Stability of Geometrically Nonlinear Timoshenko Beam with Boundary Damping

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Abstract: - This paper gives a result on the uniform exponential stability for the geometrically nonlinear Timoshenko beam with boundary damping. The proof needs an assumption on the motion of the beam. The relation between the assumption and the decay property of the beam is examined with a numerical simulation.

Key-Words: - Timoshenko beam, geometrically nonlinear elastic beam, boundary stabilization

1 Introduction

In this paper, we consider a system of partial differential equations describing the motion of a geometrically nonlinear elastic beam, and discuss the stability of the system under some boundary damping.

The dynamic stability of partial differential equations governing the motion of elastic bodies (strings, membrane, beams or plates) with boundary damping of velocity-feedback type has attracted much attention for the last few decades (see, e.g., [2] and the bibliography therein). However, most of the previous works on this subject treat those models that are derived under the condition that the deflection of the body is small enough, even though the model equations are nonlinear.

We are interested in making it clear how the known results on the stability change when we treat the large deflection of elastic bodies. As far as the author knows, the only theoretical result of this direction is for the model named Euler’s elastica [2], which establishes that the boundary damping makes the elastica uniformly exponentially stable. The elastica is a model of geometrically nonlinear, inextensible beam based on the Euler-Bernoulli theory. The end of this paper is to investigate the effect of the boundary damping for a geometrically nonlinear beam model derived based on the Timoshenko theory. We prove the uniform exponential stability of the model under an assumption, and then make some numerical observation on the decay property of the damped beam and its relation to the assumption.

2 Model Equations

Consider a system of partial differential equations

\[ \rho A u_{tt} = (N \cos \varphi - Q \sin \varphi)_x, \]
\[ \rho A v_{tt} = (N \sin \varphi + Q \cos \varphi)_x, \]
\[ \rho I \varphi_{tt} = -M_x - N(-(1 + u_x) \sin \varphi + v_x \cos \varphi) + Q((1 + u_x) \cos \varphi + v_x \sin \varphi), \]

where

\[ N = EA\{(1 + u_x) \cos \varphi + v_x \sin \varphi - 1\}, \]
\[ M = -EI \varphi_x, \]
\[ Q = GA\{- (1 + u_x) \sin \varphi + v_x \cos \varphi\}. \]

This is a model describing the large, planar motion of an elastic beam, and is derived on the basis of the finite displacement theory of Timoshenko beam (see, e.g., [5], [4]) with the linear
functions of two variables \( x \) and \( t \), \( 0 \leq x \leq L, t \geq 0 \). In this model, the centerline (i.e., the locus of the centroid of the cross section) of the beam is assumed to lie in a fixed plane with a usual Cartesian coordinate system \( OXY \), and, in the reference state of the beam, to occupy the interval \( 0 \leq X \leq L \) of the \( X \)-axis. The unknowns \( u(x, t) \) and \( v(x, t) \) denote the displacements, at time \( t \), in the \( X \)- and \( Y \)-direction, respectively, of the particle which occupies position \( (x, 0) \), \( 0 \leq x \leq L \), in the reference state: thus, the centerline at time \( t \) is described by the curve \( x \mapsto (x + u(x, t), v(x, t)) \). The unknown \( \varphi(x, t) \) denotes the angle between the normal of the plane \( \Gamma(x, t) \) and the \( X \)-axis, where \( \Gamma(x, t) \) is the cross section which at \( X = x \) and perpendicular to the centerline when the beam is in the reference state (note that the plane \( \Gamma(x, t) \), in general, isn’t perpendicular to the centerline in deformed states).

The functions \( N, Q \) and \( M \) can be interpreted as the axial force, the sharing force and the bending moment, respectively, acting on \( \Gamma(x, t) \).

The coefficients \( \rho(x), A(x) \) and \( I(x) \) in the equations are the mass density, the cross section area and the moment of inertia of the cross section; furthermore, \( E(x) \) and \( G(x) \) are the coefficients of the linear stress-strain relations, and are corresponding to Young’s modulus and sharing modulus, respectively, in the linear theory of Timoshenko beam. We assume that these coefficients are all strictly positive, smooth functions of \( x \in [0, L] \).

The boundary conditions we consider at \( x = 0 \) (a clamped end) are

\[
u(0, t) = v(0, t) = \varphi(0, t) = 0,\]

and the boundary conditions at \( x = L \) are

\[
\begin{align*}
[N \cos \varphi - Q \sin \varphi]_{x=L} &= f_{L1}, \\
[N \sin \varphi + Q \cos \varphi]_{x=L} &= f_{L2}, \\
[-M]_{x=L} &= \mu_L,
\end{align*}
\]

which represent that, on the cross section \( \Gamma(L, t) \), the beam is subjected to external forces \( f_{L1} \) and \( f_{L2} \) in \( X \)- and \( Y \)-direction, respectively, and to an external moment \( \mu_L \). In particular, we are interested in the direct velocity feedback control which is called the boundary damping, and is described by

\[
\begin{align*}
&f_{L1}(t) = -l_1 u_t(L, t), \\
f_{L2}(t) = -l_2 v_t(L, t), \\
&\mu_L(t) = -l_3 \varphi_t(L, t),
\end{align*}
\]

where \( l_i (i = 1, 2, 3) \) are non-negative constants representing feedback gains.

The purpose of this paper is to show that, under some assumption, the total energy of the beam and the solution to the initial-boundary value problem associated with (1)–(6) decays exponentially in time. The total energy of the beam at time \( t \) is given by

\[
E(t) = \frac{1}{2} \int_0^L \left( \rho A u_t^2 + \rho A v_t^2 + \rho I \varphi_t^2 \\
+ \frac{N^2}{EA} + \frac{M^2}{EI} + \frac{Q^2}{GA} \right) \, dx,
\]

and, in the undamped case (i.e., \( l_1 = l_2 = l_3 = 0 \)), satisfies \( E(t) = E(0) \) for any \( t \geq 0 \).

For the well-posedness of the initial-boundary value problem, we have only the result below of local existence in time (see [3] for the proof). While the discussion on the dynamic stability requires the result of global existence in time, we assume it in the following sections to continue the discussion.

**Proposition 1** Suppose that \( \rho, A, I, E, G \) are positive-valued \( C^2 \) functions on \([0, L] \). Let \( u_0, v_0 \) and \( \varphi_0 \) be functions in \( H^2(0, L) \), and let \( u_1, v_1 \) and \( \varphi_1 \) be those in \( H^1(0, L) \), which satisfy

\[
\begin{align*}
u_0(0) &= v_0(0) = \varphi_0(0) = 0, \\
u_1(0) &= v_1(0) = \varphi_1(0) = 0, \\
w_{10}(L) &= l_1 u_1(L) = 0, \\
w_{20}(L) &= l_2 v_1(L) = 0, \\
w_{30}(L) &= l_3 \varphi_1(L) = 0,
\end{align*}
\]

where \( w_{10}, w_{20} \) and \( w_{30} \) are the functions obtained by substituting \( u = u_0, v = v_0, \varphi = \varphi_0 \)
into $N \cos \varphi - Q \sin \varphi, N \sin \varphi + Q \cos \varphi$ and $-M$, respectively. Then there exists a $t > 0$ such that the initial-boundary value problem (1)–(6) with

$$
\begin{align*}
&u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \\
&\varphi(x,0) = \varphi_0(x), \\
&u_t(x,0) = u_1(x), \quad v_t(x,0) = v_1(x), \\
&\varphi_t(x,0) = \varphi_1(x)
\end{align*}
$$

admits a unique solution $(u,v,\varphi)$ with

$$
\begin{align*}
&u, v, \varphi \in L^\infty(0,t; H^2(0,L)), \\
&u_t, v_t, \varphi_t \in L^\infty(0,t; H^1(0,L)), \\
&u_{tt}, v_{tt}, \varphi_{tt} \in L^\infty(0,t; L^2(0,L)).
\end{align*}
$$

### 3 Exponential Stability

In what follows, we refer to $\rho, A, I, E, G$ as “physical parameters”.

The following is the main result.

**Theorem 1** Let $u, v, \varphi$ be the functions stated in Proposition 1 (the positive number $t$ in the proposition can be arbitrary for this theorem), and suppose the feedback gains $l_1, l_2, l_3$ are all positive. If there is a continuously differentiable function $\sigma(x)$ with $\sigma(0) = 0$ and

$$
\begin{align*}
&\sigma/EA', \sigma/GA', \sigma/EI' > 0, \\
&\sigma/\rho A', \sigma/\rho I' > 0, \quad 0 \leq x \leq L
\end{align*}
$$

(‘ stands for $d/dx$) such that the functions $(u,v,\varphi)$ satisfy

$$
\begin{align*}
&\int_0^L \{ \rho A u_t(x,t) \int_0^x \sigma'(\xi)(\cos \varphi(\xi,t))_t d\xi \\
&+ \rho A v_t(x,t) \int_0^x \sigma'(\xi)(\sin \varphi(\xi,t))_t d\xi \} dx \geq 0,
\end{align*}
$$

for some $T_0 > 0$, then the total energy $E(t)$ is estimated by

$$
E(t) \leq M_0 e^{-\beta t} E(T_0), \quad t \geq T_0
$$

for some constants $M_0 \geq 1$ and $\beta > 0$.

**Remark 1** In the theorem above, we may choose $\sigma(x) = x$ if the physical parameters are all constant.

**Proof of Theorem 1**

For any positive number $\epsilon$, define a functional $V$ as

$$
V(t) = E(t) + \epsilon F(t),
$$

where

$$
F(t) = \int_0^L \left\{ \rho A u_t \left[ \sigma(1+u_x) - \int_0^x \sigma' \cos \varphi d\xi \right] \\
+ \rho A v_t \left[ \sigma v_x - \int_0^x \sigma' \sin \varphi d\xi \right] \\
+ \rho I \sigma \varphi_x \right\} dx.
$$

Noting the smoothness (9) of the solution, we differentiate the functional $V(t)$ with respect to $t$, and use (1)–(3), (4), (5), (6), $\sigma(0) = 0$ to obtain

$$
\dot{V}(t) = - \left[ l_1 u_t^2 + l_2 v_t^2 + l_3 \varphi_t^2 \right]_{x=L} \\
+ \frac{1}{2} \epsilon \sigma(L) \left[ \rho A u_t^2 + \rho A v_t^2 + \rho I \varphi_t^2 \\
+ N^2 \frac{E A}{GA} + \frac{Q^2}{GA} + \frac{M^2}{EI} \right]_{x=L} \\
- \frac{1}{2} \epsilon l_1 \int_0^L \sigma'(x)(\cos \varphi(L,t) - \cos \varphi(x,t)) dx \\
- \frac{1}{2} \epsilon l_2 \int_0^L \sigma'(x)(\sin \varphi(L,t) - \sin \varphi(x,t)) dx \\
- \frac{1}{2} \epsilon l_3 \int_0^L \{ (\sigma/EA)' u_t^2 + (\sigma/GA)' v_t^2 \\
+ (\sigma/\rho A)' \varphi_t^2 + (\sigma/\rho I)' M^2 \} dx \\
- \epsilon \int_0^L \left\{ \rho A u_t \int_0^x \sigma'(\cos \varphi_\xi t) d\xi \\
+ \rho A v_t \int_0^x \sigma'(\sin \varphi_\xi t) d\xi \right\} dx
$$

(‘ denotes $d/dt$).
The terms in the right hand side of the above are estimated as follows. First, by the boundary conditions (5), (6), we have

\[(2\text{nd term}) \leq cc_1 [u_1^2 + v_1^2 + \varphi_1^2]_{x=L}
\]

\[(c_1 \text{ is a positive number depending only on the physical parameters, } l_1, l_2, l_3, \text{ and } \sigma(L)). \text{ Next, using the inequality } |\cos x - \cos y| \leq |x - y|, \text{ we have}
\]

\[(3\text{rd term+4th term}) \leq cc_2 \left( \frac{1}{4\delta} (u_1(L,t)^2 + v_1(L,t)^2) \right)
+ 2\delta \int_0^L \frac{M^2}{EI} dx \right \}
\]

\[(c_2 \text{ is a positive number depending only on } EI, l_1, l_2, \sigma', \text{ and } L) \text{ for any positive number } \delta. \text{ Finally, by (10), we have}
\]

\[(5\text{th term}) \leq -\epsilon c_3 \mathcal{E}(t)
\]

\[(c_3 \text{ is a positive constant depending only on the physical parameters, and } \sigma). \]

Now the estimations above and the assumption (11) lead to

\[\dot{V}(t; \varphi) \leq - \left\{ \min\{l_1, l_2, l_3\} - \epsilon(c_1 + c_2/(4\delta)) \right\}
\times \left[ u_1^2 + v_1^2 + \varphi_1^2 \right]_{x=L}
\]

\[- \epsilon(c_3 - 4c_2\delta)\mathcal{E}(t), \quad t \geq T_0.
\]

Hence, choosing \( \delta \) and \( \epsilon \) such that

\[\min\{l_1, l_2, l_3\} - \epsilon(c_1 + c_2/(4\delta)) > 0,
\]

\[c_3 - 4c_2\delta > 0,
\]

we have

\[\dot{V}(t) \leq -\epsilon(c_3 - 4c_2\delta)\mathcal{E}(t), \quad t \geq T_0. \quad (13)
\]

On the other hand, by Schwarz’s inequality, \(|1 - \cos x| \leq |x|, |\sin x| \leq |x|, \text{ and Theorem 2 below, we can show}
\]

\[|F(t)| \leq C\mathcal{E}(t), \quad t \geq 0
\]

\((C \text{ is a constant depending only on the physical parameters, } L, \sigma), \text{ which implies that}
\]

\[(1 - C\epsilon)\mathcal{E}(t) \leq V(t) \leq (1 + C\epsilon)\mathcal{E}(t). \quad (14)
\]

Therefore, by (13), we obtain

\[\dot{V}(t; \varphi) \leq -\beta V(t; \varphi), \quad t \geq T_0, \quad (15)
\]

where

\[\beta = \epsilon(c_3 - 4c_2\delta)/(1 + C\epsilon) > 0.
\]

If we now choose \( \epsilon \) so as to satisfy, in addition to (12),

\[1 - C\epsilon > 0,
\]

the inequality (15) and (14) imply

\[\mathcal{E}(t) \leq \frac{1 + C\epsilon}{1 - C\epsilon} e^{-\beta t} \mathcal{E}(T_0), \quad t \geq T_0.
\]

The proof is thus completed.

**Remark 2** From the argument of the proof above, we can see easily that the assumption (11) can be relaxed as follows:

\[(\text{Left-hand side of Ineq.}(11))
\]

\[\geq -c \int_0^L \left\{ (\sigma \rho A')u_1^2 + (\sigma \rho A')v_1^2 \right\} dx \quad (16)
\]

for some \( 0 \leq c < 1/2. \)

Theorem 1 asserts the exponential decay of the energy \( \mathcal{E}(t) \); in fact, it, together with the next theorem, also asserts the exponential stability of the solution to the initial boundary value problem.

**Theorem 2** Let \( X \) be the Hilbert space defined by

\[X = \left\{ (u, v, \varphi) \in (H^1(0, L))^3 \mid u(0) = v(0) = \varphi(0) = 0 \right\}
\]

with the norm

\[\|(u, v, \varphi)\|_X = \|u\|_{H^1} + \|v\|_{H^1} + \|\varphi\|_{H^1},
\]

\[\mathcal{E}(t) \leq \frac{1 + C\epsilon}{1 - C\epsilon} e^{-\beta t} \mathcal{E}(T_0), \quad t \geq T_0.
\]

\[\dot{V}(t; \varphi) \leq -\beta V(t; \varphi), \quad t \geq T_0, \quad (15)
\]

\[\beta = \epsilon(c_3 - 4c_2\delta)/(1 + C\epsilon) > 0.
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with the norm

\[\|(u, v, \varphi)\|_X = \|u\|_{H^1} + \|v\|_{H^1} + \|\varphi\|_{H^1},
\]

\[\mathcal{E}(t) \leq \frac{1 + C\epsilon}{1 - C\epsilon} e^{-\beta t} \mathcal{E}(T_0), \quad t \geq T_0.
\]
and define the functional $U$ by

$$U(u, v, \varphi) = \frac{1}{2} \int_0^L \left( \frac{N^2}{EA} + \frac{M^2}{EI} + \frac{Q^2}{GA} \right) dx,$$

for any $0 < \alpha < 1$, $\epsilon > 0$. Choosing now $\epsilon$ so as to $1 - 2\epsilon > 0$, it follows from

$$(1 - \cos \varphi)^2 \leq \varphi^2, \quad \sin^2 \varphi \leq \varphi^2$$

that

$$\text{Eq.}(19)$$

$$\geq \alpha(1 - 2\epsilon)(u_x^2 + v_x^2) - 2\alpha \left( \frac{1}{2\epsilon} - 1 \right) \varphi^2 + \frac{\varphi^2}{2L}.$$ 

Thus, noting that $\int_0^L \varphi^2 dx \leq L \int_0^L \varphi_x^2 dx$, we have

$$\tilde{U}(w) \geq \int_0^L \left\{ \alpha(u_x + 1 - \cos \varphi)^2 + \alpha(v_x - \sin \varphi)^2 + \frac{\varphi^2}{2L} + \frac{\varphi_x^2}{2} \right\} dx$$

Choosing here $\alpha$ so as to $1/(2L) - 2\alpha(1/(2\epsilon) - 1) > 0$, we arrive at the inequality $\tilde{U}(u, v, \varphi) \geq c ||(u, v, \varphi)||_{X}^2$ for some $c > 0$.

## 4 Numerical Solution

In this section, we solve numerically the initial-boundary value problem stated in Proposition 1 to observe how solutions decay under the boundary damping and when the condition (11) (or (16)) is satisfied.

### 4.1 Numerical Scheme

In what follows, we suppose the physical parameters $\rho$, $A$, $I$, $E$ and $G$ are constants, and consider the normalized form of the problem: Making the change of variables as

$$t \to Tt \quad (T = L\sqrt{\rho/E}), \quad x \to Lx,$$

$$u \to Lu, \quad v \to Lv, \quad \varphi \to \varphi,$$

$$l_1 \to \frac{\rho AL}{T} l_1, \quad l_2 \to \frac{\rho AL}{T} l_2, \quad l_3 \to \frac{\rho IL}{T} l_3,$$
and introducing the constants
\[ \gamma = \frac{G}{E}, \quad \nu = \frac{A L^2}{I}, \]
then the equations (1)–(3) are brought to
\[ u_{tt} = (N \cos \varphi - Q \sin \varphi)_x, \]
\[ v_{tt} = (N \sin \varphi + Q \cos \varphi)_x, \]
\[ \varphi_{tt} = -M_x - \nu N(-1 + u_x) \sin \varphi + v_x \cos \varphi \]
\[ + \nu Q((1 + u_x) \cos \varphi + v_x \sin \varphi) \]
for 0 ≤ x ≤ 1, t ≥ 0, where
\[ N = (1 + u_x) \cos \varphi + v_x \sin \varphi - 1, \]
\[ M = -\varphi_x, \]
\[ Q = \gamma(-1 + u_x) \sin \varphi + v_x \cos \varphi, \]
and the boundary conditions with damping are brought to
\[ u(0, t) = v(0, t) = \varphi(0, t) = 0, \]
\[ [N \cos \varphi - Q \sin \varphi]_{x=1} = -l_1 u_t(1, t), \]
\[ [N \sin \varphi + Q \cos \varphi]_{x=1} = -l_2 v_t(1, t), \]
\[ [-M]_{x=1} = -l_3 \varphi_t(1, t). \]
The total energy (7) are replaced by
\[ E(t) = \frac{1}{2} \int_0^1 (\nu u_t^2 + \nu v_t^2 + \varphi_t^2 \right)
\[ + \nu N^2 + M^2 + (\nu/\gamma)Q^2) \, dx. \]
(22)

It is easy to check that \( E(t) = E(0), \) t ≥ 0 for any solution of (20)-(21) in the undamped case \( l_1 = l_2 = l_3 = 0. \)
Furthermore, we introduce the new variables
\[ w_1 = N \cos \varphi - Q \sin \varphi, \]
\[ w_2 = N \sin \varphi + Q \cos \varphi, \]
\[ w_3 = -M, \]
\[ w_4 = u_t, \quad w_5 = v_t, \quad w_6 = \varphi_t, \quad w_7 = \varphi \]
to reformulate (20)–(21) into the 1st-order quasilinear hyperbolic system
\[ w_t = A(w)w_x + B(w)w \]
(23)
\[ (w = (w_1, w_2, \ldots, w_7)), \] where \( A \) and \( B \) are 7 × 7 matrices whose nonzero elements are
\[ \begin{bmatrix} A_{11} & A_{15} \\ A_{24} & A_{25} \end{bmatrix} = \Theta(w_7)\Sigma \Theta(-w_7), \]
\[ \begin{bmatrix} B_{16} & B_{20} \\ B_{16} & B_{20} \end{bmatrix} = \Theta(\frac{\pi}{2}) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \]
\[ \begin{bmatrix} B_{61} & B_{62} \end{bmatrix} = \nu \begin{bmatrix} \Theta(w_7)\Theta(\frac{\pi}{2}) \\ -\Theta(w_7)\Sigma \Theta(\frac{\pi}{2}) \end{bmatrix} \]
\[ \times \begin{bmatrix} \Sigma^{-1}\Theta(-w_7) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} \]
with
\[ \Theta(\varphi) = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \]
The boundary conditions (21) are rewritten as
\[ w_4 = w_5 = w_6 = w_7 = 0 \quad \text{at } x = 0, \]
\[ w_1 + l_1 w_4 = w_2 + l_2 w_5 \]
\[ = w_3 + l_3 w_6 = 0 \quad \text{at } x = 1. \]
We approximate the system (23)–(24) with a finite difference method. Since our interest is in finding the decay property of the system under the boundary damping, it is desired that the difference scheme itself has no numerical dissipation. So we adopt the Crank-Nicolson scheme. Let \( w^n_k \) be our approximation of the solution \( w \) at the grid point \((x, t) = (k \Delta x, n \Delta t)\) \( (k = 0, \ldots, K, \ n = 0, 1, \ldots)\). The scheme we use for (23) is
\[ w^{n+1}_k - (R/4)A(w^n_k)\delta_0 w^{n+1}_k \]
\[ - (\Delta t/2)B(w^n_k)w^{n+1}_k \]
\[ = w^n_k + (R/4)A(w^n_k)\delta_0 w^n_k \]
\[ + (\Delta t/2)B(w^n_k)w^n_k, \]
where \( R = \Delta t/\Delta x \) and \( \delta_0 w_k = w_{k+1} - w_{k-1} \). For the boundary conditions at \( k = 0 \) and \( K \), we
Initial conditions we examined were

\[ w_1 = w_2 = w_4 = w_5 = w_6 = 0, \]
\[ w_3 = \varphi_0 x, \quad w_7 = \varphi_0, \quad \text{at } t = 0 \]
\[ (\varphi_0(x) = \pi \sin(\pi x/2)). \]

4.2 Results

Initial conditions we examined were

\[ w_1 = w_2 = w_4 = w_5 = w_6 = 0, \]
\[ w_3 = \varphi_0 x, \quad w_7 = \varphi_0, \quad \text{at } t = 0 \]
\[ (\varphi_0(x) = \pi \sin(\pi x/2)). \]

The physical and numerical parameters were fixed as follows:

\[ \gamma = 0.37100, \quad \nu = 4.8120 \times 10^3 \]
\[ \Delta x = 6.25 \times 10^{-4}, \quad \Delta t = 5.3167 \times 10^{-3}. \]

The grid spacings \( \Delta x \) and \( \Delta t \) above were determined by taking into account the numerical dispersion the Crank-Nicolson scheme possesses.

First of all, in order to check the validity of the difference scheme we made up, we obtained numerical solutions for the undamped case (i.e., the feedback gains \( l_1, l_2, l_3 \) are all zero). In this case, the total energy \( E(t) \) defined by (22) is constant, and, from the initial conditions above, its theoretical value is \((\pi/2)^4 \approx 6.088\). For the approximate solution we obtained, the error of the total energy \( E(t) \) was

\[ |E(t) - (\pi/2)^4| < 0.018, \quad 0 \leq t \leq 1000. \]

Fig.1 (a) shows the shapes of the centerline of the vibrating beam from \( t = 0 \) to 60 (approximately a half of the period of vibration), and (b) does the kinetic energy parts of (22), i.e., \( \int_0^1 \nu u_x^2 \, dx, \int_0^1 \nu u_t^2 \, dx \) and \( \int_0^1 \varphi_t^2 \, dx \) for the same time interval. The fine motion occurring in \( \int_0^1 \varphi_t^2 \, dx \) is due to the rotation of the cross sections caused by the rotatory inertia and the sharing force.
Next, we obtained results for the boundary damping case, samples of which are shown in Fig. 2–4. In each figure, (a) shows the total energy \( E(t) \) defined by (22); (b) does the ratio \( I_1(t)/I_2(t) \) of

\[
I_1 = \int_0^L \left\{ u_t(x,t) \int_0^x (\cos \varphi(\xi,t)) \xi \, d\xi + v_t(x,t) \int_0^x (\sin \varphi(\xi,t)) \xi \, d\xi \right\} \, dx
\]

and

\[
I_2 = \int_0^L \left\{ u_t^2 + v_t^2 \right\} \, dx.
\]

We note that the assumption (11) of Theorem 1 (and its variation (16)) can be written as

\[
I_1(t)/I_2(t) \geq -c = \begin{cases} 0 & \text{for (11)} \\ -(1/2 - \eta) & \text{for (16)} \end{cases}
\]

for some \( 0 \leq \eta < 1/2 \) (see Remark 1). The horizontal dashed lines in (b) of Fig. 2–4 indicate the levels \( I_1/I_2 = 0 \) and 1/2.

These results assert that a suitable choice of the feedback gains makes the geometrically nonlinear Timoshenko beam exponentially
stable, and that the combination of Theorem 1 and the numerical evaluation of $I_1/I_2$ gives a practical criterion for exponential stability of the beam.

References


