Pressure And Entropy Variations Across The Weak Shock Wave Due To Heat Conductivity Effects

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Abstract: The nonlinear ordinary differential equations describing the normal shock wave structure problem are reduced to a system of two coupled nonlinear differential equations. An approximate analytical solution for this problem is obtained and investigated. Using this solution, all other flow variables are then given as explicit functions of the dimensionless coordinate x. The effects of heat conductivity on the distributions of velocity, pressure and entropy are discussed.

Key-words:- Pressure, Entropy, Shock wave, Heat, Fluid

1.Introduction

The structure of one-dimensional shock waves is one of the important problems in gas dynamics. This problem interested many authors [1]-[10] for some years, and the search for solutions of Navier -Stokes equations has always been their main concern.

Rankine [5] obtained an explicit solution of the Navier- Stockes equation by assuming the heat conductivity to be constant while the viscosity being neglected. Hamad [11] gave a solution for the case when heat conductivity is temperature dependent, and again the viscosity being neglected. Solutions in both cases are given in the form \( x = x(u) \), where \( x \) is the dimensionless distance coordinate and \( u \) is the dimensionless velocity. In these calculations, the other flow variables are then obtained as explicit functions of \( u \), and hence implicit functions in \( x \).

In this paper we shall seek solutions to the shock wave structure problem by assuming the heat conductivity to be temperature dependent and the viscosity being neglected. In the mean time, no rigorous procedure seems to exist for obtaining the exact solution which gives the flow variables as explicit functions of \( x \).

The aim of the present work is to give an approximate analytical solution to the basic
equations governing the system which agrees remarkably well with their numerical solution than the results obtained by Hamad.

This proposed analytic solution enables us then to obtain the properties inside the shock front in terms of the variable \( x \).

For the distributions of all the other variables inside the transition region, equations (18), (20) and (22) are obtained for the velocity, the pressure and the entropy. The effect of heat conductivity on the shock wave thickness and other characteristics are discussed.

2 Basic Equations

The fundamental equations describing the steady, one dimensional flow, parallel to \( x \)-axis of viscous, heat conducting, compressible fluid may be written in the form\[13].

\[
\frac{d}{dx'} (\rho' u') = 0 ,
\]

(1)

\[
\rho' u' \frac{du'}{dx'} + \frac{dp'}{dx'} - \frac{4}{3} \frac{d}{dx'} (\mu' \frac{du'}{dx'}) = 0 ,
\]

(2)

\[
\frac{d}{dx} \left[ \rho' u' \left( h' + \frac{u'^2}{2} \right) - \mu' \frac{d}{dx'} \left( \frac{k' h'}{\rho' c_p} - \frac{4}{3} \frac{u'^2}{2} \right) \right] = 0 .
\]

(3)

For a perfect gas, the equation of state is given by

\[
p' = \rho' RT' ,
\]

(4)

where \( \rho' \) is the density, \( u' \) the velocity, \( h' \) the enthalpy, \( p' \) the pressure, \( T' \) the absolute temperature, \( \mu' \) the coefficient of viscosity, \( k' \) the heat conductivity, \( R \) the universal gas constant, \( c_p \) and \( c_v \) the specific heats at constant pressure and constant volume, respectively.

When dealing with the structure of shock wave problem we must solve the system of equations (1)-(4) under the following boundary conditions:

\[
u' = v'_o \quad \text{and} \quad h' = h'_o , \quad \text{as} \quad x' \to -\infty ,
\]

where the subscript \( o \) corresponds to \( x' \to -\infty \), while subscript 1 corresponds to \( x' \to +\infty \).

Integrating equations (1)-(4) and using unprimed symbols for dimensionless quantities, equations (1)-(4) may be reduced to the following nondimensional equations.

\[
\rho u = 1 ,
\]

(5)

\[
\frac{4}{3} \mu u \frac{du}{dx'} = h + \gamma M^2_o u^2 - (1 + \gamma M^2_o) u ,
\]

(6)

\[
-\frac{\kappa'}{\kappa} \frac{dh}{dx'} = h + \frac{(\gamma - 1) M^2_o u^2}{2} \left[ 1 + \frac{(\gamma - 1) M^2_o}{2} \right] \frac{4}{3} (\gamma - 1) M^2_o \frac{u}{\mu} \frac{du}{dx'} ,
\]

(7)

where \( \rho = \frac{\rho'}{\rho'_o} \), \( u = \frac{u'}{u'_o} \), \( h = \frac{h'}{h'_o} \), \( p = \frac{p'}{p'_o} \), \( \mu' = \frac{\mu'}{\mu'_o} \), \( k' = \frac{k'}{\kappa' c u'_o} \) and \( M_o = \frac{u'_o}{c} \) is Mach number at \( x \to -\infty \), while \( c = \sqrt{\frac{\gamma p'_o}{\rho'_o}} \) is the sound velocity and \( \gamma = \frac{c_p}{c_v} \) is the ratio of specific heats.

In order to find the velocity \( u \) and the enthalpy \( h \) as \( x \to +\infty \), we consider

\[
\frac{du}{dx'} = \frac{dh}{dx'} = 0 \quad \text{in equations} \quad (6) \quad \text{and} \quad (7) .
\]

Consequently,
It is certainly difficult to obtain an exact analytical solution to the system of nonlinear differential equations (6)- (7) with the boundary conditions (8) and (9).

As for the temperature dependence of the heat conductivity, we will assume the following relation.

\[
\frac{h'}{h_o} = B = \left[ \frac{1 + \frac{(\gamma-1)}{2} M_o^2}{\frac{(\gamma+1)}{2} M_o^2} \right] \frac{\gamma M_o^2 - \frac{(\gamma-1)}{2}}{\left( \frac{\gamma+1}{2} \right)^2 M_o^2} .
\]

(9)

Substituting in equation (7) and putting \( u_0 = 0 \) in equation (6), we obtain \( \alpha = -2 (A+1) \), \( a_3 = -3 \beta A \), \( a_4 = \frac{1}{2} \beta^2 A \) and \( a_2 = -3 A \beta^2 (A+1) - \frac{1}{2} \beta (\beta + 6 A + 6) \).

Equations (11) and (12) are reduced to the following equation

\[
\frac{du}{dx} = \frac{l(u-1)(u-A)}{(2u - \beta)(Bu - u^2)} , \quad (17)
\]

where \( l = \frac{(\gamma+1)}{5 M_o^3 \sqrt{2\gamma}} \).

From equations (15) and (17) the reciprocal shock thickness for the case \( k = \) constant are tabulated for different values of \( M_o \) in Table I. Equation (16) can be solved exactly

The definition for the shock wave thickness was given by Prandtl \[4\] as

\[
\delta = (1 - A) \frac{du}{dx}_{\text{max}} . \quad (13)
\]

Now, it is convenient and quite natural to take the origin at the inflection point where we then have

\[
d^2u \bigg|_{x=0} = 0 , \quad \text{at } x = 0 . \quad (14)
\]

The relation between the velocity \( u_c \) at the inflection point and the velocity \( A \) at plus infinity when \( k = \) constant is

\[
u_c = \frac{1}{2} \left[ \beta + \sqrt{\beta^2 - 2\beta (A+1) + 4A} \right] , \quad (15)
\]

where \( \beta = (1 + \gamma M_o^2) \).

For \( k = h \), the velocity \( u_c \) is related to the velocity \( A \) in a form which is given by the exact solution of the quartic equation

\[
u^4 + a_1 \nu^3 + a_2 \nu^2 + a_3 \nu + a_4 = 0 \quad (16)
\]

where \( a_1 = -2 (A+1) \), \( a_3 = -3 \beta A \), \( a_4 = \frac{1}{2} \beta^2 A \) and \( a_2 = -3 A - \frac{3}{2} \beta (A+1) - \frac{1}{2} \beta (\beta + 6 A + 6) \).

3. Shock Wave Thickness
by the method of Ferrari. Now, using equations (13) and (17) the reciprocal shock wave thickness for the case \( k \neq \text{constant} \) are calculated for various values of \( M_0 \) and displayed in Table I.

Table: The reciprocal shock wave thickness \( 1/\delta \)

<table>
<thead>
<tr>
<th>( M_0 )</th>
<th>( k = \text{constant} )</th>
<th>( k \neq \text{constant} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.10</td>
<td>0.0705</td>
<td>0.0661</td>
</tr>
<tr>
<td>1.12</td>
<td>0.0830</td>
<td>0.0766</td>
</tr>
<tr>
<td>1.15</td>
<td>0.1013</td>
<td>0.0913</td>
</tr>
<tr>
<td>1.17</td>
<td>0.1135</td>
<td>0.1005</td>
</tr>
<tr>
<td>1.19</td>
<td>0.1258</td>
<td>0.1095</td>
</tr>
<tr>
<td>1.20</td>
<td>0.1331</td>
<td>0.1139</td>
</tr>
</tbody>
</table>

4. Structure of Weak Shock Wave

If instead of considering the shock wave as a surface of discontinuity, we shall look at it as a transition region in which the variation of the flow variables is continuous. The width of this transition region is considered to be extended from minus infinity to plus infinity. However, we shall show in the present work that the main change takes place in a region with finite length.

Although no exact analytical solution of equation (17) in the form \( u = u(x) \) is known, we can give an approximate analytical solution similar to that considered by Thompson et al. [8].

We suggest the following formula for the velocity

\[
\begin{align*}
  u &= \beta_1 + \beta_2 \tanh \beta_4 x + \beta_3 \sech \beta_4 x, \\
  \beta_4 &= \frac{1}{\beta_2} \left( \frac{du_c}{dx} \right) \\
\end{align*}
\]

where \( \beta_1, \beta_2, \beta_3 \) and \( \beta_4 \) are four unknown constants to be determined from the boundary conditions on \( u \) as \( x \rightarrow \pm \infty \) and at \( x = 0 \).

These constants are therefore

\[
\begin{align*}
  \beta_1 &= \frac{1}{2}(A + 1), \\
  \beta_2 &= \frac{1}{2}(A - 1), \\
  \beta_3 &= (u_c - \beta_1), \\
  \beta_4 &= \frac{1}{\beta_2} \left( \frac{du_c}{dx} \right) \\
\end{align*}
\]

where \( \frac{du_c}{dx} \) is given from equation (17) by putting \( u = u_c \).

This solution satisfies the boundary conditions and is to be exact at the origin.

From equation (15) and (16) we see that \( u_c \) exists if \( \beta < 2A \). Now, since the inflection point is essential for the existence of a shock wave, the previous condition means that a continuous solution joining \( u = 1 \) and \( u = A \) is only possible. But if \( \beta > 2A \) then a discontinuity must occur, and it should disappear if we consider the effect of compression viscosity[2].

Before employing our analytical solution, we verified first the accuracy of the numerical solution with the exact one obtained by Rankine where the origin is located at the inflection point. The results of these calculations are given in Table II, which shows that they are almost identical.
It is clear that the numerical solution gives results consistent with those obtained exactly for the case \( k = \text{constant} \). This is encouraging to compare our proposed analytical solution with the numerical solution for the case \( k \neq \text{constant} \). Also the results obtained by Hamad are compared with the numerical solution where the origin is located at the inflection point, as shown in Figures 1–3. From these figures, it is clear that our proposed solution (18) gives results in better agreement with the numerical solution rather than that obtained by Hamad.

Table II: A comparison between the exact solution and the numerical solution with \( k = \text{constant} \).

<table>
<thead>
<tr>
<th>( M_0 = 1.15 )</th>
<th>( M_0 = 1.2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 ) exact solution</td>
<td>( x_1 ) exact solution</td>
</tr>
<tr>
<td>-8.175</td>
<td>-8.174</td>
</tr>
<tr>
<td>-5.244</td>
<td>-5.242</td>
</tr>
<tr>
<td>-3.432</td>
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<td>0.0</td>
</tr>
<tr>
<td>0.755</td>
<td>0.757</td>
</tr>
<tr>
<td>1.950</td>
<td>1.952</td>
</tr>
<tr>
<td>3.680</td>
<td>3.682</td>
</tr>
<tr>
<td>( u ) numerical solution</td>
<td>( U ) numerical solution</td>
</tr>
<tr>
<td>0.98</td>
<td>0.98</td>
</tr>
<tr>
<td>0.96</td>
<td>0.96</td>
</tr>
<tr>
<td>0.94</td>
<td>0.94</td>
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<tr>
<td>0.92</td>
<td>0.92</td>
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<tr>
<td>0.85</td>
<td>0.84</td>
</tr>
<tr>
<td>0.83</td>
<td>0.80</td>
</tr>
</tbody>
</table>

The velocity profiles are presented in the following cases:

(i) \( M_0 = 1.1, \ \gamma = 5/3 \) with \( k = \text{constant} \) and \( k \neq \text{constant} \) as shown in Fig. 4.

(ii) \( M_0 = 1.19, \ \gamma = 5/3 \) with \( k = \text{constant} \) and \( k \neq \text{constant} \) as shown in Fig. 5.

In these Figures, it is obvious that the transition region increases as \( k \) is not constant, while \( M_0 \) is constant, but it decreases as \( M_0 \) increases.

After having \( u \) from equation (18) and using equation (5), we can obtain the density as explicit function of \( x \) in the form

\[
\rho = \left[ \beta_1 + \beta_2 \tanh \beta_4 x + \beta_3 \sec h^2 \beta_4 x \right]^{-1} \tag{19}
\]

Also for the pressure, we have

\[
p = \rho h = \gamma M_0^2 \left[ \beta - (\beta_1 + \beta_2 \tanh \beta_4 x + \beta_3 \sec h \beta_4 x) \right] \tag{20}
\]

The pressure distributions are presented in the following cases:

(i) \( M_0 = 1.12, \ \gamma = 5/3 \) with \( k = \text{constant} \) and \( k \neq \text{constant} \) as shown in Fig. 6.

(ii) \( M_0 = 1.17, \ \gamma = 5/3 \) with \( k = \text{constant} \) and \( k \neq \text{constant} \) as shown in Fig. 7.

(iii) \( M_0 = 1.12, \ M_0 = 1.17, \ M_0 = 1.2, \ \gamma = 5/3 \) with \( k \neq \text{constant} \) as shown in Fig. 8.

From these Figures, one can also conclude that the transition region for \( k \neq \text{constant} \) is greater than for \( k = \text{constant} \). In addition, it is clear that if \( M_0 \) increases, then pressure increases and the transition region decreases.

Furthermore, it will be of interest to determine the entropy \( S \). In general, for a perfect gas,

\[
S = c_p \ln (h) \ R \ln (p) \tag{21}
\]
Using equations (12), (18) and (20), we obtain the non-dimensional entropy

\[
\frac{\bar{S}}{\bar{S}} = \frac{S}{c_v} = \\
= \ln[ \gamma M_o \beta_1 + \beta_2 \tanh \beta_4 x + \beta_3 \sec h^2 \beta_4 x]^r \\
- [\gamma M_o^2 (\beta_1 + \beta_2 \tanh \beta_4 x + \beta_3 \sec h^2 \beta_4 x)^{r+1}] .
\]

(22)

The entropy profiles are then presented with:

(i) \( M_o = 1.1, M_o = 1.19, \gamma = 5/3 \) and \( k \neq \) constant as shown in Fig.9.

(ii) \( M_o = 1.19, \gamma = 5/3 \) with \( k = \) constant and \( k \neq \) constant as shown in Fig.10.

In Figure 9, it is clear that the maximum value of the entropy increases as \( M_o \) increases, but the transition region decreases as \( M_o \) increases. In the meanwhile, Figure 10 shows that the transition region for \( k \neq \) constant is greater than for \( k = \) constant. Also, the entropy increases with \( x \) until it reaches its maximum value at the point \( x \approx -0.9 \), then it decreases to its expected value as \( x \to +\infty \).

5. Results

We summarize now the main results obtained as follows:

1- From Prandtl definition (13) of the shock wave thickness we see that the velocity at the inflection point which satisfies the nonlinear differential equation (14) plays an important role in the computations of the transition region. The formula (15) for \( u_c \) is used to calculate the shock wave thickness for the case \( k = \) constant. But for the case \( k \neq \) of equation (16) such that \( A < u_c < 1 \) is used to compute the thickness of the shock wave. The results displayed in Table I show that for the same value of \( M_o \), the shock wave thickness \( \delta \) becomes greater when \( k \) is not constant.

2- In search for an analytical solution of the nonlinear differential equation (17), one is not able to find an exact solution in the form

\( u = u(x) \). The proposed approximate analytical solution (18) satisfies the boundary condition (8) and matches the exact solution for equation (17) at the point \( x = 0 \). The solution (18) yields results in good agreement with the numerical solution than those obtained by Hamad solution as shown in Figures 1-3 for the case \( k \neq \) constant. The numerical results are consistent with the exact results for the case \( k = \) constant.

3- We derived expressions for the distributions of velocity, pressure and entropy as explicit functions of \( x \) which are given by equations (18), (20) and (22).

4- Figures 4-10 where easily obtained from these expressions, and the main conclusion would read that as \( M_o \) increases then both pressure and entropy increases, but the
transition region decreases. Also, the transition region for the flow variables increases as \( k \) is not constant while \( M_0 \) is constant.

**References**

Fig. 1: A comparison between 
- Solid line: proposed solution
- Dotted line: numerical solution

for $M_e = 1.17$, $\gamma = 1.4$, $\alpha = \text{const.};$

- a) $\nu = \text{const.}$
- b) $\nu \neq \text{const.}$

Fig. 2: Comparison between 
- Solid line: proposed solution
- Dotted line: numerical solution

for $M_e = 1.17$, $\gamma = 1.4$, $\alpha = \text{const.};$

- a) $\nu = \text{const.}$
- b) $\nu \neq \text{const.}$

Fig. 3: The velocity distribution inside the transition region.

for $M_e = 1.17$, $\gamma = 1.4$, $\alpha = \text{const.};$

- Solid line: $\nu = \text{const.}$
- Dotted line: $\nu \neq \text{const.}$

Fig. 4: The velocity distribution inside the transition region.

for $M_e = 1.17$, $\gamma = 1.4$, $\alpha = \text{const.};$

- Solid line: $\nu = \text{const.}$
- Dotted line: $\nu \neq \text{const.}$

Fig. 5: The pressure distribution inside the transition region.

for $M_e = 1.17$, $\gamma = 1.4$, $\alpha = \text{const.};$

- Solid line: $\nu = \text{const.}$
- Dotted line: $\nu \neq \text{const.}$
Fig 7. The pressure distribution inside the transition region
for $M_e = 1.5$, $\gamma = \frac{5}{3}$; $\kappa = \text{const.}$.

Fig 8. The pressure distribution inside the transition region
for $x \neq \text{const.}$, $\gamma = \frac{5}{3}$; $M_e = 1.19$.

Fig 9. The entropy distribution inside the transition region
for $M_e = 1.19$, $\gamma = \frac{5}{3}$; $\kappa = \text{const.}$.

Fig 10. The entropy distribution inside the transition region
for $M_e = 1.19$, $\gamma = \frac{5}{3}$; $\kappa = \text{const.}$.