A Differential Game for Air-Land Combat Operations

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Abstract: The purpose of this paper is to interpret the potential strategies on Air-Land Combat Operations which is based on Lanchester's Combat Law as a differential game. The major concern is how to employ and distribute combined strategies when offensive and defensive positions are taken in air and army forces on both sides. These decisions making for whether to concentrate air fire to attack the opponent's air forces or to interdict its land troops, will correspond to the interactive movements taken by the ground troops which will eventually decide to advance, to withdrawal or even to delay in order to accomplish the mission in an Air-Land Combat Operation.

Keywords: Air-Land Combat Operations, Differential Game, Lanchester's Combat Law.

1 Introduction

During World War I, since the aircraft designed by Lanchester was adopted by the British military, Lanchester turned his attention to the war and thoroughly investigated data in all kinds of battles in history to study issues about how aircraft affect war and their influence in achieving victory. The result was displayed by a set of differential equations widely known as the Lanchester's Combat Law [1] which described the relationship of attrition, presented in the Engineering Journal in 1914. Therefore, the study of the air force collaborating with the army in military action was initially explored. However, at that time the air force was just launching development and it was not yet regarded as the main force, but instead it was employed as the means of reconnaissance as well as for fire adjustment. Until 1939, the boom in aeronautics corresponded with the scholars' advocation on influential air power, so the significant role of air forces was causing more concern than ever. Some documents like 'The Command of the Air' of Giulio Douhet [2], 'Winged Defense' [3] and 'Skyways: A book on Modern Aeronautics' [4] of William Mitchell were quite fundamental. The outbreak of World War II facilitated the combining of the procedures of the air forces and army forces. Air forces, thereafter, were not only working for rigid fire support but also carrying out cooperative missions like acquiring air domination, blocking the battle zone, covering, and so on.

In World War II, Isaacs [5] who started to apply the Lanchester's Combat Law in differential game theory, discussed the problem of weapons distribution under a fixed support ratio for accomplishing long term and short term missions. It led to a solution for those strategic problems that occurred in the process of allocating forces. Due to Isaacs, who laid the foundations of decision making from an Air-Land operational viewpoint, models combining differential game theory and Lanchester Combat Law have become worthwhile. Some representative models are: (i) Isbell and Marlow [6], who discussed the fire programming problems. (ii) A model dealing with the differential game problem of artillery and air force support, presented by Weiss [7], discusses the optimal strategy determined by the weapon range, cost and killing rate. (iii) A model handling the problem of allocating support fire that mainly consists of artillery in combat was developed by Kawara [8]. (iv) Berkovitz and Dresher [9], [10] consider a problem in tactical air war which is concerned with the allocation at each strike of the tactical forces among these competing air tasks such as counter-air, air-defense, and support of ground operations. (v) Bellman and Dreyfus [11] use a dynamic programming approach to treat a tactical air-warfare model. (vi) Different models applying differential games to study various scenarios of military conflicts and tactical allocation problems occurring in Lanchester-type equations of warfare were constructed by Taylor [12], [13] in his research reports published during 1974 to 1983.

In addition, a dynamic model of a missile war, developed by Chattopadhyay [14], studied the subject of optimal targeting and firing strategies in a missile war. Austin et al. [15] presented game theory for automated maneuvering during air-air combat; Stephane [16] demonstrated differential game and symbolic programming to calculate a guaranteed aircraft evasion in modern aerial duels; Grimm et al. [17] discussed open-loop guidance for pre-launch maneuvering in medium-range air combat; Smith et al. [18] yielded classifier systems in combat: two-sided learning of maneuvers for advanced fighter aircraft; Hsia et al. [19], [20] discussed issues about guarding a territory using gaming, the major concern of these studies being attack, defense and pursuit speed; Chen [21] developed an optimal control problem in determining the optimal reinforcement schedules for the Lanchester equations.

2 Air-Land Combat Operation Model

For most scholars mentioned above like Isbell, Weiss, Kawara, Berkovitz, Bellman and Taylor etc, their differential games for land-air battles were based on the issues concerning fire support distribution in air-to-air, air-to-land or ground battles. But attrition and advance rate occurring in the engagement process of ground battles were often ignored. This model, hence, contains these crucial factors and discusses the different strategies employed by the engaged forces to implement Air-Land depth operations and eventually to earn a total victory.

2.1 Scenarios and Assumptions

In order to describe the model in a feasible way, assumptions are confined to the following situations: First, Orange represents the side attacking and Blue defending. This battle is assumed to trigger an operation involving vast air forces and army forces. Second, the air forces and army forces on both sides are completely involved in the operation. There are not any kinds of reinforcement followed up. Third, the air forces on both sides decide to adopt a concentration strategy to attack the opponent's air or army forces in order to avoid dividing forces. Meanwhile, the army will focus on fighting the opponent's army. In reality, both the air and the army can be split to attack the air forces and the land troops simultaneously; on the other hand, they may conceal and choose not to commit into the operation. In accordance with this, there will be 256 ways to form the combined strategies. The model thus may fall into a complicated and chaotic status. Therefore, the strategy is strictly confined to concentration. Fourth, both army troops can advance with relative velocity to acquire territory as the combat power of air forces and army forces on both sides is proceeding at a certain degree of interaction. But air forces cannot advance for acquiring any territory.

Fifth, if Orange's ground troops are terminated in the first place, his air forces must attack Blue's army as the top priority. Such action will prevent and slow down the possibility of a high velocity advance performed by the opponent's army, and vice versa. Sixth, this model uses perfect information; that is, both sides are sufficiently aware of the combat rules, the objective function, and attrition and advance movement, but do not know which strategy the opponent is going to take. Seventh, we only discuss deterministic models, excluding stochastic models. Eighth, combat power is calculated by using the force strength method of Dupuy [22]. Ninth, combat power attrition is calculated according to Lanchester's Square Laws which will correspond to the concentration strategy mentioned above in assumption third.

2.2 Operation Parameters and Notations

The parameters and notations used are described as follows:

$x_1(t)$, $x_2(t)$: the combat power of Army forces on the sides of Orange and Blue at time t,

$x_3(t)$, $x_4(t)$: the combat power of Air forces on the sides of Orange and Blue at time t,

t_0 : time at which x_i is initialised, that is $x_i(t_0)= x_{i0}$, $i \in \{1,2,3,4\}$,

T: time to end the battle,

t_i : time at which $x_i(t)$ is the first annihilated, that is $x_i(t_i)=0$, $0<t \leqq t_i<T$, $i \in \{1,2,3,4\}$,

t'_j : time at which $x_j(t')$ is the second annihilated, that is $x_j(t'_j)=0,\ t_i < t' \le t'_j \le T, i \ne j \in \{1,2,3,4\}$

T_{ij} : from initial to t_i and to t_j, total time to end the battle is $T_{ij} = t_i + t_j$,

x_{kj} : when $x_j(t_j)=0$ the survival combat power of the rest of the engaged forces at time t_j, that is, $x_k(t_j)= x_{kj}>0$, $k \ne j$, $k, j \in \{1,2,3,4\}$,

E_{ij} : the part from $x_i(t_i)=0$ to $x_j(t_j)=0$ of the terminal state, $i, j \in \{1,2,3,4\}, i \ne j$,

k_{ij} : the attrition rate at which one side of i th forces can kill one side of the j th forces,

ψ, ϕ :the control variable of surviving combat power of Blue, Orange air forces attacking the opponent army forces,

u: the maximum relative advance velocity of Orange and Blue army forces,

v_1,v_2: the constant for army and air forces which describes how their combat power effectiveness affects the advance,

r: the fixed ratio combat power (Offense:Defense, generally, it is 3:1) which allows Orange to advance.

V_1, V_2 : represent the relative velocity factors of air forces and army forces individually, which will affect the advance of the ground troops on both sides.

2.3 Objective Function and Dynamic Equations

According to Dupuy's study [23], when a side possessed air superiority, it could not only reinforce the support for its air and army forces but also devitalize the opponent's combat power on the air, army and even supplies. It could also enable the troops with their relative maneuvering. When the opponent increased his forces, the side with air superiority could reduce his vulnerability. Therefore, in this model we assume that the advance rate is considerably related with the current survival combat power occurring among the engaged army and air forces on both sides. Under such a condition, air forces are expected to destroy and block the opponent's air and army forces, but the army forces will focus their attack on the opponent's ground troops to acquire as much territory as they can. The objective function and dynamic equations will thus be :

$$\begin{aligned}
&\min_{\psi}\max_{\phi} J(X,\phi,\psi,t) \\
&= \min_{\psi}\max_{\phi} \{V_1 \cdot [x_1(T) - r \cdot x_2(T)] + V_2 \cdot [x_3(T) - x_4(T)]\} \\
&\begin{cases}
\dot{x}_1(t) = f_1(X,\phi,\psi,t) = -k_{21} \cdot x_2(t) - \psi \cdot k_{41} \cdot x_4(t) & x_1(0) = x_{10} \\
\dot{x}_2(t) = f_2(X,\phi,\psi,t) = -k_{12} \cdot x_1(t) - \phi \cdot k_{32} \cdot x_3(t) & x_2(0) = x_{20} \\
\dot{x}_3(t) = f_3(X,\phi,\psi,t) = -(1-\psi) \cdot k_{43} \cdot x_4(t) & x_3(0) = x_{30} \\
\dot{x}_4(t) = f_4(X,\phi,\psi,t) = -(1-\phi) \cdot k_{34} \cdot x_3(t) & x_4(0) = x_{40}
\end{cases} \\
&and \quad \{x_1(t) \ge 0,\, x_2(t) \ge 0,\, x_3(t) \ge 0,\, x_4(t) \ge 0\}
\end{aligned} \qquad (1)$$

Thereby, the schematic diagram of the Air-Land Combat Operation Model is presented in Fig.1.

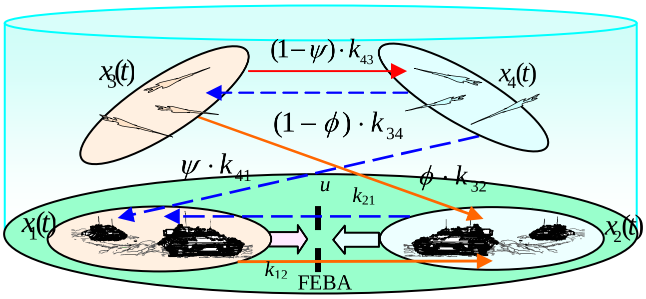

Fig.1: Air-Land Combat Operation Model

2.4 Definitions of Terminal States

Taking the crucial factors of combat power, space and time into consideration, the operation will be terminated as long as any two forces are annihilated; that is, their values become zero. In some way, it is necessary to simplify the potential encountering of too many complex combinations for constructing a feasible model; we, therefore, exclude other terminal states. The definitions of terminal states for this model are presented below:

(a) If Orange's air and army forces are both destroyed, Blue obtains the final victory, and vice versa. It can be as follows;

E_{13}: $x_1(t_1)=0$, $x_2(T)>0$, $x_3(t'_3)=0$, $x_4(T)>0$, $0<t_1<t'_3 \leqq T$, Blue wins.

E_{31}: $x_1(t'_1)=0$, $x_2(T)>0$, $x_3(t_3)=0$, $x_4(T)>0$, $0<t_3<t'_1 \leqq T$, Blue wins.

E_{24}: $x_1(T)>0$, $x_2(t_2)=0$, $x_3(T)>0$, $x_4(t'_4)=0$, $0<t_2<t'_4 \leqq T$, Orange wins.

E_{42}: $x_1(T)>0$, $x_2(t'_2)=0$, $x_3(T)>0$, $x_4(t_4)=0$, $0<t_4<t'_2 \leqq T$, Orange wins.

(b) If Orange's army forces are destroyed first and then Blue's army forces are ruined, under such circumstances, there are no ground troops left for acquiring territory but only air forces remaining for further operation. Therefore, the side possesses air superiority will win the battle at the end. Thus;

E_{12} : $x_1(t_1)=0$, $x_2(t'_2)=0$, $x_3(T)>0$, $x_4(T)>0$, $0<t_1<t'_2 \leqq T$,
if $x_4(T) > x_3(T)$, Blue wins;
if $x_3(T) > x_4(T)$, Orange wins.

E_{21} : $x_1(t'_1)=0$, $x_2(t_2)=0$, $x_3(T)>0$, $x_4(T)>0$, $0<t_2<t'_1 \leqq T$,
if $x_3(T) > x_4(T)$, Orange wins;
if $x_4(T) > x_3(T)$, Blue wins.

(c) Suppose Orange's air forces are destroyed first, after that the Blue's army forces are ruined. It is obvious that Blue has lost all his ground forces to get any territory; hence Orange owns the victory, and vice versa.

E_{32} : $x_1(T)>0$, $x_2(t'_2)=0$, $x_3(t_3)=0$, $x_4(T)>0$, $0<t_3<t'_2 \leqq T$, Orange wins.

E_{41} : $x_1(t'_1)=0$, $x_2(T)>0$, $x_3(T)>0$, $x_4(t_4)=0$, $0<t_4<t'_1 \leqq T$, Blue wins.

3 Strategy Analysis

3.1 Categories of Strategies

In Moffat's [24] paper "The system dynamics of future warfare", five forms of Air-Land operation models and the movement of the Forward Line of Own Troops (FLOT) were mentioned. However, as the duration of the operation progresses, it can't provide sufficient research to identify an approach of using a proper strategy to allocating air forces and their interaction with the ground troops in advance and attrition. The model we construct tries to develop a further discussion oriented to the basis of such an air-land interaction. All in all, we consider only 1 or 0 as the possible option for ϕ and ψ. Thus, we can classify four main strategies as follows:

Strategy-1: If both Blue and Orange decide to put all their air forces into attacking the opponent's army forces, that is, $\phi=1, \psi=1$, such a strategy will lead to two possible outcomes. One is that Orange's army is totally destroyed or the other possibility, Blue's army is destroyed. Under such circumstances, Orange (or Blue) will continue attacking the opponent's army to prevent a high velocity advance. On the other hand, Blue (or Orange) will shift the target to the opponent's air forces. At this stage, the original strategy has switched to $\phi=1, \psi=0$ (or $\phi=0, \psi=1$). The outcome will eventually become a termination state; either E_{13} (E_{24}) or E_{12}(E_{21}). That is to say, either the Orange's (or Blue's) air forces are destroyed or the Blue's (or Orange's) army forces are destroyed, as shown below in Fig.2.

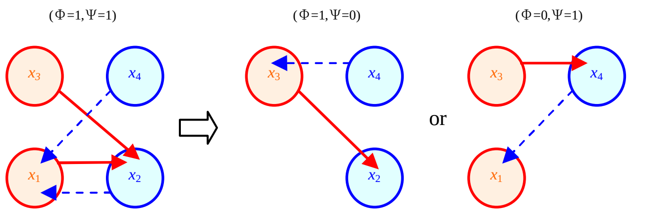

Fig.2: Different situations in strategy-1.

Strategy-2: Both Blue and Orange decide to put all their air forces to attacking opponent's air forces, that is, $\phi = 0$, $\psi = 0$. This strategy will develop into two feasible situations; either Orange's or Blue's air forces are destroyed first. Then, Blue's (or Orange's) air forces will transfer the target to the opponent's army in order to exhaust and block its advance. Therefore, Orange (or Blue) losing the support of air forces (being destroyed) will change his strategy to $\phi = 0$, $\psi = 1$ (or $\phi = 1$, $\psi = 0$). The terminal state will be E_{31} (E_{42}) or E_{32} (E_{41}). That is, Orange's (or Blue's) army forces are destroyed. On the other hand, the strategy can develop another two possible outcomes; either Orange's or Blue's army forces are destroyed first. Such a condition is quite similar to situations in strategy-1, and the original strategy will shift to $\phi = 1$, $\psi = 0$ (or $\phi = 0$, $\psi = 1$). The terminal state will turn out to be either E_{13} (E_{24}) or E_{12} (E_{21}). That is, Orange's (or Blue's) air forces are terminated, as shown below in Fig.3.

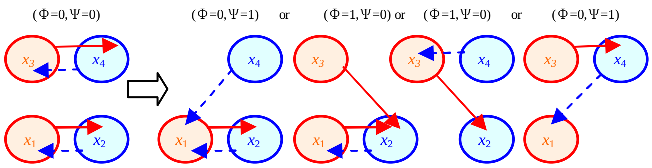

Fig.3: Different situations in strategy-2.

Strategy-3: Orange decides to put all his air forces into attacking the opponent's army forces, and Blue air forces concentrate their attack on the opponent's air forces, that is $\phi = 1$, $\psi = 0$. This strategy can lead to the feasible outcome, Blue's (or Orange's) army forces are terminated first. Under such circumstances, Blue's (or Orange's) air forces will transfer his focus to attack opponent's army in order to exhaust and block its advance. Orange (or Blue) will concentrate his attack on to the opponent's air forces and shift the strategy to $\phi = 0$, $\psi = 1$ (or $\phi = 1$, $\psi = 0$). At the end, the terminal state will be either E_{21} (E_{12}) or E_{24} (E_{13}); that is, Orange's (or Blue's) army forces are terminated or Blue's (or Orange's) air forces are terminated. However, when Orange's air forces are destroyed first, Blue's air forces will focus the target to attack the opponent's army. Therefore, the strategy is switched to $\phi = 0$, $\psi = 1$. The terminal state will be either E_{31} or E_{32}; that is, either Orange's or Blue's army forces are terminated at the end, as shown below in Fig.4.

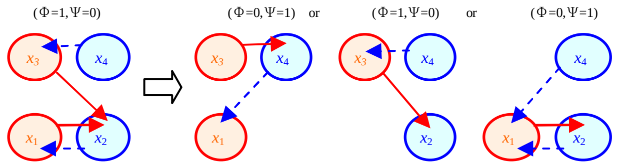

Fig.4: Different situations in strategy-3.

Strategy-4: If Orange attacks Blue's air forces and Blue attacks Orange's army forces, that is, $\phi = 0$, $\psi = 1$. Such strategy will lead to similar situations as they have developed in strategy-3. The strategy is then transferred to $\phi = 1$, $\psi = 0$ or $\phi = 0$, $\psi = 1$ or $\phi = 1$, $\psi = 0$. The terminal state will be E_{12} (E_{21}) or E_{13} (E_{24}) or E_{42} (or E_{41}), as shown below in Fig.5.

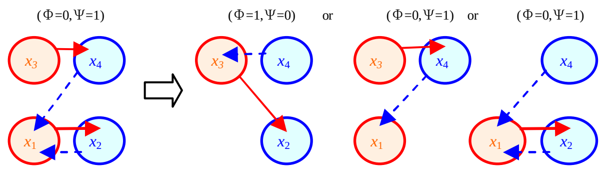

Fig.5: Different situations in strategy-4.

3.2 Deterministic Approach for the Strategy Solutions

In general, people may require an algebra solution for a problem of a differential game by means of the approaches developed by Isaacs [5] ,Friedman [25] and etc. On the other hand, some may attempt to obtain a numerical solution on account of the dynamic programming [26], the linear feedback solution [27], the gradient technique [28] and so on. In this model, due to the assumptions mentioned in section 2.1 and 2.4 as well as the four strategies in section 3.1, we intend to adopt the deterministic approach to analyze the terminal results for these strategies. In this regard of strategy-1, if both Blue and Orange decide to put all their air forces into attacking the opponent's army forces, then $\phi = 1$, $\psi = 1$, by equations (1) we will obtain

$$\begin{cases} \dot{x}_1(t) = -k_{21} \cdot x_2(t) - k_{41} \cdot x_4(t) & x_1(0) = x_{10} \\ \dot{x}_2(t) = -k_{12} \cdot x_1(t) - k_{32} \cdot x_3(t) & x_2(0) = x_{20} \\ \dot{x}_3(t) = 0 & x_3(0) = x_{30} \\ \dot{x}_4(t) = 0 & x_4(0) = x_{40} \end{cases} \qquad (2)$$

As a result, we will have

$$
\begin{cases}
x_1(t)=x_{10}Cosh(\sqrt{k_{12}k_{21}}\cdot t)-x_{20}\sqrt{\dfrac{k_{21}}{k_{12}}}Sinh(\sqrt{k_{12}k_{21}}\cdot t) \\
\qquad +x_{30}\dfrac{k_{32}}{k_{12}}[Cosh(\sqrt{k_{12}k_{21}}\cdot t)-1]-x_{40}\dfrac{k_{41}}{\sqrt{k_{12}k_{21}}}Sinh(\sqrt{k_{12}k_{21}}\cdot t) \\
x_2(t)=-x_{10}\sqrt{\dfrac{k_{21}}{k_{12}}}Sinh(\sqrt{k_{12}k_{21}}\cdot t)+x_{20}Cosh(\sqrt{k_{12}k_{21}}\cdot t) \\
\qquad -x_{30}\dfrac{k_{32}}{\sqrt{k_{12}k_{21}}}Sinh(\sqrt{k_{12}k_{21}}\cdot t)+x_{40}\dfrac{k_{41}}{k_{21}}[Cosh(\sqrt{k_{12}k_{21}}\cdot t)-1] \\
x_3(t)=x_{30} \\
x_4(t)=x_{40}
\end{cases}
\tag{3}
$$

There are two possible outcomes emerging from the strategy, one will be

(a). $x_1(t)$ terminated first;that is,$x_1(t_1)$=0, from (3) we can get the time

$$
t_1=\frac{\ln\left\{\dfrac{x_{30}k_{32}\sqrt{k_{21}}-\sqrt{[\sqrt{k_{12}}(x_{20}k_{21}+x_{40}k_{41})]^2-[\sqrt{k_{21}}(x_{10}k_{12}+x_{30}k_{32})]^2+(x_{30}k_{32}\sqrt{k_{21}})^2}}{\sqrt{k_{12}}(x_{20}k_{21}+x_{40}k_{41})-\sqrt{k_{21}}(x_{10}k_{12}+x_{30}k_{32})}\right\}}{\sqrt{k_{12}k_{21}}}
\tag{4}
$$

only under ondition-1: $\sqrt{k_{12}}(x_{20}k_{21}+x_{40}k_{41})>\sqrt{k_{21}}(x_{10}k_{12}+x_{30}k_{32})$

When we exam the condition, we can recognize the following facts. As Blue owns the total superiority of air and army forces, Orange's army will be the first annihilated. Later, Blue's air will shift his fire target to attack the opponent's air and Orange's air will be destroyed. Putting the result from (3) into the objective function and we'll reach the value:

$$
\begin{aligned}
J_1(t,\phi,\psi)&=\int_0^{t_1}\{V_1\cdot[x_1(t)-r\cdot x_2(t)]+V_2\cdot[x_3(t)-x_4(t)]\}dt \\
&=\Big\{t_1\frac{[x_{30}k_{21}(V_2k_{12}-V_1k_{32})+x_{40}k_{12}(rV_1k_{41}-V_2k_{21})]}{k_{12}k_{21}} \\
&\quad+\frac{V_1(x_{10}r-x_{20})[Cosh(\sqrt{k_{12}k_{21}}\cdot t_1)-1]}{k_{12}}-V_1\Big\langle\ k_{12}k_{21}(x_{30}k_{32}r-x_{40}k_{41}) \\
&\quad\cdot[1-Cosh(\sqrt{k_{12}k_{21}}\cdot t_1)]-[k_{12}k_{21}(x_{10}+x_{20}r)+(x_{30}k_{21}k_{32}+x_{40}k_{12}k_{41}r)] \\
&\quad\cdot\sqrt{k_{12}k_{21}}\cdot Sinh(\sqrt{k_{12}k_{21}}\cdot t_1)\ \Big\rangle\Big\}\Big/\sqrt{k_{12}^5k_{21}^5}
\end{aligned}
\tag{5}
$$

Meanwhile, for blocking Blue army forces to advance, Orange air forces will concentrate his attack at the opponent's army forces, and Blue air forces alter the target to Orange air forces, then $\tilde{\phi}=1$, $\tilde{\psi}=0$. And equations (2) become

$$
\begin{cases}
\dot{x}_1(t')=0 & x_1(t_1)=0 \\
\dot{x}_2(t')=-k_{12}\cdot x_1(t')-k_{32}\cdot x_3(t') & x_2(t_1)=x_{21} \\
\dot{x}_3(t')=-k_{43}\cdot x_4(t') & x_3(t_1)=x_{31}=x_{30} \\
\dot{x}_4(t')=0 & x_4(t_1)=x_{41}=x_{40}
\end{cases}
\tag{6}
$$

where

$$
\begin{aligned}
x_2(t_1)=x_{21}&=-x_{10}\sqrt{\frac{k_{21}}{k_{12}}}Sinh(\sqrt{k_{12}k_{21}}\cdot t_1)+x_{20}Cosh(\sqrt{k_{12}k_{21}}\cdot t_1) \\
&\quad-x_{30}\frac{k_{32}}{\sqrt{k_{12}k_{21}}}Sinh(\sqrt{k_{12}k_{21}}\cdot t_1)+x_{40}\frac{k_{41}}{k_{21}}[Cosh(\sqrt{k_{12}k_{21}}\cdot t_1)-1]
\end{aligned}
\tag{7}
$$

The result for (6) will be

$$
\begin{cases}
x_1(t')=0 \\
x_2(t')=x_{21}+\dfrac{k_{32}t'(-2x_{30}+x_{40}k_{43}t')}{2} \\
x_3(t')=x_{30}-x_{40}k_{43}t' \\
x_4(t')=x_{40}
\end{cases}
\tag{8}
$$

There are two terminal results under this stage. The first one, when $x_3(t')=0$ the battle is over; thus, we can have $t'_3=\dfrac{x_{30}}{x_{40}k_{43}}$ and the total termination time will be $T_{13}=t_1+t'_3=t_1+\dfrac{x_{30}}{x_{40}k_{43}}$.

Exercise the result from (8) in the objective function of (1). Its value yields

$$
\begin{aligned}
J'_3(t',\phi,\psi)&=\int_{t_1}^{t_3}\{V_1\cdot[x_1(t')-r\cdot x_2(t')]+V_2\cdot[x_3(t')-x_4(t')]\}dt' \\
&=-(x_{21}rV_1-x_{30}V_2+x_{40}V_2)t'_3+\frac{(x_{30}rV_1k_{32}-x_{40}V_2k_{43})}{2}t'^2_3-\frac{(x_{40}rV_1k_{32}k_{43})}{6}t'^3_3 \\
&\quad+\frac{t_1}{6}\{6x_{21}rV_1-3x_{30}(k_{32}rV_1t_1+2V_2)+x_{40}[rV_1k_{43}k_{32}t_1^2+3V_2(2+k_{43}t_1)]\}
\end{aligned}
\tag{9}
$$

The total value will be $J_{13}(t,\phi,\psi)=J_1(t,\phi,\psi)+J'_3(t',\phi,\psi)$

The second terminal result, when $x_2(t'_2)$=0, the battle is over. According to (8), we can get time

$$
t'_2=\frac{x_{30}\sqrt{k_{32}}\pm\sqrt{x_{30}^2k_{32}-2x_{40}x_{21}k_{43}}}{x_{40}\sqrt{k_{32}}k_{43}}
\tag{10}
$$

Only under condition-2: $\dfrac{x_{30}}{x_{40}k_{43}}>\dfrac{2x_{21}}{x_{30}k_{32}}$. Such condition indicates that Orange should possess the air superiority and then it is possible for Orange to destroy Blue's Army. And the total termination time will be $T_{12}=t_1+t'_2=t_1+\dfrac{x_{30}\sqrt{k_{32}}+\sqrt{x_{30}^2k_{32}-2x_{40}x_{21}k_{43}}}{x_{40}\sqrt{k_{32}}k_{43}}$.

Similar to the preceding calculation, we put the result (8) into the objective function of (1) and it yields:

$$
\begin{aligned}
J'_2(t',\phi,\psi)&=\int_{t_1}^{t_2}\{V_1\cdot[x_1(t')-r\cdot x_2(t')]+V_2\cdot[x_3(t')-x_4(t')]\}dt' \\
&=-(x_{21}rV_1-x_{30}V_2+x_{40}V_2)t'_2+\frac{(x_{30}rV_1k_{32}-x_{40}V_2k_{43})}{2}t'^2_2-\frac{(x_{40}rV_1k_{32}k_{43})}{6}t'^3_2 \\
&\quad+\frac{t_1}{6}\{6x_{21}rV_1-3x_{30}(k_{32}rV_1t_1+2V_2)+x_{40}[rV_1k_{43}k_{32}t_1^2+3V_2(2+k_{43}t_1)]\}
\end{aligned}
\tag{11}
$$

and the total value will become
$J_{12}(t,\phi,\psi)=J_1(t,\phi,\psi)+J'_2(t',\phi,\psi)$

(b). On the other hand, when $x_2(t_2)$=0 and it is terminated first., by equations (3) we can solve

$$
t_2=\frac{\ln\left\{\dfrac{-x_{40}\sqrt{k_{12}}k_{41}+\sqrt{[\sqrt{k_{21}}(x_{10}k_{12}+x_{30}k_{32})]^2-[\sqrt{k_{12}}(x_{20}k_{21}+x_{40}k_{41})]^2+(x_{40}\sqrt{k_{12}}k_{41})^2}}{\sqrt{k_{21}}(x_{10}k_{12}+x_{30}k_{32})-\sqrt{k_{12}}(x_{20}k_{21}+x_{40}k_{41})}\right\}}{\sqrt{k_{12}k_{21}}}
\tag{12}
$$

Only under the condition-3:

$\sqrt{k_{21}}(x_{10}k_{12}+x_{30}k_{32})>\sqrt{k_{12}}(x_{20}k_{21}+x_{40}k_{41})$

This condition can demonstrate a similar fact as we have clarified for condition-1 but by exchanging the positions between Orange and Blue. Then, to exercise the objective function in (1), we have:

$$
\begin{aligned}
J_2(t,\phi,\psi) &= \int_0^{t_2} \{V_1 \cdot [x_1(t) - r \cdot x_2(t)] + V_2 \cdot [x_3(t) - x_4(t)]\} dt \\
&= \Big\{ t_2 \frac{[x_{30}k_{21}(V_2k_{12} - V_1k_{32}) + x_{40}k_{12}(rV_1k_{41} - V_2k_{21})]}{k_{12}k_{21}} \\
&\quad + \frac{V_1(x_{10}r - x_{20})[Cosh(\sqrt{k_{12}k_{21}} \cdot t_2) - 1]}{k_{12}} - V_1 \Big\langle \, k_{12}k_{21}(x_{30}k_{32}r - x_{40}k_{41}) \\
&\quad \cdot [1 - Cosh(\sqrt{k_{12}k_{21}} \cdot t_2)] - [k_{12}k_{21}(x_{10} - x_{20}r) + (x_{30}k_{21}k_{32} - x_{40}k_{12}k_{41}r)] \\
&\quad \cdot \sqrt{k_{12}k_{21}} \cdot Sinh(\sqrt{k_{12}k_{21}} \cdot t_2) \Big\rangle \Big\} \Big/ \sqrt{k_{12}^5 k_{21}^5}
\end{aligned}
\tag{13}
$$

For blocking Orange army forces to advance, Blue air forces will concentrate his attacking at the opponent's army forces, and Orange air forces change the target to fire Blue air forces, then $\tilde{\phi} = 0$, $\tilde{\psi} = 1$. So, equations (2) become

$$
\begin{cases}
\dot{x}_1(t') = -k_{21} \cdot x_2(t') - k_{41} \cdot x_4(t') & x_1(t_2) = x_{12} \\
\dot{x}_2(t') = 0 & x_2(t_2) = 0 \\
\dot{x}_3(t') = 0 & x_3(t_2) = x_{32} = x_{30} \\
\dot{x}_4(t') = -k_{34} \cdot x_3(t') & x_4(t_2) = x_{42} = x_{40}
\end{cases}
\tag{14}
$$

and the solution is

$$
\begin{cases}
x_1(t') = x_{12} + \dfrac{k_{41}t'(-2x_{40} + x_{30}k_{34}t')}{2} \\
x_2(t') = 0 \\
x_3(t') = x_{30} \\
x_4(t') = x_{40} - x_{30}k_{34}t'
\end{cases}
\tag{15}
$$

When $x_4(t'_4) = 0$, then $t'_4 = \dfrac{x_{40}}{x_{30}k_{34}}$, the total termination time yields $T_{24} = t_2 + t'_4 = t_2 + \dfrac{x_{40}}{x_{30}k_{34}}$.
Meanwhile, the value of the objective function will be

$$
\begin{aligned}
J'_4(t',\phi,\psi) &= \int_{t_2}^{t'_4} \{V_1 \cdot [x_1(t') - r \cdot x_2(t')] + V_2 \cdot [x_3(t') - x_4(t')]\} dt' \\
&= (x_{12}V_1 + x_{30}V_2 - x_{40}V_2)t_4 + \frac{(x_{30}V_2k_{34} - x_{40}V_1k_{41})}{2} t_4'^2 - \frac{(x_{30}V_1k_{34}k_{41})}{6} t_4'^3 \\
&\quad + \frac{t_2}{6} \{6x_{12}V_1 - 3x_{40}(k_{41}V_1t_2 + 2V_2) + x_{30}[V_1k_{34}k_{41}t_2^2 + 3V_2(2 + k_{34}t_2)]\}
\end{aligned}
\tag{16}
$$

and its total value is
$J_{24}(t,\phi,\psi) = J_2(t,\phi,\psi) + J'_4(t',\phi,\psi)$.

The other possible terminal result is when $x_1(t'_1) = 0$, and on account of (15) we can obtain

$$
t'_1 = \frac{x_{40}\sqrt{k_{41}} + \sqrt{x_{40}^2 k_{41} - 2x_{30}x_{12}k_{34}}}{x_{40}\sqrt{k_{32}}\, k_{43}}
\tag{17}
$$

Only under the condition-4: $\dfrac{x_{40}}{x_{30}k_{34}} > \dfrac{2x_{12}}{x_{40}k_{41}}$.
Similar to the condition 2, it explains that Blue should possess the air superiority and then it is possible for Blue to destroy Orange's Army. The total termination time is $T_{21} = t_2 + t'_1 = t_2 + \dfrac{x_{40}\sqrt{k_{41}} + \sqrt{x_{40}^2 k_{41} - 2x_{30}x_{12}k_{34}}}{x_{40}\sqrt{k_{32}}\, k_{43}}$

And its value

$$
\begin{aligned}
J'_1(t',\phi,\psi) &= \int_{t_2}^{t'_1} \{V_1 \cdot [x_1(t') - r \cdot x_2(t')] + V_2 \cdot [x_3(t') - x_4(t')]\} dt' \\
&= (x_{12}V_1 + x_{30}V_2 - x_{40}V_2)t'_1 + \frac{(x_{30}V_2k_{34} - x_{40}V_1k_{41})}{2} t_1'^2 - \frac{(x_{30}V_1k_{34}k_{41})}{6} t_1'^3 \\
&\quad + \frac{t_2}{6} \{6x_{12}V_1 - 3x_{40}(k_{41}V_1t_2 + 2V_2) + x_{30}[V_1k_{34}k_{41}t_2^2 + 3V_2(2 + k_{34}t_2)]\}
\end{aligned}
\tag{18}
$$

where the total value will become
$J_{21}(t,\phi,\psi) = J_2(t,\phi,\psi) + J'_1(t',\phi,\psi)$

Due to the similar process to achieving the solutions for strategies 2, 3, and 4, as well as the limitation of the paper, these results are available by making contact with the authors through e-mail.

4 Conclusions

According to the discussion mentioned above, we can shed the following essential perceptions. First, that one side holds the air dominance is the case. When its army is superior or equal to the opponent's, it's better that the air force can use concentrated fire to exhaust the opponent's air combat power, and then destroy the opponent's army forces. However, when his army is inferior, the optimal strategy is to use massed air fire to damage the opponent's army forces first, and then destroy the air forces. Second, that both sides grasp the balance of air dominance is the case. To the side which owns army superiority, the priority for the air force's mission will be to tear down the opponent's army first and then the air force. When both sides possess an even army capacity, the air forces on both sides will adopt a similar strategy; that is, to defeat the opponent's air first and then the army. As to the side with army inferiority, its air should strike the opponent's air as the priority and then deal with the army. Third, that one side is air inferiority is the case. If its army is better or equal than the opponent's, its air force should concentrate fire to destroy opponent's air and then the army. However, if its army is also inferior, the priority of air mission is to attack the opponent's army and then its air.

The objective function contained in the model indicates the relative advance velocity which is happened upon the survival combat power on both sides. It has highlighted the significant insight that possessing air dominance can definitely affect the advance velocity of the ground troops. That is, the advance velocity is constrained not only by the relative combat power from the army but also from the air forces. Apparently, the side that holds air superiority can certainly detain the opponent's army advance and compel its retreat; eventually, it will also increase his army effectiveness in terms of advance velocity. This model, therefore, illustrates an active maneuvering of an operational process and presents a practical air-land combat operation model. In the coming future, we expect to develop another paper which will integrate these results we have acquired from this model into a computer simulation in which we may be likely to perceive some significant insights in terms of optimal strategies for the nature of the air-land combat operations.

References

[1] Lanchester, F.W., "The Dawn of the Fourth Arm-NO.V. The Principle of concentration", *Engineering* 98, 1914, pp.422–423.

[2] Douhet, G., "*The Command of the Air*", NewYork, 1921 (1942 Trans).

[3] Mitchell, W.,"*Winged Defense*", Putman's Son,1925.

[4] Mitchell, W., "*Skyways: A Book on Modern Aeronautics*", Lippineott, 1930.

[5] Isaacs, R., "*Differential Games*", John Wiley & Sons, Inc., New York, 1965.

[6] Isbell, J.R. and Marlow, W.H., "Attrition Games," Naval Res. Log. Quat. 3, 1956, pp71-94.

[7] Weiss, H.K., "Some Differential Games of Tactical Interest and the Value of a Supporting Weapon System",*Operations Research* 7, 1959, pp.180-196.

[8] Kawara, Y., "An Allocation Problem of Support Fire in Combat as a Differential Game", *Operations Research* 21, 1973, pp. 942–951.

[9] Berkovitz, L. and Dresher, M., "A Game theory Analysis of Tactical Air War", *Operations. Research.* 7, 1959, pp.599-620.

[10] Berkovitz, L. and Dresher, M., "Allocation of Two Types of Aircraft in Tactical Air War: A Game Theoretic Analysis", *Operations. Research.* 8, 1960, pp.694-706.

[11] Bellman, R., Dreyfus, S.,"On A Tactical Air-Warfare Model of Mengel", *Operations. Research.* 6, 1957, pp.65-78.

[12] Taylor, J.G.., "*Application of Differential Games to Problems of Military Conflict: Tactical Allocation Problems*", Part II-III Technical Report (AD-758 663, AD-778 630)NPIS, Monterey,California,1972.

[13] Taylor, J.G.., "*Lanchester Models of Warfare*", Vol I-II, Operations Research Society of America Arlington, VA, 1983.

[14] Chattopadhyay, R., "Differential Game Theoretic Analysis of a Problem of Warfare", *Naval. Res. Log.* 16, 1969, pp.435–441.

[15] Austin, F., Carbone, G., Flaco, M., and Hinz, H., "Game Theory for Automated Maneuvering During Air-Air Combat", Journal of Guidance, Control, and Dynamics, Vol, No. 6, 1990, pp.1143-1149.

[16] Stephane, L.M., "Differential Games and Symbolic Programming to Caculate a Guaranteed Aircraft Evasion in Modern Aerial Duels", Proceeding of the 33rd Conference on Decision and Control. Lake Buena Vista, FL-December.1994, pp.3868-3870.

[17] Grimm, W., Prasad, U.R., Well, K.H., "Open-Loop Guidance For Pre-Launch Maneuvering In Medium-Range Air Combat", Proceeding of the 27th Conference on Decision and Control. Austin, Texas-December. 1998, pp.1442-1447.

[18] Smith, R.E., Dike, D.K., Mehra,R.K., Ravichandran,B., El-Fallah, A., "Classifier systems in combat: Two-sided learning of maneuvers for advanced fighter aircraft", Comput. Methods Appl. Mech. Engrg. 186, 2000, pp421-437.

[19] Hsia, K.H. and Hsieh, J.G., "A first approach to fuzzy differential game problem: guarding a territory", Fuzzy Sets and Systems, Vol.55,1993, pp.157-167.

[20] Chu, H.J., Hsieh, J.G., Lee, Y.S. and Hsia, K.H., "A pursuit-evasion game with one curvature constraint", Optimal Control Applications & Methods, Vol.21, 2000,pp.113-131.

[21] Chen, H. M.,"An optimal control problem in determining the optimal reinforcement schedules for the Lanchester equations", Computers & Operations Research. 30, 2003, pp.1051-1066.

[22] Dupuy, T.N., "*Numbers, Predictions and War: Using History to Evaluate Combat factors and Predict the Outcome of Battles*", HERO Books, Fairfax, Virginia. 1985, pp.43-51; pp.185-203; p.77.

[23] Dupuy, T.N.,"*Numbers, Predictions and War: Using History to Evaluate Combat factors and Predict the Outcome of Battles*", HERO Books, Fairfax, Virginia, 1985, pp.77-94.

[24] Moffat, J., "The System Dynamics of Future Warfare", *EJOR*. 90, 1996, pp.609-618.

[25] Friedman, A., "*Differential Games*", John wiley & Sons, Inc., New York,1971.

[26] Vaught,K.N.,"*A computational method for nolinear differential game*", Ph.D. Dissertation,University of California, Los Angeles,1971.

[27] Gaskell, S., "C*omputational algorithms for optimal solution of non-linear differential game*", Ph.D. Dissertation,University of Pennsylvania, phiadel-phia,1970.

[28] Koberts, D.A. & Montgomeery, R.C."*Development and application of a gradient method for solving differential games*", NASA-TN-D-6502, National Aeronautics and Space Administration, Washington, D.C,1971.