation (8) is in the span of $\Theta$, $\xi_1$, $\eta_1$, $\xi_2$, and $\eta_2$. We write*

$$\alpha = d_2\eta_2 + d_1\eta_1 + e_0\Theta + e_1\xi_1 + M_\sigma\xi_2,$$

*Furthermore, we set*

$$G_0 = -Y(d_2) + (B_1 + P)d_2 + d_1,$$
$$G_1 = -\frac{1}{K_0}(Y(G_0) - BG_0 - e_0), \quad \text{and}$$
$$J_1 = Y(G_1) - E_1G_1 - e_1.$$

# 4 Open problems

Due to the results of [5] the hyperbolic equation is *Darboux integrable (at level $(p, q)$)* if and only if there are nonnegative integers $p$ and $q$ such that $H_p = 0$ and $K_q = 0$. We say that the hyperbolic equation is *semi-Darboux integrable* if $H_p = 0$ for some $p \geq 0$ or $K_q = 0$ for some $q \geq 0$.

1. *Find an algorithm that will decide whether a hyperbolic equation (2) is Darboux integrable (at any level). More interestingly, find an algorithm that will decide whether a hyperbolic equation (2) is semi-Darboux integrable.* The first result in this direction is due to Lie [6]. Lie showed that there are only two Darboux integrable equations of type $u_{xy} = f(u)$; namely the wave equation $u_{xy} = 0$ and the Lioville equation $u_{xy} = e^u$. Moutard's theorem (see [2] chapter XV) characterizes hyperbolic equations (2) which solutions $u(x, y)$ can be written in an explicit form depending on two arbitrary functions $\phi(x)$, $\psi(y)$ and their derivatives. Goursat [3] classified all equations of type (11) Darboux integrable at levels (0,1) and (1,1). His findings were rederived by Vessiot [9] in a more systematic way. A new approach to this classical problem was developed by Vassiliou [8]. A recent breakthrough is a paper [7] by Sokolov and Zhiber in which the authors succeeded in classifying all nonlinear Darboux integrable equations of type (11).

2. *Find an algorithm that will determine if a given hyperbolic equation (2) will admit an equation in involution (of any order).*

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