A Simplified Approach of Machines Interference Problem

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Abstract: This paper presents a simplified analytical approach of machines interference problem. For a group of S systems with repair served by M repairpersons two indicators must be evaluated: the systems availability and the percentage of working time for repairpersons. To evaluate these indicators two different approaches are available: analytical approach based on Markov models and simulation approach. To avoid the complexity of classic Markov models this paper proposes an approximate method that allows evaluating with accuracy the indicators previously defined. Simulation results obtained based on stochastic Petri net models confirm the efficiency of the analytical method proposed in this paper.

Key Words: Systems with repair, Availability, Markov chains, Approximation, Simulation, Stochastic Petri net

1 Introduction

The standard model for analyzing the availability of a system with repair is a Markov chain [1]. If a Markov model has m states and only steady state probabilities are required, m linear equations must be solved. For any sizable practical problem m becomes very large and the solution time becomes very long. If there are n elements in a system, the Markov model may contain as many as 2ⁿ states and, consequently, the resulting set of 2ⁿ coupled linear equations is formidable.

The two approaches available to deal with this problem are to either tolerate the largeness (largeness tolerance) or avoid it (largeness avoidance) [2]. In this work we chose the second approach.

2 Problem Description

Consider S identical systems with repair served by M repairpersons. System comprises n modules noted by m₁, m₂, ..., mₙ. The reliability model is given in Fig.1.

\[ \lambda_i \mu_i \]

Fig.1 Reliability model

As Fig.1 shows spare modules were not considered in this work. We assume that times to module failures and module repair are independent exponentially distributed random variables. The mean time to failure module \( m_i \) (MTTFₗ) is \( 1/\lambda_i \) and the mean time to repair module \( m_i \) (MTTRₗ) is \( 1/\mu_i \). In other words, \( \lambda_i \) is the failure rate for module \( m_i \), and \( \mu_i \) the repair rate. In order to evaluate the systems availability and the percentage of working time for repairpersons a discrete-time Markov model can be used. The system is assumed to be either up or down, with no partial or intermediate states. We also assume that a down module is perfectly recovered by repairing.

3 Example of Classic Approach

Consider two identical systems (S=2) served by a repairperson. Each system comprises three modules A, B, C for which failure rates \( \lambda_A, \lambda_B, \lambda_C \) and repair rates \( \mu_A, \mu_B, \mu_C \) are known. Because there are three identical systems, we consider only the states presented in Table 1. A down module is marked by underline. Symbol ‘’ indicates the module under repairing. For example, into the state (AB,A’B,AB) one system is up, module m₁ is down, module m₂ is down, and so on, module mₙ is down. When the number of systems is big the space of states becomes very large. Even if the method to be applied is simple in essence, we must have in view the complexity of Markov models [3]. The following section will show that classic approach is difficult to apply even for simple cases.
The transition matrix $M$ for this Markov model is given in Fig. 3. First, we must determine the steady state probabilities.

Let $p_i$ be the steady probability of state $S_i$, $i=1$ to 13, \( P=[p_1, p_2, \ldots, p_{13}]^T \), and \( O=[0, 0, \ldots, 0]^T \). The set of linear equations (1) gives the steady probability values.

\[
\begin{align*}
M \cdot P &= 0 \\
p_1 + p_2 + \cdots + p_{13} &= 1
\end{align*}
\]

The systems availability is given by Eq. (2) and the percentage of working time for repairpersons by Eq. (3).

\[
A=(p_1+2(p_2+p_3)+(p_4+p_5+p_6+p_7))/3 \cdot 100 \% \quad (2)
\]

\[
PWT = (1-p_1) \cdot 100 \% \quad (3)
\]

Even for this simple case, the Markov chain has thirteen states being quite complicated. For the case with four systems the Markov chain will have twenty-one states.
Table 1 States of Markov model presented in Fig. 2

<table>
<thead>
<tr>
<th>$S_i$</th>
<th>System states</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>3(AB)</td>
<td>all systems are up</td>
</tr>
<tr>
<td>$S_2$</td>
<td>2(AB),AB</td>
<td>two systems are up and one down</td>
</tr>
<tr>
<td>$S_3$</td>
<td>2(AB),A</td>
<td>one system is up and two are down</td>
</tr>
<tr>
<td>$S_4$</td>
<td>AB,2(AB)</td>
<td></td>
</tr>
<tr>
<td>$S_5$</td>
<td>AB,AB',AB</td>
<td></td>
</tr>
<tr>
<td>$S_6$</td>
<td>AB,AB',AB</td>
<td></td>
</tr>
<tr>
<td>$S_7$</td>
<td>AB,2(AB)</td>
<td></td>
</tr>
<tr>
<td>$S_8$</td>
<td>3(AB)</td>
<td>all systems are down</td>
</tr>
<tr>
<td>$S_9$</td>
<td>2(A'B),AB</td>
<td></td>
</tr>
<tr>
<td>$S_{10}$</td>
<td>A'B,2(AB)</td>
<td></td>
</tr>
<tr>
<td>$S_{11}$</td>
<td>2(AB),AB'</td>
<td></td>
</tr>
<tr>
<td>$S_{12}$</td>
<td>AB,2(AB')</td>
<td></td>
</tr>
<tr>
<td>$S_{13}$</td>
<td>3(AB)</td>
<td></td>
</tr>
</tbody>
</table>

4 A simplified Modeling Approach

To avoid the complexity of Markov model we propose a reduced model, focused on system states and not on module states, as in classic approach. In other words, we try to merge from the beginning the states in which the same number of systems down, indifferent of the modules out of order. For the example previously discussed, a reduced Markov model has four states, as shown in Table 1:
- all systems are up ($S_1$),
- two systems are up and one down ($S_2$ or $S_3$),
- one system is down and two are up ($S_4$ or $S_7$),
- all three systems are down ($S_8$ or $S_{13}$).

This reduced Markov model is presented in Fig.4, where $ds$ is the number of systems down.

![Fig. 4 Simplified graph of states for $S=3$](image)

Eqs. (2) and (3) show that the reduced Markov model allows to evaluate the systems availability and the percentage of working time for the repairpersons.

Now, the main problem is how can we determine, or even estimate with accuracy, the transition rates in the reduced Markov model? Suppose we have a repairable system with the reliability model given in Fig.1. The extended graph of states for this system is presented in Fig.5, and a reduced graph in Fig.6.

![Fig. 5 Extended graph of states for $S=1$](image)

![Fig. 6 Simplified graph of states for $S=1$](image)

The failure rate of the system is $\lambda$ and the repair rate is $\mu$. It is easily to demonstrate that time to system failure has also an exponential distribution with parameter $\lambda$, given by

$$\lambda = \sum_{i=1}^{n} \lambda_i.$$  (4)

Mean time to repair the system (MTTR) depends both on the repair rates $\mu_1+\mu_n$ and the failure rates $\lambda_1, \lambda_n$. When the system is down the probability that module $m_i$ is failed is $\pi_i = \lambda_i / (\lambda_1 + \lambda_2 + \cdots + \lambda_n) = \lambda_i / \lambda$.

Because mean time to repair module $m_i$ (MTTR$_i$) is $1/\mu_i$, the following equation can be used to determine MTTR.

$$MTTR = \sum_{i=1}^{n} \pi_i \cdot MTTR_i = \sum_{i=1}^{n} \frac{\pi_i}{\mu_i} = \frac{1}{\lambda} \sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}.$$  (5)

If we assume an exponential distribution for time to repair system then repair rate $\mu$ is equal to $1/\text{MTTR}$, and is given by

$$\mu = \frac{1}{\text{MTTR}} = \frac{1}{\frac{1}{\lambda} \sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}} = \frac{\sum_{i=1}^{n} \lambda_i}{\sum_{i=1}^{n} \lambda_i/\mu_i}.$$  (6)

5 Analytical Result

Consider $S$ identical systems served by $M$ repair persons. The reliability model for a system is given in section 2. To estimate the systems availability and the percentage of working time for repairpersons we propose the simplified Markov chain presented in Fig. 7. The number of systems up is noted by $us$.

![Fig. 7 Simplified graph of states for $S=3$](image)
Fig. 7 Reduced Markov model for S systems and M repairpersons

\[
\begin{array}{ccccccc}
\mu & 2\mu & \mu & \mu & \mu & \mu & 0 \\
\mu & \mu & \mu & \mu & \mu & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(S-1)\lambda & (S-2)\lambda & (S-3)\lambda & (S-4)\lambda & (S-5)\lambda & (S-6)\lambda & 0 \\
(S-M)\lambda & (S-M-1)\lambda & (S-M-2)\lambda & (S-M-3)\lambda & (S-M-4)\lambda & (S-M-5)\lambda & 0 \\
\lambda & \lambda & \lambda & \lambda & \lambda & \lambda & 0 \\
\end{array}
\]

Fig. 8 Transition matrix \( M \) of Markov model presented in Fig. 7

To determine the values of steady state probabilities the set of linear equations (7) can be used.

\[
\begin{align*}
-S\lambda p_0 + \mu p_1 &= 0 \\
S\lambda p_0 - ((S-1)\lambda + \mu) p_1 + 2\mu p_2 &= 0 \\
(S-1)\lambda p_1 - ((S-2)\lambda + 2\mu) p_2 + 2\mu p_3 &= 0 \\
(S-2)\lambda p_2 - ((S-3)\lambda + 3\mu) p_3 + 3\mu p_4 &= 0 \\
(S-M+1)\lambda p_{M-1} - ((S-M)\lambda + M\mu) p_M + M\mu p_{M+1} &= 0 \\
-(\lambda + M\mu) p_{S+1} + M\mu p_S &= 0 \\
p_0 + p_1 + \cdots + p_{S-1} + p_S &= 1
\end{align*}
\]

(7)

The set of Eqs. (7) is rewritten in (8) where \( \rho = \lambda / \mu \).

\[
\begin{align*}
p_k &= p_0 \frac{\rho^k}{k!} \prod_{i=0}^{k-1} (S - i), k = 1 + M \\
p_k &= p_M \left( \frac{\rho}{M} \right)^{k-M} \prod_{i=M+1}^{k} (S - i), k = M + 1 + S \\
p_M &= V = \frac{\rho^M}{M!} \prod_{i=0}^{M} (S - i) \\
p_0 &= \frac{1}{1 + \sum_{k=1}^{M} \left( \frac{\rho^k}{k!} \prod_{i=0}^{k-1} (S - i) \right) + \sum_{k=M+1}^{\infty} \left( \frac{\rho^k}{M!} \prod_{i=M+1}^{k} (S - i + 1) \right)}
\end{align*}
\]

(9)

(10)

(11)

(12)

Eq. (13) gives the systems availability, and Eq. (14), the percentage of working time for repairpersons. The following case study confirms these analytical results.
\[ A = \left( p_0 + \sum_{i=1}^{S-1} \left( \frac{S-i}{S} \cdot p_i \right) \right) \cdot 100\% \]  
(13)

\[ PWT = \left( \sum_{i=1}^{M-1} \left( \frac{i}{M} \cdot p_i \right) + \sum_{i=M}^{S} p_i \right) \cdot 100\% \]  
(14)

6 Case Study

Consider \( S \) weaving machines served by \( M \) weavers. Weaving is a stochastic process because the warp yarns and the filling yarn break off at aleatory moments. To model weaving as a discrete event process four primary random variables must be considered [4]:

- Time to break a warp yarn - let \( \lambda_1 \) be the break rate of warp yarns.
- Time to break the filling yarn - let \( \lambda_2 \) be the break rate of the filling yarn.
- Time to remedy a broken warp yarn - let \( \mu_1 \) be the remedy rate for a warp break.
- Time to remedy a broken filling yarn - let \( \mu_2 \) be the remedy rate for a filling break.

Suppose these random variables have exponential distribution laws. Take the following parameters, as presented in [5] and [6]: \( \lambda_1 = 1/775 \text{s}^{-1}, \lambda_2 = 1/1300 \text{s}^{-1}, \mu_1 = 1/70.4 \text{s}^{-1}, \mu_2 = 1/84.56 \text{s}^{-1} \).

Table 2 Approximate results for case study

<table>
<thead>
<tr>
<th>(M,S)</th>
<th>Machine availability %</th>
<th>Percentage of working time %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>analytical results</td>
<td>simulation results</td>
</tr>
<tr>
<td>S=1</td>
<td>86.51</td>
<td>86.54</td>
</tr>
<tr>
<td>S=2</td>
<td>84.97</td>
<td>84.90</td>
</tr>
<tr>
<td>S=3</td>
<td>83.14</td>
<td>83.10</td>
</tr>
<tr>
<td>S=4</td>
<td>80.99</td>
<td>80.90</td>
</tr>
<tr>
<td>S=5</td>
<td>78.47</td>
<td>78.37</td>
</tr>
<tr>
<td>S=6</td>
<td>75.55</td>
<td>75.50</td>
</tr>
<tr>
<td>M=2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S=4</td>
<td>86.20</td>
<td>86.15</td>
</tr>
<tr>
<td>S=5</td>
<td>85.45</td>
<td>85.42</td>
</tr>
<tr>
<td>S=6</td>
<td>84.22</td>
<td>84.16</td>
</tr>
<tr>
<td>S=8</td>
<td>82.41</td>
<td>82.35</td>
</tr>
<tr>
<td>S=10</td>
<td>79.91</td>
<td>79.88</td>
</tr>
<tr>
<td>M=3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S=6</td>
<td>86.43</td>
<td>86.41</td>
</tr>
<tr>
<td>S=9</td>
<td>86.07</td>
<td>86.04</td>
</tr>
<tr>
<td>S=12</td>
<td>85.31</td>
<td>85.27</td>
</tr>
</tbody>
</table>

Table 2 presents numerical results for the availability of weaving machines and the percentage of working time, for many pairs \((M,S)\).

Table 2 gives approximate analytical results, obtained with Eqs. (9)+ (14), and simulation results. The simulation program used in this work is based on modeling of the weaving process by a color stochastic Petri net [5]. We can remark a good accordance between analytical and simulation results. For comparison, Table 3 gives exact results for simple cases when a weaver serves one, two and three weaving machines, respectively. The case with three weaving machines was treated in section 1. Consequently, for \( M=1 \) and \( S=3 \) we solved first the set of Eqs. (1) and then we applied Eq. (2) and (3).

Table 3 Exact results for case study

<table>
<thead>
<tr>
<th>(M,S)</th>
<th>Machine availability %</th>
<th>Percentage of working time %</th>
</tr>
</thead>
<tbody>
<tr>
<td>S=1</td>
<td>86.51</td>
<td>13.49</td>
</tr>
<tr>
<td>S=2</td>
<td>84.96</td>
<td>26.49</td>
</tr>
<tr>
<td>S=3</td>
<td>83.13</td>
<td>38.87</td>
</tr>
</tbody>
</table>

Numerical results presented in this section show the efficiency of this simplified analytical method.

7 Limits of this Approach

Eqs. (9)+ (14) give exactly results only if the all repair rates \( (\mu_1, \mu_2, \ldots, \mu_n) \) are identical. Of course, in practical applications this condition is not satisfied and, consequently, we can obtain only approximate results. The question is how accurate are these results? In reduced Markov models, time to repair a system is a mix of different random variables with exponential distributions. If this new variable has also a nearly exponential distribution the analytical method we proposed gives very good results. For the weaving process presented in the case study, the empirical distribution for time to remedy a broken yarn is very close by the negative-exponential function \( 1-e^{-\mu t} \), with \( \mu \) given by Eq. (6).

When the empirical distribution for time to repair a system is far to an exponential distribution, the results given by Eqs. (9)+ (14) could have low accuracy.

Consider the reliability model given in section 2. Let \( \mu_{\text{max}} = \max \{ \mu_i \} \), \( i=1-n \), and \( \mu_{\text{min}} = \min \{ \mu_i \} \). If the ratio \( \mu_{\text{max}}/\mu_{\text{min}} \) is not greater than three then the results given by Eqs. (9)+ (14) could have low accuracy.

Let \( \mu_{\text{max}} = \mu_{\text{min}} \), \( i=1-n \), and \( \mu_{\text{max}} = \mu_{\text{min}} \). If the ratio \( \mu_{\text{max}}/\mu_{\text{min}} \) is not greater than three then the results given by Eqs. (9)+ (14) could have low accuracy.
For example, consider again the weaving process, but with $\mu_2=1/(70.4*3)$, so that the ratio $\mu_1/\mu_2$ is equal to 3. Fig. 9 presents the empirical distribution, $F_d(t)$, for time to remedy a broken yarn, and the negative-exponential function used in the approximate approach. Table 4 gives the results with regard to machine availability and percentage of working time. The differences between analytical and simulation results are up to 1.42% for availability, and up to 1.60% for percentage of working time (when $M=1$ and $S=5$).

Remark: The accuracy of analytical results in Table 4 is better when the problem is more complex.

![Fig.9 Empirical distribution function, $F_d(t)$, for time to remedy a broken yarn ($\mu_1=3\mu_2$)](image)

### Table 4 Results for the weaving process ($\mu_1=3\mu_2$)

<table>
<thead>
<tr>
<th>(M,S)</th>
<th>Machine availability %</th>
<th>Percentage of working time %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>analytical results</td>
<td>simulation results</td>
</tr>
<tr>
<td>$S=1$</td>
<td>79.79</td>
<td>79.77</td>
</tr>
<tr>
<td>$S=2$</td>
<td>76.66</td>
<td>76.05</td>
</tr>
<tr>
<td>$S=3$</td>
<td>72.91</td>
<td>71.80</td>
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<td>$S=4$</td>
<td>68.53</td>
<td>67.28</td>
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<td>$S=5$</td>
<td>63.61</td>
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<td>$S=6$</td>
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<td>76.59</td>
<td>76.21</td>
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<td>$S=8$</td>
<td>72.91</td>
<td>72.14</td>
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<td>$S=10$</td>
<td>67.84</td>
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<td>$S=12$</td>
<td>61.68</td>
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<td>$S=9$</td>
<td>77.93</td>
<td>77.60</td>
</tr>
<tr>
<td>$S=12$</td>
<td>74.85</td>
<td>74.35</td>
</tr>
</tbody>
</table>

### 8 Conclusions

This paper deals with the machines interference problem and presents a simplified analytic method based on discrete-time Markov models.

For a group of $S$ systems with repair served by $M$ repairpersons, in which all random variables are exponentially distributed, we give relationships to evaluate with accuracy the systems availability and the percentage of working time for repairpersons (Eqs. (9)-(14) in section 5).

A realistic case study concerning a weaving process is also presented in this paper. In this case study, the approximate results given by Eqs. (9)-(14) are very closed by the simulation results.

When the module repair rates ($\mu_1,\ldots,\mu_n$) are very scattered the accuracy of results could be low. In this case the user must check first if the empirical distribution for time to repair a system can be approximated by a negative-exponential function.

Generally, when the problem is more complex ($n$, $S$ or $M$ are greater) the accuracy is higher.

### References:


