On the Minimal Rank Completion Problem for Pattern Matrices (*)

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Abstract: This paper deals with the minimal rank completion problem when the partial matrix P has the specified entries equal to zero, and the remaining entries are nonzero variables of a field. We give an upper and a lower bound for the minimal rank of P and study when these bounds coincide. In this case, the minimal rank of P is characterized.
Finally, we obtain completions with minimal rank for some classes of pattern matrices as the class of pattern block band matrices which includes the pattern tridiagonal matrices.

Key-Words: Partial matrix, completion problem, minimal rank, pattern block band matrices.

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1 Introduction

Matrix completion problems involve study of *partial matrices*, that is, rectangular matrices some of whose entries are specified, and the remainder of whose entries are free variables of some indicated set. By a *completion* of a partial matrix we consider a specification of the free variables yielding a conventional matrix.

Consider a (zero) pattern matrix P, that is, P is a partial matrix whose specified entries are equal to zero, and the remaining entries are nonzero variables of a field. In this paper we ask for those completions of P with the lowest possible rank.

2 Definitions

We recall some concepts given in [2, 4]. For a positive integer n we denote by $\langle n\rangle$ the set $\{1, 2 \ldots, n\}$. Let A be an $m \times n$ matrix and let $\alpha \subseteq \langle m\rangle$ and $\beta \subseteq \langle n\rangle$. We denote by $A[\alpha \mid \beta]$ the submatrix of A whose rows are indexed by α and whose columns are indexed by β in the order listed.

Two matrices A and B are said to be *permutationally equivalent* if there exist permutation matrices P_1 and P_2 such that $A = P_1 B P_2$. We recall the *permanent* of an $n \times n$ matrix $A = (a_{ij})$ by $\text{per}(A) = \sum_{\sigma\in S_n} \left(\prod_{i=1}^{n} a_{i\sigma(i)}\right)$, where the summation extends over all permutations of $\langle n\rangle$.

Definition 1 *Let $G = \{V(G), E(G)\}$ be a bipartite graph where the vertex-set is $V(G) = V_R(G) \cup V_C(G)$, $V_R(G) = \{v_i, i \in \langle m\rangle\}$, and $V_C(G) = \{w_j, j \in \langle n\rangle\}$.*

A set of r edges $(v_{i_1}, w_{j_1}), \ldots, (v_{i_r}, w_{j_r})$ in $E(G)$ is said to be an r-matching *(between $\{v_{i_1}, \ldots, v_{i_r}\}$ and $\{w_{j_1}, \ldots, w_{j_r}\}$) if $v_{i_1}, \ldots, v_{i_r}$ are distinct and $w_{j_1}, \ldots, w_{j_r}$ are distinct.*

An r-matching is said to be a constrained r-matching *if it is the only r-matching in G between $\{v_{i_1}, \ldots, v_{i_r}\}$ and $\{w_{j_1}, \ldots, w_{j_r}\}$.*

Definition 2 *Any $m \times n$ matrix $A = (a_{ij})$ over a field $\mathbb{F}$ has associated a* bipartite graph $G_A = \{V(G_A), E(G_A)\}$. *The vertex-set $V(G_A)$ has $m+n$ vertices denoted by $V(G_A) = \{v_1, v_2 \ldots, v_m, w_1, w_2, \ldots, w_n\}$ and it is divided into two disjoint subsets $V(G_A) = V_R(G_A) \cup V_C(G_A)$ where $V_R(G_A) = \{v_i, i \in \langle m \rangle\}$ is associated with the rows of A, and $V_C(G_A) = \{w_j, j \in \langle n \rangle\}$ is associated with the columns of A. The edge-set of G_A is the set $E(G_A) = \{(v_i, w_j) \mid a_{ij} \neq 0\}$.*

Definition 3 *Let $\mathbb{F}$ be an arbitrary field. A matrix P is said to be a* (zero) pattern matrix*, with respect to $\mathbb{F}$, if each nonzero element is an independent indeterminate over $\mathbb{F}$. We denote the nonzero indeterminate entries by stars. Note that a pattern matrix P is a special* partial matrix *and it is also called a* generic matrix *with respect to $\mathbb{F}$ (see [1, pp. 294]). We use the same notation $P[\alpha \mid \beta]$ for a* subpattern *of P as we introduce above for a standard submatrix.*

The pattern of a matrix A over $\mathbb{F}$, pattern(A), *is the pattern matrix obtained by replacing every nonzero entry of A by an indeterminate element of $\mathbb{F}$. If pattern$(A) = P$, then A is also called a* completion matrix *of P. Note that* pattern$(A) =$ pattern(B) *if and only if $G_A = G_B$, therefore any $m \times n$ pattern matrix $P = (p_{ij})$ over $\mathbb{F}$ is associated with a bipartite graph G_P such that, $(v_i, w_j) \in E(G_P)$ if and only if the entry p_{ij} of P is an unspecified element of $\mathbb{F}$.*

The minimal rank *of a pattern matrix P over $\mathbb{F}$, $\mathrm{mr}(P)$, is defined by the number $\mathrm{mr}(P) = \min\{rank(A) \mid \text{pattern}(A) = P\}$. Similarly, $\mathrm{MR}(P) = \max\{rank(A) \mid \text{pattern}(A) = P\}$ denotes the* maximal rank *of P over $\mathbb{F}$.*

An $n \times n$ pattern matrix P is said to be formally singular (nonsingular) *over $\mathbb{F}$ if every matrix over $\mathbb{F}$ with pattern P is singular (nonsingular).*

3 A lower bound for the minimal rank

In this section we obtain a lower bound for the minimal rank of a pattern matrix P. We consider some results given by Hershkowitz and Schneider in [4, Section 3] and we extend them to a field with only two elements by using a different approach. The implication (1) $\Rightarrow$ (2) in Lemma 1 is proved by [1, Theorem 1.4.2] where a $(0,1)$-matrix A of order n with permanent equal to 1 is considered.

Lemma 1 *Let $\mathbb{F}$ be an arbitrary field. Let P be an $m \times n$ pattern matrix, and let r be a nonnegative integer with $1 \leq r \leq \min\{m, n\}$. Consider the following conditions:*

1. *G_P has a constrained r-matching.*
2. *P has an $r \times r$ subpattern permutationally equivalent to a triangular pattern with nonzero diagonal entries.*
3. *P has an $r \times r$ subpattern formally nonsingular over $\mathbb{F}$.*

We have, (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). Furthermore, if $\mathbb{F}$ is a field of at least three elements, then the three conditions are equivalent.

It is known (see [4, Example 3.6]) that condition (3) does not imply conditions (1) and (2) when $\mathbb{F}$ is the field $\{0, 1\}$ with two elements. Moreover, by Lemma 1 we obtain next Theorem.

Theorem 1 *Let $\mathbb{F}$ be an arbitrary field. Let P be an $m \times n$ pattern matrix, and let r be a nonnegative integer, $1 \leq r \leq \min\{m, n\}$. If P satisfies any condition of Lemma 1, then every matrix A over $\mathbb{F}$ with pattern P has rank$(A) \geq r$. Then, $\mathrm{mr}(P) \geq r$.*

Definition 4 *Let $\mathbb{F}$ be an arbitrary field. We define the* maximal triangle size *of an $m \times n$ pattern matrix P and denote it by $\mathrm{MT}(P) = r$, if r is the maximal nonnegative integer satisfying the condition (2) of Lemma 1.*

Now, we obtain a lower bound for the minimal rank of a pattern matrix P over $\mathbb{F}$.

Corollary 1 *Let $\mathbb{F}$ be an arbitrary field. Let P be an $m \times n$ pattern matrix, then $\mathrm{MT}(P) \leq \mathrm{mr}(P)$.*

The converse of Theorem 1 holds for the cases $r = 1$ or $r = \min\{m, n\}$ (see [4, Theorem 3.4, Corollary 3.5]).

Lemma 2 *Let $\mathbb{F}$ be an arbitrary field. Let P be an $m \times n$ pattern matrix, and let r be a nonnegative integer with $1 \leq r \leq \min\{m, n\}$. Consider the following conditions:*

1. G_P has an r-matching.

2. P has an $r\times r$ subpattern which is not formally singular over $\mathbb{F}$.

We have, (1) $\Leftarrow$ (2). Furthermore, if $\mathbb{F}$ is a field of at least three elements, then the two conditions are equivalent.

Proof:
(2) $\Rightarrow$ (1) Let $Q = P[i_1,\ldots,i_r \mid j_1,\ldots,j_r]$ be a subpattern of P which is not formally singular over $\mathbb{F}$, then there exists an $r\times r$ matrix A over $\mathbb{F}$ with pattern$(A) = Q$ and $\det(A)\neq 0$. We can suppose that $a_{i_1j_1}a_{i_2j_2}\ldots a_{i_rj_r}\neq 0$, that is, $(v_{i_1},w_{j_1}),\ldots,(v_{i_r},w_{j_r})$ is an r-matching in G_P. When $\mathbb{F}$ is a field with at least three elements, (1) $\Rightarrow$ (2) is established in [4, Lemma 3.7]. □

It is known (see [4, Example 3.8]) that condition (1) does not imply condition (2) in Lemma 2 when $\mathbb{F}$ is the field $\{0,1\}$ with two elements.

The next result follows easily from Lemma 2. See also [4, Lemma 3.7] for the necessary condition in the particular case $r=m=n$.

Theorem 2 *Let $\mathbb{F}$ be a field of at least three elements. Let P be an $m\times n$ pattern matrix, and let r be a nonnegative integer with $1\leq r\leq \min\{m,n\}$. P satisfies any condition of Lemma 2 if and only if there exists a matrix A over $\mathbb{F}$ with pattern P and rank$(A)\geq r$. Then,* $\mathrm{MR}(P)\geq r$.

By Theorem 2 we note that $\mathrm{MR}(P)=r$, if and only if, r is the maximal nonnegative integer satisfying any condition of Lemma 2. Note that from the conditions of Theorems 1 and 2 the results given in [4, Theorem 3.9] can be obtained.

4 An upper bound for the minimal rank

From Section 3 we have the inequalities, $\mathrm{MT}(P)\leq \mathrm{mr}(P)\leq \mathrm{MR}(P)$. Observe that $\mathrm{MR}(P)$ is, in general, a bad upper bound for $\mathrm{mr}(P)$, therefore we introduce a better upper bound for the minimal rank of P as the next Proposition 2 shows.

Definition 5 *Let $\mathbb{F}$ be an arbitrary field. Let P be an $m\times n$ pattern matrix. We call* a complete subpattern *of P to any subpattern of P with all nonzero entries.* A covering by complete subpatterns *of P is a collection of complete subpatterns which cover the nonzero elements of P. The covering is* minimum *if no covering has a smaller cardinality. The* minimum covering number *of P,* $\mathrm{bi}(P)$, *is the cardinality of a minimum covering of P. This definition appears in the bipartite graph theory as the* biclique covering number *[3, pp. 116] and [5, pp. 30]. We denote by $bi(P)=bi(G_P)$ the biclique covering number of the bipartite graph G_P associated with P. Recall that* $\mathrm{bi}(G_P)$ *is the smallest number of bicliques (i.e. complete bipartite subgraphs) which cover the edges of G_P.*

Proposition 1 *Let P be an $m\times n$ pattern matrix over a field $\mathbb{F}$ of at least* $\mathrm{bi}(P)+1$ *elements. Then* $\mathrm{mr}(P)\leq \mathrm{bi}(P)$.

Proof: We construct a completion matrix A_P of P with $\mathrm{rank}(A_P)\leq \mathrm{bi}(P)$, then $\mathrm{mr}(P)\leq \mathrm{rank}(A_P)\leq \mathrm{bi}(P)$ and the proposition follows.

Let $\mathrm{bi}(P)=s$, then there exist s complete subpatterns of P which cover the nonzero entries of P, and we obtain a decomposition of P in $m\times n$ pattern matrices $P_k=(p_{ij}^{(k)})$ with $\mathrm{bi}(P_k)=1$, $k\in\langle s\rangle$.

For each k, let $A_k=(a_{ij}^{(k)})$ be the (0,1)-completion matrix of P_k, that is,

$$a_{ij}^{(k)}=\begin{cases}1, & \text{if } p_{ij}^{(k)}=*\\ 0, & \text{if } p_{ij}^{(k)}=0\end{cases}$$

Note that $\mathrm{rank}(A_k)=1$. Let $A_P=\sum_{k=1}^{s}A_k$, then $\mathrm{pattern}(A_P)=P$ because $\mathbb{F}$ is a field of at least $s+1$ elements and every entry $a_{ij}\in A_P$ satisfies $0\leq a_{ij}\leq s$. Furthermore, $\mathrm{rank}(A_P)\leq \sum_{k=1}^{s}\mathrm{rank}(A_k)=s$ and the result follows. □

We remark that Proposition 1 does not hold in general for a field of at most $\mathrm{bi}(P)$ elements as one can see in [4, Example 3.6]. In that case we have $\mathrm{bi}(P)=2$, but $\mathrm{mr}(P)=3$ over the field $\{0,1\}$. Nevertheless, the minimum number of elements of $\mathbb{F}$ depends on the structure of the given pattern matrix $P=(p_{ij})$. This is, if each nonzero entry $p_{ij}\in P$ satisfies $p_{ij}^{(k)}\neq 0$, for some integer k, $k\in\langle k_{ij}\rangle$, $1\leq k_{ij}\leq s$, and let $t\geq k_{ij}$, for all $i\in\langle m\rangle$, $j\in\langle n\rangle$. Then the matrix A_P on Proposition 1 can be a completion matrix of P if $\mathbb{F}$ contains at least the elements of the set $\{0,1,2,\ldots,t+1\}$.

Proposition 2 *Let $\mathbb{F}$ be a field of at least three elements. Let P be an $m \times n$ pattern matrix, then* $\mathrm{bi}(P) \leq \mathrm{MR}(P)$.

Proof: Let $\mathrm{MR}(P) = r$, then by Theorem 2, G_P has an r-matching but G_P does not have any t-matching for $t > r$. By Frobenius-König Theorem (e.g. [4, Theorem 3.1]) there exists r lines (rows and/or columns) that contain all nonzero elements of P. Suppose, for instance, that $i_1, i_2, \ldots, i_{r_1}$ rows and $j_1, j_2, \ldots, j_{r_2}$ columns cover the nonzero entries of P, with $r_1 + r_2 = r$ and $r_i \geq 0$. Then, we construct the $r_1 + r_2$ complete subpatterns of P,

$$P[i_p \mid \beta],\ p \in \langle r_1 \rangle, \text{ and } p_{i_p j} \neq 0,\ \forall j \in \beta \subseteq \langle n \rangle$$

$$P[\alpha \mid j_p],\ p \in \langle r_2 \rangle, \text{ and } p_{i j_p} \neq 0,\ \forall i \in \alpha \subseteq \langle m \rangle.$$

By Definition 5, $\mathrm{bi}(P) \leq r$ and the result follows. □

We remark that Proposition 2 does not hold for the field $\{0, 1\}$ (see [4, Example 3.8]) because $\mathrm{bi}(P) = 3$, but $\mathrm{mr}(P) = \mathrm{MR}(P) = 2$.

Theorem 3 *Let P be an $m \times n$ pattern matrix over a field $\mathbb{F}$ of at least* $\mathrm{bi}(P) + 1$ *elements, then*

$$\mathrm{MT}(P) \leq \mathrm{mr}(P) \leq \mathrm{bi}(P) \leq \mathrm{MR}(P).$$

Remark 1 *Note that Theorem 3 is a direct consequence of Propositions 1 and 2 when the field has at least 3 elements. However, in the cases* $\mathrm{bi}(P) = 0$ *or* 1 *the result is clear and no change needed. In fact,* $\mathrm{bi}(P) = 0$ *if and only if* $P = 0$, *and* $\mathrm{bi}(P) = 1$ *if and only if P is permutationally equivalent to a pattern matrix with only one $r \times s$ complete subpattern for some $r \in \langle m \rangle$, $s \in \langle n \rangle$, and the remaining entries are zero. For this pattern matrix P, if $\mathbb{F}$ is a field of two elements then* $\mathrm{MT}(P) = \mathrm{mr}(P) = \mathrm{bi}(P) = \mathrm{MR}(P) = 1$ *and Theorem 3 holds.*

Remark 2 *Let P be a pattern matrix. We are looking for the equality* $\mathrm{MT}(P) = \mathrm{bi}(P)$ *of Theorem 3, because in this case the minimal rank of P is characterized. For the trivial cases,* $\mathrm{MT}(P) = 1$ *or* $\mathrm{MT}(P) = \min\{m, n\}$ *the equality holds. In fact,* $\mathrm{MT}(P) = 1$ *when P is a complete pattern or when P has at least one nonzero row, and every other nonzero row has the same pattern to this one, so P[nonzero rows | nonzero columns] is a complete subpattern that covers all nonzero entries of P, therefore* $\mathrm{bi}(P) = 1$.

The case $\mathrm{MT}(P) = \mathrm{bi}(P) = \min\{m, n\} = m$ *occurs when P has an $m \times m$ diagonal (or triangular) subpattern with nonzero diagonal entries. Furthermore, in the next section we introduce some classes of pattern matrices P where the equality* $\mathrm{MT}(P) = \mathrm{mr}(P) = \mathrm{bi}(P)$ *holds.*

5 Applications: Pattern Block Band Matrices

From now on, let P be a pattern matrix over a field $\mathbb{F}$ of at least $\mathrm{bi}(P) + 1$ elements.

Definition 6 *We call the $n \times n$ pattern matrix $P = (p_{ij})$ with $p_{ij} = 0$, if and only if, $\mid i - j \mid > h$, $0 \leq h \leq n - 1$, the* pattern band matrix *of* bandwidth $2h + 1$.

The trivial cases when $h = 0$ and $h = n - 1$ have been studied in Remark 2. From now on, we consider that h satisfies $1 \leq h \leq n - 2$, and obtain the following result.

Proposition 3 *Let P be the $n \times n$ pattern band matrix of bandwidth $2h + 1$, then* $\mathrm{mr}(P) = n - h$.

Proof: Note that $\mathrm{MT}(P) \geq n - h$ by considering the $(n - h) \times (n - h)$ triangular subpattern of P given by $P[1, 2, \ldots, n-h \mid h+1, \ldots, n]$. Otherwise, we consider the position of the nonzero entries of P, and denote by $B_k = P[k, k+1, \ldots, k+h \mid k, k+1, \ldots, k+h]$, $k \in \langle n-h \rangle$, the $(h+1) \times (h+1)$ complete subpatterns of P which cover the nonzero elements of P, then $\mathrm{bi}(P) \leq n - h$. By Theorem 3, we have $\mathrm{mr}(P) = n - h$. □

Remark 3 *Let P be the $n \times n$ pattern band matrix of bandwidth $2h + 1$, and let $\{B_k\}_{k=1}^{n-h}$ the $(h+1) \times (h+1)$ complete subpatterns in a minimum cover of P of Proposition 3. Then a minimal rank completion matrix $A_P = (a_{ij})_{i,j=1}^{n}$ of P is obtained in a similar way as Proposition 1 shows, that is,*

$$a_{ij} = \begin{cases} 0, & \text{if } a_{ij} \notin B_k, \forall k \in \langle n-h \rangle \\ 1, & \text{if } a_{ij} \in B_k, \text{ for one } k \in \langle n-h \rangle \\ \vdots & \vdots \\ n-h, & \text{if } a_{ij} \in \bigcap_{k=1}^{n-h} B_k \end{cases}$$

In fact, by Proposition 1 we have $\mathrm{rank}(A_P) \le \mathrm{bi}(P) = n - h$. *By Proposition 3 we obtain* $\mathrm{rank}(A_P) \ge \mathrm{mr}(P) = n-h$, *then* $\mathrm{rank}(A_P) = n-h$ *and* A_P *is a minimal rank completion matrix of the pattern band matrix* P *of bandwidth* $2h+1$.

Remark 4 *Let* P *be an* $n \times n$ *real pattern band matrix of bandwidth* $2h+1$. *If* h *is a positive integer satisfying*

$$\frac{n-1}{2} < h \le n-2$$

then the study of $\mathrm{mr}(P)$ *is equivalent to considering the smaller pattern band matrix of size* $(2(n-h)-1) \times (2(n-h)-1)$ *obtained by deleting the rows and columns with indices* $\{n-h, n-h+1, \dots, h\}$ *of* P *because they are duplicate rows (and columns).*

Definition 7 *We call an* $n \times n$ *pattern matrix* $P = (p_{ij})$ *a* pattern block band matrix *if*

1. $p_{ii} \neq 0$, *for all* $i \in \langle n \rangle$,
2. $p_{ij} \neq 0$ *if and only if* $p_{ji} \neq 0$,
3. $p_{ij} = 0$ *implies* $p_{ik} = 0$, *for* $k > j$.

The bandwidth *is* $2h+1$, *where* h *is the smallest number such that* $| i - j | > h$ *implies* $p_{ij} = 0$ *for all* $i, j \in \langle n \rangle$.

Note that the class of pattern block band matrices is the class of pattern matrices with block diagonal nonzero entries, and the rest of entries are zero. Furthermore, the diagonal blocks could have the same sizes, so this class includes the class of pattern band matrices where the diagonal blocks are all $(h+1) \times (h+1)$ *(see Definition 6).*

Remark 5 *In Propositions 4 and 5 we have a pattern block band matrix* P *with* $\mathrm{bi}(P) = s$, *and consider the position of the nonzero entries of* P, *then we denote by* $\{B_k\}_{k=1}^{s}$ *the* $w_k \times w_k$ *complete subpatterns which cover the nonzero elements of* P *in a similar way as the proof of Proposition 3 introduces for the case of pattern band matrices. Furthermore, we suppose that each* B_k *has intersection with* B_{k+1} *in a* $n_k \times n_k$ *complete subpattern for any* $n_k \in \langle w_k - 1 \rangle$, $k \in \langle s-1 \rangle$. *It is easy to obtain that* $h = \max_{1 \le k \le s}\{w_k\} - 1$.

Proposition 4 *Let* P *be an* $n \times n$ *pattern block band matrix of bandwidth* $2h+1$. *Then* $\mathrm{MT}(P) = \mathrm{mr}(P) = \mathrm{bi}(P)$.

Proof: Let $\mathrm{bi}(P) = s$. By Theorem 3 we only need to prove that $\mathrm{MT}(P) \ge s$, that is, we try to construct an $s \times s$ triangular subpattern of P with nonzero diagonal entries. By Remark 5, the triangular subpattern is given by the s rows

$$\{1, w_1 - n_1 + 1, \dots, \sum_{k=1}^{s-1} w_k - \sum_{k=1}^{s-1} n_k + 1\}$$

and the s columns of P

$$\{w_1, w_1 + w_2 - n_1, \dots, \sum_{k=1}^{s} w_k - \sum_{k=1}^{s-1} n_k\}$$

i.e., we consider the first row and the last column of each B_k, $k \in \langle s \rangle$. $\square$

Remark 6 *Let* P *be a pattern block band matrix. By Remark 3, it is easy to construct a minimal rank completion matrix of* P. *Otherwise, not only we can obtain* $\mathrm{bi}(P)$ *by graph theory, but also Proposition 5 gives the possible positive integers for* $\mathrm{bi}(P)$. *Note that the upper bound* $n-h$ *is the result when a pattern band matrix is considered (Proposition 3).*

Proposition 5 *Let* P *be an* $n \times n$ *pattern block band matrix of bandwidth* $2h+1$, *then*

$$\frac{n-1}{h} \le \mathrm{bi}(P) \le n-h, \quad h \in \langle n-2 \rangle.$$

Proof: Denote $s = \mathrm{bi}(P)$. By Remark 5 one can obtain the following relation between the size of P and the size of each B_k, $k \in \langle s \rangle$

$$\sum_{k=1}^{s} w_k - \sum_{k=1}^{s-1} n_k = n$$

Since $h = \max_{1 \le k \le s}\{w_k\} - 1$, then $\sum_{k=1}^{s} w_k \le s(h+1)$. From $\sum_{k=1}^{s-1} n_k \ge s-1$, we have

$$n = \sum_{k=1}^{s} w_k - \sum_{k=1}^{s-1} n_k \le s(h+1) - (s-1) = sh + 1$$

that is, $\frac{n-1}{h} \le \mathrm{bi}(P)$.

Now, let $w_{k_1} = \max_{1\leq k\leq s}\{w_k\} = h+1$, then

$n = \sum_{k=1}^{s} w_k - \sum_{k=1}^{s-1} n_k =$

$\sum_{k=1}^{k_1-1}(w_k - n_k) + h + 1 + \sum_{k=k_1}^{s-1}(w_{k+1} - n_k)$

$\geq (k_1 - 1) + h + 1 + (s - k_1) = h + s.$ □

Example 1 *Let P be the 11×11 real pattern block band matrix of bandwidth 9,*

$$P = \begin{pmatrix} * & * & * & & & & & & & & \\ * & * & * & * & * & & & & & & \\ * & * & * & * & * & * & * & & & & \\ & * & * & * & * & * & * & & & & \\ & * & * & * & * & * & * & * & & & \\ & & * & * & * & * & * & * & & & \\ & & * & * & * & * & * & * & * & * & \\ & & & & * & * & * & * & * & * & \\ & & & & & & * & * & * & * & \\ & & & & & & * & * & * & * & * \\ & & & & & & & & & * & * \end{pmatrix},$$

where the zero entries of P are not introduced. We can see that $\mathrm{bi}(P) = 6$, and

$$w_1 = 3, w_2 = 4, w_3 = 5, w_4 = 4, w_5 = 4, w_6 = 2$$
$$n_1 = 2, n_2 = 3, n_3 = 3, n_4 = 2, n_5 = 1.$$

By using the proof of Proposition 4, we construct the triangular subpattern $Q = P[1,2,3,5,7,10 \mid 3,5,7,8,10,11]$ of P, then $6 \leq \mathrm{MT}(P) \leq \mathrm{mr}(P) \leq \mathrm{bi}(P) = 6$. By Remarks 3 and 6, one can obtain the following completion matrix A_P of P (the zero entries are not introduced)

$$A_P = \begin{pmatrix} 1 & 1 & 1 & & & & & & & & \\ 1 & 2 & 2 & 1 & 1 & & & & & & \\ 1 & 2 & 3 & 2 & 2 & 1 & 1 & & & & \\ & 1 & 2 & 2 & 2 & 1 & 1 & & & & \\ & 1 & 2 & 2 & 3 & 2 & 2 & 1 & & & \\ & & 1 & 1 & 2 & 2 & 2 & 1 & & & \\ & & 1 & 1 & 2 & 2 & 3 & 2 & 1 & 1 & \\ & & & & 1 & 1 & 2 & 2 & 1 & 1 & \\ & & & & & & 1 & 1 & 1 & 1 & \\ & & & & & & 1 & 1 & 1 & 2 & 1 \\ & & & & & & & & & 1 & 1 \end{pmatrix}$$

As rank$(A_P) = 6$, then A_P is a minimal rank completion of P. Finally, for this example, the inequality in Proposition 5 is

$$\frac{10}{4} \leq 6 \leq 7$$

The lower bound $\mathrm{bi}(P) = 3$ is obtained if we consider an example with $w_1 = w_2 = 5$, $w_3 = 3$ and $n_1 = n_2 = 1$. Otherwise, the upper bound $\mathrm{bi}(P) = 7$ appears for $w_k = 5$, $k \in \langle 7\rangle$ and $n_j = 4$, $j \in \langle 6\rangle$, which represents an 11×11 pattern block band matrix of bandwidth 9.

6 Conclusion

In this work we obtain a lower and an upper bounds for the minimal rank of any rectangular pattern matrix. We apply these bounds to characterize the minimal rank for some classes of square pattern matrices as the class of pattern block band matrices which includes the class of pattern band matrices, and illustrate our theoretical results in the final example.

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