Sensitivity Problems for Impulsive Differential Inclusions*

T. F. FILIPPOVA
Department of Optimal Control
Institute of Mathematics and Mechanics of the Russian Academy of Sciences
16 S.Kovalevskaya Str., GSP-384, Ekaterinburg 620219
RUSSIA

Abstract: - The paper deals with the state estimation problem for impulsive control system described by differential inclusions with measures. The problem is studied under uncertainty conditions with set–membership description of uncertain variables which are taken to be unknown but bounded with given bounds (e.g., the model may contain unpredictable errors without their statistical description). Basing on the techniques of approximation of the discontinuous generalized trajectory tubes by the solutions of usual differential systems without measure terms we study the dependence of trajectory tubes of the impulsive differential inclusion on system parameters that define the constraints on initial data, a variation of impulses, restrictions on measurable controls and may be treated, e. g., as errors of the system modelling. So the main topic of the paper is to study the problems of the sensitivity of the considered differential system (and its set-valued solutions) with respect to possible errors.

Key-Words: - Differential equations, differential inclusions, impulsive control, set-valued solutions.

1 Introduction

In this paper the impulsive control problem for a dynamic systems with unknown but bounded initial states is studied. Such problems arise from mathematical models of dynamical and physical systems for which we have an incomplete description or a loose mode of time dependence of their generalized coordinates [1, 2, 3, 4, 5, 6, 7, 8].

We discuss an approach based on ideas of well known discontinuous time substitution [9]. Using the techniques of differential inclusions theory [10, 11, 12, 2] we study the dependence of set-valued solutions (trajectory tubes) of the impulsive differential inclusion on system parameters.

There is a long list of publications devoted to impulsive control optimisation problems, among them we mention here only the results related to the present investigation [13, 14, 6, 15, 16, 17]. The question arises how the results of classical control theory established for uncertain dynamical systems can be extended to the case of impulsive systems. Our study combines both approaches mentioned above and presents new results related to sensitivity analysis for differential uncertain systems of impulsive structure.

In this paper we consider a dynamic control system described by a differential equation with a usual control function $u(\cdot)$ and a measure (or impulsive control component) $v(\cdot)$:

$$dx(t) = f(t, x(t), u(t))dt \ + \qquad (1)$$

$$+ B(t, x(t), u(t))dv(t), \;\; x \in R^n, \;\; t_0 \leq t \leq T,$$

with unknown but bounded initial condition

$$x(t_0 - 0) = x^0, \;\; x^0 \in X^0. \qquad (2)$$

Here $u(t)$ is a usual (measurable) control with constraint

$$u(t) \in U, \quad U \subset R^m, \qquad (3)$$

and $v(t)$ is an impulsive control function which is continuous from the right, with constrained variation

$$\mathrm{Var}_{t \in [t_0, T]} \, v(t) \leq \mu, \qquad (4)$$

where μ is a given positive number.

So we consider here the case when the system control variable consists of two parts $w = \{u, v\}$ with the first component u being of the ordinary type and

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the second one v being the measure (or the impulsive control). We assume also that $f(t, x, u)$ and $n \times k$-matrix $B(t, x, u)$ are continuous in their variables.

One of the principal points of interest of the theory of control under uncertainty conditions [1, 4, 5] is to study the set of all solutions

$$x[t] = x(t, t_0, x^0, u, v)$$

to (1) - (2). The guaranteed estimation problem consists in describing the set

$$X[\cdot] = X(\cdot\,, t_0, X^0) = \bigcup_{\{u(\cdot),\, v(\cdot)\}} \{\ x[\cdot]\ \mid x[t] = x(t, t_0, x^0, u, v),\ x^0 \in X^0\ \} \tag{5}$$

of solutions to the system (1) - (2) under constraints (3) - (4) and the t – cross-section (t – cut) $X[t]$ of the $X[\cdot]$. The t – cut $X[t]$ is actually the attainability set (the reachable set) of the system at instant t from the initial set-valued "state" X^0. The set $X[t]$ may be treated also as the unimprovable set-valued estimate of the unknown state $x(t)$ of the system (1) - (2) under restrictions (3) - (4).

The mathematical background for investigations of set-valued estimates $X[t]$ ranging from theoretical schemes to numerical techniques may be found in [1, 4, 5, 18].

Thus, in this paper we actually apply the set-membership (bounding) approach to the description and to the studies of the information states for a nonlinear impulsive system with hard bounds on the uncertain initial states.

2 Problem Formulation

2.1 Uncertain Impulsive Systems with Parameter Disturbance

In this section we consider a dynamic system of a simpler type described by a differential equation with a measure

$$dx(t) = f(t, x(t), u(t))dt\ +\ B(t)dv(t), \tag{6}$$

with unknown but bounded initial condition

$$x(t_0 - 0) = x^0,\ \ x^0 \in X^0_\lambda. \tag{7}$$

Here $x \in R^n$, $t_0 \le t \le T$ and we assume that matrix function B in (1) depends on time only and $u(t)$ is a usual (measurable) control with the parameterized constraint

$$u(t) \in U_\lambda, \quad U_\lambda \subset R^m,$$

and $v(t)$ is an impulsive control function which is continuous from the right, with

$$\mathrm{Var}_{t\in[t_0,T]}\, v(t) \le \mu_\lambda.$$

Here λ is a finite dimensional parameter ($\lambda \in R^k$) that will be taken as tending to some fixed value λ_0 (obviously without loss of generality we may take $\lambda_0 = 0$).

We will study the dependence of trajectory tubes $X[\cdot]$ of the impulsive differential system on system parameters λ that define the constraints on initial data, a variation of impulses, restrictions on measurable controls. These parameter disturbances may be treated, e. g., as errors of the system modelling or as hard bounds on admissible system noises.

So the main topic of the paper is to study the problems of the sensitivity of the considered differential system (and its set-valued solutions) with respect to possible errors of modelling.

2.2 Reformulation of the Problem in Terms of the Differential Inclusions Theory

Along with the system (6)–(7) let us consider a new system, namely, a differential inclusion of the following type

$$dx(t) \in F_\lambda(t, x(t))dt + B(t)dv(t), \tag{8}$$

with the initial condition

$$x(t_0 - 0) = x^0,\ \ x^0 \in X^0_\lambda. \tag{9}$$

Here we use the notation

$$F_\lambda(t, x) = f(t, x, U_\lambda) = \cup\{\ f(t, x, u) \mid u \in U_\lambda\ \}$$

for the set-valued map F_λ in (8).

The introduction of this differential inclusion (8) which we will study further may be motivated by the well known results given by the control theory [10] and also by results of the theory of differential inclusions [11, 12].

Let $X_\lambda(\cdot, t_0, X^0_\lambda)$ be the set of all solutions to the inclusion (8) that emerge from X^0_λ (the trajectory tube related to all initial state vectors x^0 and all admissible impulse controls $v(t)$ and defined as in (5)). Denote

$$X_\lambda[t] = X_\lambda(t, t_0, X^0_\lambda)$$

to be its cross-section at instant t. The set $X_\lambda[t]$ is actually the reachable set of the impulsive differential inclusion (8) - (9) (or, equivalently, of the impulsive control system (6) - (7)) from the initial set X^0_λ taken at instant t.

So the main problem given in section 2.1 may be reformulated in terms of the differential inclusions theory that is in order to answer the main questions we need to find first the type of the dependance of set-valued solutions $X_\lambda[\cdot]$ of (8) on the variation of parameter vectors λ.

3 Problem Solution

3.1 Basic Assumptions

Assume that F_λ is a continuous multivalued map ($F_\lambda : [t_0, T] \times R^n \to \text{conv}R^n$) that satisfies the Lipschitz condition with constants $L_1, L_2 > 0$, namely

$$h(F_\lambda(t_1, x), F_\lambda(t_2, y)) \le L_1|t_1 - t_2| + L_2 \parallel x - y \parallel,$$

$$\forall x, y \in R^n, \;\; \forall t_1, t_2 \in [t_0, T]$$

where $\text{conv}R^n$ denotes the space of all compact and convex subsets of R^n and $h(A, B)$ is the Hausdorff distance for A, $B \subseteq R^n$, i.e.

$$h(A, B) = \max\,\{h^+(A, B), h^-(A, B)\},$$

with $h^+(A, B), h^-(A, B)$ being the Hausdorff semidistances between sets A, B,

$$h^+(A, B) = \sup\{d(x, B) \mid x \in A\}, h^-(A, B) =$$

$$h^+(B, A), \; d(x, A) = \inf\,\{\parallel x - y \parallel \; \mid y \in A\}.$$

Assume also the Lipschitz continuity of the matrix function $B(t)$

$$\parallel B(t_1) - B(t_2) \parallel \le L_3|t_1 - t_2|, \; \forall t_1, t_2 \in [t_0, T]$$

and also the so-called extendability condition ([10])

$$F_\lambda(t, x) \subset c(1 + ||x||)S,$$

$$S = \{\; x \in R^n \;\mid\; ||x|| \le 1 \;\}.$$

Definition 1. A function $x[t] = x(t, t_0, x^0)$ ($x^0 \in X^0_\lambda, t \in [t_0, T]$) will be called a solution (a trajectory) of the differential inclusion (8) if for all $t \in [t_0, T]$

$$x[t] = x^0 + \int_{t_0}^{t} \psi(t)dt + \int_{t_0}^{t} B(t)dv(t), \qquad (10)$$

where $\psi(\cdot) \in L^n_1[t_0, T]$ is a selector of F_λ, i.e.

$$\psi(t) \in F_\lambda(t, x[t]) \;\; a.e.$$

The last integral in (10) is taken as the Riemann-Stieltjes one. Following the scheme of the proof of the well-known Caratheodory theorem we can prove the existence of solutions $x[\cdot] = x(\cdot, t_0, x^0) \in BV^n[t_0, T]$ for all $x^0 \in X^0$ where $BV^n[t_0, T]$ is the space of n-vector functions with bounded variation at $[t_0, T]$.

3.2 Discontinuous Replacement of Time

Let us introduce a new time variable ([9, 17, 15]):

$$\eta(t) = t + \int_{t_0}^{t} dv(t),$$

and a new state coordinate

$$\tau(\eta) = \inf\,\{\; t \mid \eta(t) \ge \eta \;\}.$$

Consider the following auxiliary differential inclusion

$$\frac{d}{d\eta}\begin{pmatrix} z \\ \tau \end{pmatrix} \in G_\lambda(\tau, z) \qquad (11)$$

with the initial condition

$$z(t_0) = x^0, \;\; \tau(t_0) = t_0, \;\; t_0 \le \eta \le T + \mu_\lambda.$$

Here

$$G_\lambda(\tau, z) = \bigcup_{0 \le \nu \le 1} \left\{(1 - \nu)\begin{pmatrix} F_\lambda(\tau, z) \\ 1 \end{pmatrix} + \right.$$

$$\left. + \,\nu \begin{pmatrix} B(\tau) \\ 0 \end{pmatrix} \right\}. \qquad (12)$$

Let us prove two auxiliary results connected with two properties of the system (11).

Lemma 1. The map $G_\lambda(\tau, z)$ is convex and compact valued

$$G_\lambda : [t_0, T + \mu_\lambda] \times R^n \to \text{conv}R^{n+1}$$

and $G_\lambda(\tau, z)$ is Lipschitz continuous in both variables τ, z.

Proof. Let us fix some admissible λ, τ and z. and prove first that $G_\lambda(\tau, z)$ is a convex set. Take any g_1 and g_2 such that $g_i \in G_\lambda(\tau, z)$ $(i = 1, 2)$. Then from (12) we have that there exist numbers ν_1, ν_2 ($\nu_i \in$

$[0,1]$, $i = 1,2$) and vectors f_1, f_2 ($f_i \in F_\lambda(\tau, z)$, $i = 1,2$) such that

$$g_i = (1-\nu_i)\begin{pmatrix} f_i \\ 1 \end{pmatrix} + \nu_i \begin{pmatrix} B(\tau) \\ 0 \end{pmatrix}.$$

We need to prove that for any α ($\alpha \in [0,1]$)

$$\alpha g_1 + (1-\alpha) g_2 \in G_\lambda(\tau, z).$$

Denote $\nu_* = \alpha\nu_1 + (1-\alpha)\nu_2$ and

$$f_* = \frac{\alpha(1-\nu_1)}{(1-\nu_*)} f_1 + \frac{(1-\alpha)(1-\nu_2)}{(1-\nu_*)} f_2. \qquad (13)$$

For any $\alpha \in [0,1]$ we have $\nu_* \in [0,1]$ and the coefficients in (13) determine the convex combination of f_1, f_2. From the convexity of $F_\lambda(\tau, z)$ we conclude that $f_* \in F_\lambda(\tau, z)$ and therefore

$$\alpha g_1 + (1-\alpha) g_2 = (1-\nu_*)\begin{pmatrix} f_* \\ 1 \end{pmatrix} +$$

$$+ \nu_* \begin{pmatrix} B(\tau) \\ 0 \end{pmatrix} \in G_\lambda(\tau, z).$$

Let us prove now that the set $G_\lambda(\tau, z)$ is compact in R^{n+1}. It is easy to check that $G_\lambda(\tau, z)$ is bounded (because $F_\lambda(\tau, z)$ is assumed to be bounded). So we need to prove only that $G_\lambda(\tau, z)$ is closed, i.e. if we have

$$g_n = (1-\nu_n)\begin{pmatrix} f_n \\ 1 \end{pmatrix} + \nu_n \begin{pmatrix} B(\tau) \\ 0 \end{pmatrix},$$

with $f_n \in F_\lambda(\tau, z)$, $\nu_n \in [0,1]$ and $g_n \to g$ ($n \to \infty$) then the limit g should belong to $G_\lambda(\tau, z)$. Indeed, basing on the compactness of $F_\lambda(\tau, z)$ and $[0,1]$ we can extract converging subsequences

$$f_{n_k} \to f_* \in F_\lambda(\tau, z), \;\; \nu_{n_k} \to \nu_* \in [0,1].$$

Since $g_{n_k} \to g$ ($k \to \infty$), we have

$$g = (1-\nu_*)\begin{pmatrix} f_* \\ 1 \end{pmatrix} + \nu_* \begin{pmatrix} B(\tau) \\ 0 \end{pmatrix},$$

therefore $g \in G_\lambda(\tau, z)$. The Lipschitz continuity of $G_\lambda(\tau, z)$ follows directly from our basic assumptions given in the section 3.1. So Lemma 1 is proved.

In addition to the above assumptions we will assume further that the initial problem constraints depend continuously on a parameter λ ($\lambda \in R^k$) in such a way that

$$\lim_{\lambda \to 0} h(X^0_\lambda, X^0) = 0,$$

$$\lim_{\lambda \to 0} h(U_\lambda, U) = 0, \;\; \lim_{\lambda \to 0} \mu_\lambda = \mu.$$

The next auxiliary property provides the continuous dependance of the set-valued right-hand side $G_\lambda(\tau, z)$ of the differential inclusion (11) on a parameter λ.

Lemma 2. Under the above assumptions we have

$$\lim_{\lambda \to 0} h(G_\lambda(\tau, z), G_0(\tau, z)) = 0, \;\; \forall (\tau, z) \in R^{n+1}.$$

Proof. It is the direct consequence of the continuity of set-valued function $F_\lambda(\tau, z)$.

3.3 Main Results

Denote $w = \{z, \tau\}$ the extended state vector of the system (11) and consider trajectory tube of this differential inclusion (which has no measure or impulse components):

$$W_\lambda[\eta] = \bigcup_{w^0) \in X^0_\lambda \times \{t_0\}} w(\eta, t_0, w^0), \;\; t_0 \le \eta \le T + \mu_\lambda.$$

From Lemmas 1-2 and from the properties of trajectory tubes of ordinary differential inclusions [10, 4] we can conclude that the following result is valid.

Theorem 1. The limit equality

$$\lim_{\lambda \to 0} h(W_\lambda[T + \mu_\lambda], W_0[T + \mu]) = 0.$$

is true.

The next lemma explains the construction of the auxiliary differential inclusion (11).

Lemma 3. The set $X_\lambda[T]$ is the projection of $W_\lambda[T + \mu_\lambda]$ at the subspace of variables z:

$$X_\lambda[T] = \pi_z W_\lambda[T + \mu_\lambda].$$

The proof of this Lemma follows from the definition of the system (11).

Combining Theorem 1 and Lemma 3 we have the main result of this section concerning the continuity of the time cross-sections of trajectory tubes of the impulsive differential system on parameters.

Theorem 2. The following equality

$$\lim_{\lambda \to 0} h(X_\lambda[T], X_0[T]) = 0.$$

is true.

Remark. If we have the additional constraint on the range of vector measure $v(dt)$ defined by a closed and convex cone $K \subset R^k$ then instead of the auxiliary system (11) we should consider the following system

$$\frac{d}{d\eta}\begin{pmatrix} z \\ \tau \end{pmatrix} \in G_\lambda(\tau, z) \tag{14}$$

with

$$G_\lambda(\tau, z) = \bigcup_{0 \le \nu \le 1} \left\{ (1-\nu)\begin{pmatrix} F_\lambda(\tau, z) \\ 1 \end{pmatrix} + \right.$$

$$\left. + \nu \begin{pmatrix} B(\tau)(K \cap S) \\ 0 \end{pmatrix} \right\}.$$

The main continuity results given by Theorems 1 and 2 are valid in this case also (but the proofs of analogies of Lemmas 1-3 become more complicated).

3.4 The Reparametrization Approach

We study in this section the following measure differential inclusion

$$dx(t) \in F(t, x(t))dt + \tag{15}$$

$$+ \mathcal{G}(t, x(t))v(dt), \;\; \forall t \in [0,1]$$

$$x(0) = x_0, \;\; v(dt) \in \mathcal{K}$$

where

$$F : [0,1] \times R^n \to \text{conv}(R^n),$$

$$\mathcal{G} : [0,1] \times R^n \to R^{n\times k}, \;\; \mathcal{K} = C^*([0,1]; K)$$

and K is a positive pointed convex closed cone in R^k.

The first question that arises here is how to define the solution $x(\cdot)$ to (15) or to its differential inclusion interpretation (as it was done in Definition 1 above):

$$x(t) = x(0) + \int_0^t f(\tau, x(\tau))d\tau +$$

$$+ \int_0^t G(\tau, x(\cdot))v(d\tau), \;\; \forall t \in [0,1],$$

where f and G are suitable selections of F and $\mathcal{G}$.

The main problem in this context is to define correctly the interaction between the evolving trajectory and the impulsive integrating measure in (15). We underline here that the trajectories $x(t)$ are discontinuous and belong to a space of functions with bounded variation. Among many results related to treatment of dynamic systems of this kind let us mention the results devoted to a precise definition of a solution to (1) especially for the case $B = B(t, x)$ [17] and a long list of publications concerning the optimality conditions (e.g., [19, 15, 20, 16]).

The approach presented in [16, 7] enables a definition of a solution concept which ensures the well posedness of the control problem. The technique to study the parameter continuity conditions is based now on the reparameterization procedure that reduces the original problem to an auxiliary conventional one.

Definition 2 [16, 7]. The *reparameterized system* is

$$\dot{y}(s) \in F(\theta(s), y(s))\dot{\theta}(s) + G(\theta(s), y(s))\dot{\gamma}(s),$$

being $\dot{\gamma}(s)$ the variation rate of the control measure in the reparameterized time.

For a given v and a pair of measurable selections (f, G) of $(F, \mathbf{G})$, we have a set of reparametrized trajectories satisfying :

$$\mathcal{F}_{v,f,G} = \Big\{ y(\cdot) : \dot{y}(s) = f(\theta(s), y(s))\dot{\theta}(s) +$$

$$+ G(\theta(s), y(s))\dot{\gamma}(s), \dot{\gamma}(s) \in K,$$

$$(\dot{\theta}(s), \dot{\gamma}(s)) \in \Omega, \quad \text{a.e. in } [0,1] \Big\},$$

where

$$\Omega := \{ w \in R^+ \times K : \sum_{i=0}^q w_i = 1 \},$$

$\gamma(0) = 0$ and $\gamma(\eta(t)) = v([0,t])$, $\forall t \in [0,1]$.

Then, we apply existing conditions to this new auxiliary problem and express them in terms of the data of the original problem as it was done in the above section.

4 Conclusions

We considered here the continuous dependence of the trajectory tubes (and therefore of the reachable sets) of the impulsive control system on the variation of finite dimensional parameter that defines the system constraints.

The results were achieved by some consequent steps:

– first, we modify the control problem and introduce instead of it the impulsive differential inclusion with set-valued solutions;

– second, we invent and study the ordinary differential inclusion (in the extended state space) connected with the impulsive inclusion from step 1;

– third, we analyze the properties of set-valued solutions of the system of step 2, then we return backward and obtain the results concerning the solution of the initial impulsive problem.

In case when the singular system component depends on the state we apply the time reparametrization techniques which uses the concept of “robust solution” to measure driven differential inclusions.

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