Computer generation of Generalized Numerical Ranges of Linear Operators in Indefinite Inner Product Spaces

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Abstract: – One of the possible generalizations of the classical numerical range of a given linear operator in an indefinite inner product space is the so-called J, C-tracial range. We present an algorithm to generate this generalized numerical range and give simple examples for illustration.

Key Words: – Indefinite inner product spaces, generalized numerical ranges.

1 Introduction

A complex matrix of order n is called *pseudo-unitary matrix of signature* (k,l) if the corresponding linear transformation preserves the quadratic hermitian form

$$q(x) = |x_1|^2 + \cdots + |x_k|^2 \\ -|x_{k+1}|^2 - \cdots - |x_n|^2, \quad k+l=n.$$

The group of pseudo-unitary matrices of signature (k,l) is denoted by $U_{k,l}$.

Let $\{E_{11}, E_{12}, \cdots, E_{nn}\}$ denote the standard basis in M_n the algebra of $n \times n$ complex matrices, that is, E_{ij} is the matrix with entries 0 except the (i,j) entry which is 1. In the sequel the matrix

$$\sum_{j=1}^{k} E_{jj} - \sum_{j=k+1}^{n} E_{jj} \in U_{k,l}$$

will be denoted by J. It can be easily seen that U is a pseudo-unitary matrix in $U_{k,l}$ if and only if $U^*JU = J$.

The *J-numerical range* of $A \in M_n$ is defined and denoted by

$$V_J(A) = \left\{ \frac{x^*Ax}{x^*Jx} : \quad x \in \mathbb{C}^n, \ x^*Jx \neq 0 \right\}.$$

If $J \in U_{n,0}$, $V_J(A)$ reduces to the *classical numerical range* usually denoted by $W(A)$

$$W(A) = \left\{ \frac{x^*Ax}{x^*x} : \quad x \in \mathbb{C}^n, \ x^*x \neq 0 \right\} \\ = \{x^*Ax : \quad x \in \mathbb{C}^n, \ x^*x = 1\}.$$

This concept is very useful for studying matrices and operators, and has a lot of applications in different areas.

We consider the sets

$$V_J^+(A) = \{x^*Ax : \quad x \in \mathbb{C}^n, \ x^*Jx = 1\}$$

and

$$V_J^-(A) = \{x^*Ax : \quad x \in \mathbb{C}^n, \ x^*Jx = -1\}$$

which have been studied by other researchers ([1,2,3]).

Evidently we have $V_{-J}^+(A) = V_J^-(A)$ and

$$V_J(A) = V_J^+(A) \cup -V_{-J}^+(A).$$

Motivated by theory and applications, there are many generalizations of the classical numerical range, such as the *C-numerical range of A* defined for $A, C \in M_n$ by

$$W_C(A) = \{\mathrm{Tr}(CU^*AU) : \ U^*U = I\}. \qquad (1)$$

This concept motivates the definition of the *J, C-tracial range of A* for $A, C \in M_n$

$$V_{J,C}(A) = \{\mathrm{Tr}(CU^*AU) : \ U^*JU = J\}.$$

It is clear that for $U = [x_1, \cdots, x_n]$, we have $U^*JU = (x_j^*Jx_j)$. If $U \in U_{k,l}$ and $J = (l_{ij})$, then

$$x_i^*Jx_j = l_{ij}, \quad i,j = 1, \cdots, n. \qquad (2)$$

Let $C = \sum_{i=1}^n c_i E_{ii}$. It can be easily checked that

$$V_{J,C}(A) = \{\sum_{i=1}^{n} c_i x_i^* A x_i : \ x_i \in \mathbb{C}^n \text{ satisfying (2)}\}.$$

If $C = E_{jj}$, it follows that

$$V_{J,E_{jj}}(A) = \{x^*Ax : \ x \in \mathbb{C}^n, x^*Jx = \pm 1\} = V_J^{\pm}(A).$$

2 Problem Formulation

The characterization of $V_{J,C}(A)$ seems to be a difficult problem. Our aim is to present some of its basic properties as well as its characterization for matrices of small orders. We also present an algorithm for generating $V_J(A)$ and give some examples for illustration. Since the pseudo-unitary group $U_{k,l}$ in M_n is connected and the mapping from $U_{k,l}$ to $\mathbb{C}$ defined by $R \to \mathrm{Tr}(CR^*AR)$ is continuous, $V_{J,C}(A)$ is connected for any $A, C \in M_n$. The following properties of $V_{J,C}(A)$ can be easily verified.

3 Problem solution

3.1 Basic properties

Proposition 3.1.1

(i) For any $T \in U_{k,l}$ *and for any* $A \in M_n$, $V_{J,C}(T^*AT) = V_{J,C}(A)$.

(ii) For any $A, C \in M_n$, $V_{J,C^*}(A^*) = \overline{V_{J,C}(A)}$.

(iii) For any $A, C \in M_n$,

$$V_{J,C}(A) = V_{J,A}(C),$$

that is, the roles of A *and* C *in* $V_{J,C}(A)$ *are symmetric.*

(iv) For any $\alpha, \beta \in \mathbb{C}$ *and for any* $A, C \in M_n$, $V_{J,C}(\alpha J + \beta A) = \alpha \mathrm{Tr}(JC) + \beta V_{J,C}(A)$.

Proposition 3.1.2 *Let $A \in M_n$ and $\xi \in \mathbb{C}$. The following conditions are equivalent:*

(a) For all $C \in M_n$,

$$V_{J,C}(A) = \{\xi \mathrm{Tr}(JC)\};$$

(b) $A = \xi J$.

Proof $(b) \Rightarrow (a)$ If $A = \xi J$ for some $\xi \in \mathbb{C}$, then

$$\begin{aligned} V_{J,C}(A) &= \{\mathrm{Tr}(CU^*(\xi J)U) : U^*JU = J\} \\ &= \{\xi \mathrm{Tr}(CU^*JU) : U^*JU = J\} \\ &= \{\xi \mathrm{Tr}(JC)\}, \quad \forall C \in M_n. \end{aligned}$$

$(a) \Rightarrow (b)$ Firstly, let us consider the case $\xi = 0$, that is, $V_{J,C}(A) = \{0\}$ for all $C \in M_n$. Suppose that $A \neq 0$. We shall prove that there exists a matrix $C \in M_n$ such that $V_{J,C}(A) \neq \{0\}$, a contradiction.

Consider $C = E_{jj}$. Then $V_{J,C}(A) = V_J^+(A)$ or $V_{J,C}(A) = V_{-J}^+(A)$. Since $A \neq 0$, it can easily be seen that $V_J^+(A) \neq \{0\}$ and $V_{-J}^+(A) \neq \{0\}$.

Now, we consider the case $\xi \neq 0$. Assume that $V_{J,C}(A) = \{\xi \mathrm{Tr}(JC)\}$ for all $C \in M_n$. By Proposition 3.1.1 *iv*, we have

$$V_{J,C}(A - \xi J) = V_{J,C}(A) - \xi \mathrm{Tr}(JC) = \{0\}, \\ \forall C \in M_n.$$

By the first part of the proof we can conclude that $A - \xi J = 0$. □

We denote by H_n the real space of $n \times n$ Hermitian matrices.

Proposition 3.1.3 *For any $C \in H_n$, $V_{J,C}(A) \subset \mathbb{R}$ if and only if $A \in H_n$.*

Proof: $(\Leftarrow)$ Let A and C be Hermitian matrices. If $z \in V_{J,C}(A)$, then there exists a pseudo-unitary matrix U such that

$$\begin{aligned} z &= \mathrm{Tr}(CU^*AU) = \mathrm{Tr}(C^*U^*A^*U) \\ &= \mathrm{Tr}(CU^*AU)^* = \bar{z}, \end{aligned}$$

and so $V_{J,C}(A) \subset \mathbb{R}$.

$(\Rightarrow)$ Let $z \in V_{J,C}(A)$. Since we are assuming that $V_{J,C}(A) \subset R$, then

$$z = \mathrm{Tr}(CU^*AU) = \overline{\mathrm{Tr}(CU^*AU)} = \mathrm{Tr}(CU^*AU)^* \\ = \mathrm{Tr}(CU^*A^*U), \quad \forall U \in U_{k,l}.$$

Thus

$$\mathrm{Tr}(CU^*(A - A^*)U) = 0, \quad \forall C \in H_n.$$

Recalling that $\mathrm{Tr}(PQ)$ is an inner product in H_n, we conclude that $A - A^* = 0$. □

A matrix $A \in M_n$ is said *J-normal* if $AJA^* = A^*JA$. A matrix $A \in M_n$ is said *essentially J-Hermitian* if $\mu A + \nu J$ is Hermitian for some $0 \neq \mu, \nu \in \mathbb{C}$, that is, A is J-normal with collinear eigenvalues.

Proposition 3.1.4 *For any $C \in H_n$, $V_{J,C}(A)$ is a subset of a straight line if and only if A is essentially J- Hermitian.*

Proof: $(\Leftarrow)$ If A is essentially J-Hermitian, then $\mu A + \nu J$ is Hermitian for some $0 \neq \mu, \nu \in \mathbb{C}$. By Proposition 3.1.1 *iv* and Proposition 3.1.3 the implication follows.

$(\Rightarrow)$ By Proposition 3.1.1 *iv*, we may rotate and translate $V_{J,C}(A)$ so that it becomes a subset of the real line. That is, there exist $\mu, \nu \in \mathbb{C}$, such that $V_{J,C}(\mu A + \nu J) \subset \mathbb{R}$. By Proposition 3.1.3, $\mu A + \nu J$ is Hermitian and so A is essentially J-Hermitian. □

3.2 The 2×2 case

The elliptical range theorem is an important result in the theory of the classical numerical range and its generalizations. In the studies in this area, a useful technique is reducing the problems to the case of matrices of order 2. For instance, convexity results are proved using such reduction techniques. The elliptical range theorem asserts that the numerical range of a 2×2 matrix A with eigenvalues λ_1 and λ_2 is an elliptical disk with foci λ_1 and λ_2, and with the length of minor axis equal to $\sqrt{\mathrm{Tr}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2}$. In Theorem 3.2.2, we give a detailed description of the J, C-tracial range of order 2 matrices and $J \in U_{1,1}$. In particular, it is shown that except for the degenerate cases when these generalized numerical ranges are a subset of a line, a half plane or the whole complex plane, they are bounded by a branch of a hyperbola.

Lemma 3.2.1 will be used in the proof of Theorem 3.2.2.

Lemma 3.2.1 (Hyperbolic range theorem) *The $V_J(A)$ numerical range of a 2×2 matrix A such that the eigenvalues of JA are λ_1 and λ_2, and $A \neq \lambda J$, is bounded by a hyperbola with foci λ_1 and λ_2, and with the length of imaginary axis equal to*

$$\sqrt{-\mathrm{Tr}(A^*JAJ) + |\lambda_1|^2 + |\lambda_2|^2}.$$

In particular, for the degenerate cases of the hy-

perbola, $V_J(A)$ is a singleton, a line, a subset of a line or the whole complex plane.

Proof. Suppose $A \neq \lambda J$, for some $\lambda \in \mathbb{C}$. (If $A = \lambda J$, it is obvious that $V_J(A) = \{\lambda\}$.)

Firstly, consider $\mathrm{Tr}A = 0$. Taking into account Proposition 3.1.1 *iv*, by a translation we can take A in the form

$$A = \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}.$$

If $c = d = 0$, $V_J(A)$ reduces to the origin. If c or d is different from 0, it can be easily checked that $V_J(A)$ is the complex plane.

Now, consider the case $\mathrm{Tr}A \neq 0$. Using a multiplication by a unimodular complex number, every $v \in \mathbb{C}^2$ can be brought to the form

$$\begin{pmatrix} 1 \\ z \end{pmatrix}$$

for some $z \in \mathbb{C}$. On the other hand, performing a suitable transformation of the form $A' = \frac{2}{\mathrm{Tr}A}(A - \mathrm{Tr}(AJ)J)$, by Proposition 3.1.1 *iv* we may assume without loss of generality that the principal entries of A are equal to 1. Moreover, we may concentrate on the $V_J(M)$ numerical range of the 2×2 matrix M

$$M = \begin{pmatrix} 1 & de^{i\alpha} \\ fe^{i\alpha} & 1 \end{pmatrix},$$

by performing a multiplication on the left and on the right of A' by a convenient unimodular pseudo-unitary diagonal matrix. Thus, we consider the set

$$V_J(M) = \left\{ \frac{\begin{pmatrix} 1 & \bar{z} \end{pmatrix} M \begin{pmatrix} 1 \\ z \end{pmatrix}}{1 - z\bar{z}} : \quad z \in \mathbb{C},\ 1 - z\bar{z} \neq 0 \right\}. \tag{3}$$

Let $z = \rho e^{i\phi}$, where $1 - \rho^2 \neq 0$ and $D = \frac{1+\rho^2}{1-\rho^2}$. After some computations, it can be seen that the set of points in (3) is given by

$$\frac{((x-D)\sin\alpha - y\cos\alpha)^2}{(f-d)^2} + \frac{((x-D)\cos\alpha + y\sin\alpha)^2}{(d+f)^2} = \frac{D^2 - 1}{4}. \tag{4}$$

Using a standard procedure for evaluating the envelope of the family of curves given in (4), we obtain the general equation of a conic

$$Ax^2 + Cy^2 + 2Bxy + F = 0, \tag{5}$$

where

$$A = d^2 + f^2 - 2df\cos 2\alpha;$$
$$B = -2df\sin 2\alpha;$$
$$C = -4 + d^2 + f^2 + 2df\cos 2\alpha;$$
$$F = 1/4((d^2 - f^2)^2 + 4(d^2 + f^2)^2 - 8df\cos 2\alpha).$$

Reducing the conic in (3.3) to its principal axis, we get the equation

$$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = \frac{1}{4}, \tag{6}$$

where

$$a^2 = 2 - (d^2 + f^2) + 2\sqrt{1 + d^2f^2 - 2df\cos 2\alpha};$$
$$b^2 = -2 + (d^2 + f^2) + 2\sqrt{1 + d^2f^2 - 2df\cos 2\alpha} \tag{7}$$

and

$$c = \sqrt{a^2 + b^2}$$

is the focal length. We now discuss the equation (6). We can assume that one of the following cases holds:

Case 1: $d = f$ and $\alpha = 0$.

Under the hypothesis, we have $b = 0$ in (7). As can be easily seen this case corresponds to the case of the matrix M being Hermitian. After some routine computations, we conclude that $A = B = F = 0$ in (3) and so $V_J(M)$ is contained in the real line. If $a^2 > 0$, then

$$V_J(M) =]-\infty, -a/2] \cup [a/2, +\infty[.$$

If $a^2 \leq 0$, then $V_J(M)$ is the whole real line.

Case 2: $d \neq f$ or $\alpha \neq 0$.

In this case $b \neq 0$. If $a^2 > 0$, (6) represents a hyperbola. If $a^2 = 0$, the hyperbola in (6) degenerates in a line, and $V_J(M)$ is the whole complex plane. If $a^2 < 0$, (6) reduces to an imaginary ellipse and $V_J(M)$ is again the complex plane.

We observe that the eigenvalues of JM, say, λ_1', λ_2', are of the form $\pm\sqrt{1 - dfe^{2i\alpha}}$ and that

$$\mathrm{Tr}(MJM^*J) = 2 - d^2 - f^2.$$

Notice also that

$$b^2 = |\lambda_1'|^2 + |\lambda_2'|^2 - \mathrm{Tr}(MJM^*J),$$

and that the direction of the real axis of the hyperbola is given by the vector

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 + df\cos(2\alpha) + \sqrt{1 + d^2f^2 - 2df\cos(2\alpha)} \\ 2df\sin(2\alpha) \end{pmatrix}.$$

This completes the proof of the Lemma. □

Theorem 3.2.2 *Let $C = \mathrm{diag}(\gamma_1, \gamma_2) \in M_2$ and let (λ_1, λ_2) be the eigenvalues of JA, $A \in M_2$. The $V_{J,C}(A)$ numerical range of A is bounded by a branch of hyperbola with foci $\gamma_1\lambda_1 - \gamma_2\lambda_2$ and $\gamma_1\lambda_2 - \gamma_2\lambda_1$, and with the length of imaginary axis equal to*

$$\mathrm{Tr}\, C\sqrt{-\mathrm{Tr}(A^*JAJ) + |\lambda_1|^2 + |\lambda_2|^2}.$$

In particular, for the degenerate cases of the hyperbola, $V_{J,C}(A)$ is a singleton, a line, a subset of a line, a half-plane or the whole complex plane.

Proof. For any $A \in M_2$, it is not difficult to verify that

$$V_{J,C}(A) = (\gamma_1 + \gamma_2)V_{J,E_{11}}(A) - \gamma_2 \mathrm{Tr}(JA),$$

and that

$$V_{J,C}(A) = (\gamma_1 + \gamma_2)V_{J,E_{22}}(A) + \gamma_1 \mathrm{Tr}(JA).$$

Using Lemma 3.2.1 and performing some computations the Theorem follows. □

3.3 Algorithm and Examples

In this section, we describe an algorithm for generating the $V_J(A)$ range.

Before we describe the algorithm, we give the following definition:

Definition *Consider a set of points $\{P_1, \cdots, P_n\}$ and let $\zeta_i = \pm 1$ be the sign associated with the point P_i. The pseudo-convex hull of the points $P_1, \cdots, P_n$ is defined by*

$$\mathrm{Co}(P_1, \cdots, P_n) = \left\{ \frac{\sum_{i=1}^n \zeta_i q_i P_i}{\sum_{i=1}^n \zeta_i q_i} : q_i \geq 0, \sum_{i=1}^n \zeta_i q_i \neq 0 \right\}.$$

Step 1. Choose some matrix $A \in M_n$.

Step 2. Determine the matrix $A(\theta) = A\mathrm{e}^{i\theta}$, $\theta \in [0, 2\pi]$, and its Hermitian part $\mathcal{R}e(A(\theta)) = \frac{1}{2}(A(\theta) + A(\theta)^*)$.

Step 3. Determine the eigenvectors $v_k(\theta)$ of $J\mathcal{R}e(A(\theta))$,

$$J\mathcal{R}e(A(\theta))v_k(\theta) = \lambda_k v_k(\theta), \quad k = 1, \cdots, n.$$

Step 4. For each eigenvector $v_k(\theta)$, determine the interval $I_k \subseteq [0, 2\pi]$ such that $v_k(\theta)^* J v_k(\theta) \neq 0$ for $\theta \in I_k$.

Step 5. For each eigenvector $v_k(\theta)$, draw the curves with parametric equations

$$x = Re\frac{v_k(\theta)^* A v_k(\theta)}{v_k(\theta)^* J v_k(\theta)}, \quad y = Im\frac{v_k(\theta)^* A v_k(\theta)}{v_k(\theta)^* J v_k(\theta)},$$
$$\theta \in I_k, \quad k = 1, \cdots, n.$$

(For each curve, the sign of $v_k(\theta)^* J v_k(\theta)$ should be retained.) These are the generating curves of the boundary of $V_J(A)$, which is the pseudo-convex hull of the curves.

Example 1

Let

$$A = \begin{pmatrix} 1 & 0 & a \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

Thus

$$\mathcal{R}e(A(\theta)) = \begin{pmatrix} \cos\theta & 0 & \frac{1}{2}a\mathrm{e}^{i\theta} \\ 0 & b\cos\theta & 0 \\ \frac{1}{2}a\mathrm{e}^{-i\theta} & 0 & \cos\theta \end{pmatrix}.$$

The eigenvectors of $J\mathcal{R}e(A(\theta))$, $J = \mathrm{diag}(1, -1, 1)$, are

$$v_1(\theta) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_2(\theta) = \frac{1}{\sqrt{2}}\begin{pmatrix} -\mathrm{e}^{i\theta} \\ 0 \\ 1 \end{pmatrix},$$
$$v_2(\theta) = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{e}^{i\theta} \\ 0 \\ 1 \end{pmatrix}.$$

We have

$$v_1^*(\theta)Jv_1(\theta) = -1, \quad v_2^*(\theta)Jv_2(\theta) = v_3^*(\theta)Jv_3(\theta) = 1,$$

$$v_1^*(\theta)Av_1(\theta) = b, \quad v_2^*(\theta)Av_2(\theta) = 1 - \frac{1}{2}a\mathrm{e}^{-i\theta},$$
$$v_3^*(\theta)Av_3(\theta) = 1 + \frac{1}{2}a\mathrm{e}^{-i\theta}.$$

Hence, the singleton $\{-b\}$ and the circle $(x-1)^2 + y^2 = \frac{1}{4}a^2$ (more precisely, two superimposed circles) are the generating curves of the boundary of $V_J(A)$, which is the pseudo-convex hull of these curves. If $1 - \frac{1}{2}|a| \leq -b \leq 1 + \frac{1}{2}|a|$, $V_J(A)$ reduces to the whole complex plane.

Example 2

Let, again,

$$A = \begin{pmatrix} 1 & 0 & a \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b \in \mathbb{R},$$

$$\mathcal{R}e(A(\theta)) = \begin{pmatrix} \cos\theta & 0 & \frac{1}{2}a\mathrm{e}^{i\theta} \\ 0 & b\cos\theta & 0 \\ \frac{1}{2}a\mathrm{e}^{-i\theta} & 0 & \cos\theta \end{pmatrix}.$$

However, now take $J = \mathrm{diag}(-1, 1, 1)$. The eigenvectors of $J\mathcal{R}e(A(\theta))$, are

$$v_1(\theta) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_2(\theta) = \begin{pmatrix} -\frac{a}{2}\mathrm{e}^{i\theta} \\ 0 \\ f(\theta) \end{pmatrix},$$

$$v_3(\theta) = \begin{pmatrix} -\frac{a}{2}\mathrm{e}^{i\theta} \\ 0 \\ f(\theta) \end{pmatrix},$$

where $f(\theta) = \cos\theta + \sqrt{\cos^2\theta - \frac{a^2}{4}}$. We have

$$v_1^*(\theta)Jv_1(\theta) = 1,$$

$$v_2^*(\theta)Jv_2(\theta) = 2f(\theta)\sqrt{\cos^2\theta - \frac{a^2}{4}},$$

$$v_3^*(\theta)Jv_3(\theta) = -2f(\theta)\sqrt{\cos^2\theta - \frac{a^2}{4}},$$

$$v_1^*(\theta)Av_1(\theta) = b,$$

$$v_2^*(\theta)Av_2(\theta) = f(\theta)\left(2\cos\theta - \frac{a^2}{2}\mathrm{e}^{-i\theta}\right),$$

$$v_3^*(\theta)Av_3(\theta) = f(\theta)\left(2\cos\theta - \frac{a^2}{2}\mathrm{e}^{-i\theta}\right).$$

Therefore, the singleton $\{b\}$ and the two branches of the hyperbola

$$\frac{x^2}{1 - \frac{a^2}{4}} - \frac{y^2}{\frac{a^2}{4}} = 1$$

are the generating curves of the boundary of $V_J(A)$, which is the pseudo-convex hull of these curves. (The + sign is associated to one of the branches and the – sign to the other.) If $\frac{a^2}{4} \geq 1$, or if $-1+\frac{a^2}{4} \geq b$, $V_J(A)$ reduces to the whole complex plane.

4 Conclusions

The previous developments give light to the production of algorithms and computer programs for generating generalized numerical ranges in indefinite inner product spaces. The design of an efficient algorithm or computer program to generate $V_{J,C}(A)$ for general C and A seems the next step of this research program.

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