On a Reliability Polynomial of Fibonacci and Some Other Graphs

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Abstract: This paper investigates the reliability polynomial of a Fibonacci graph. Trying to find an efficient way of computing the coefficients of this polynomial, we show that for \(1 \leq k \leq 3\) the absolute value of its \(k\)-th coefficient is equal to the number of \(k\)-edge mincuts in this Fibonacci graph. Further, we generalize this result to some other directed acyclic graphs.

Key-Words: coefficients of reliability polynomial, Fibonacci graph, mincut, operating path, probabilistic graph, reliability, reliability polynomial, size of mincut, st-connectedness, st-dag

1 Introduction

We consider the well-known problem of computing the probability that there exists an operating path from a source to a target in a stochastic network (probabilistic graph). The problem and its generalizations concerning directed and undirected graphs belong to the class of network reliability problems. Network reliability has been considered in a large number of papers. The problem is NP-complete in network size in the general case (see [1], [2], [6], [8], [9]). In this paper, we investigate the problem under special conditions in relation to a special graph called a Fibonacci graph.

The input to network reliability problems is a probabilistic graph \(G=(V,E)\), where \(V\) is a set of vertices and \(E\) is a set of edges, representing pairs of vertices. If the pairs are ordered (i.e., the pair \((v,w)\) is different from the pair \((w,v)\)), then we call the graph directed (digraph). All edges of a probabilistic graph can fail randomly and independently of one another, according to certain known probabilities. Hence, each edge \(e \in E\) is characterized by a known failure probability \(p_e\) and by an operation probability \(q_e = 1 - p_e\).

A two-terminal directed acyclic graph (st-dag) has only one source – \(s\) and only one target – \(t\). In an st-dag, every vertex lies on some path from \(s\) to \(t\).

For a probabilistic graph \(G\) and a specified vertex \(s\) and \(t\) of \(G\), we define the two-terminal reliability to be the probability that there exists an operating path (a path of operating edges) between \(s\) and \(t\). We call such a state a system operation and corresponding event is \(EP(s,t)\). A state when no operating path exists between \(s\) and \(t\) is said to be a system failure. The path is taken to be directed when \(G\) is directed, and there is always an operating path from \(s\) to itself. In the directed case, the problem of computing the probability \(Pr[EP(s,t)]\) is usually called st-connectedness.

We define a cutset or simply a cut to be a set of edges whose failure implies system failure. A size of a cut is a number of edges in the cut. A mincut is a minimal cut (the deletion of any edge from a mincut turns it to a non-cut).

There are two important facets of the st-connectedness problem. Firstly, the probability \(Pr[EP(s,t)]\) should be computed numerically. Secondly, the symbolic expression for \(Pr[EP(s,t)]\) is to be generated. In such a case, the probabilities \(p_e\) and \(q_e\) related to the corresponding edges \(e \in E\) become the parameters of the expression.

In the special case, when all edge failure probabilities \(p_e\) are equal to the same value \(p\) and, consequently, all edge operation probabilities are equal to the same value \(q = 1 - p\), the probability of system operation is called the reliability polynomial, and is denoted \(Rel(s, t, p)\). Letting \(\sigma_k\) be the number of \(k\)-edge cuts (leaving \(m - k\) operational edges) the probability of system operation for an \(m\)-edge st-dag can be presented as

![Fig.1. An n-vertex Fibonacci graph.](image-url)
\[ \text{Rel}(s, t, p) = 1 - \sum_{k=1}^{n} \sigma_k p^k (1 - p)^{n-k}. \]  

(1)

Actually, in (1) the two-terminal reliability is a polynomial only in \( p \). Generally, to compute \( \sigma_k \) is a problem of exponential time complexity. As follows from formula (1), the degree of the reliability polynomial does not exceed the number of edges in the graph. For this reason, the size of the symbolic expression for \( \text{Rel}(s, t, p) \) is \( O(m) \) for any \( m \)-edge st-dag.

The notion of a Fibonacci graph \( (FG) \) was introduced in [3]. In such an st-dag, two edges leave each of its \( n \) vertices except the two final vertices \((n-1) \) and \( n \). Two edges leaving the \( i \) vertex \((1 \leq i \leq n-2)\) enter the \( i+1 \) and the \( i+2 \) vertices. The single edge leaving the \( n-1 \) vertex enters the \( n \) vertex. No edge leaves the \( n \) vertex. This graph is illustrated in Fig. 1. It can be easily shown that the number of edges in an \( n \)-vertex \( FG \) is \( m = 2n - 3 \).

The paper [4] presents a method for the solution of the st-connectedness problem for an \( n \)-vertex \( FG \). It is shown that this problem can be solved numerically in \( O(n^2) \) time. Also, we proved in [4] that the total number of mincuts in an \( n \)-vertex \( FG \) equals \( \left[ \frac{n^2}{4} \right] \). In [5], we compute numbers of mincuts of all sizes in \( FG \). It is shown that an \( n \)-vertex \( FG \) has mincuts of all sizes from 2 to \( \left[ \frac{n+1}{2} \right] \). In this paper, we investigate the coefficients of \( \text{Rel}(s, t, p) \) of an \( n \)-vertex \( FG \).

### 2 Coefficients of a Reliability Polynomial of a Fibonacci Graph

The coefficients of the reliability polynomial \( \text{Rel}(s, t, p) \) of an \( FG \) can be computed by formula (1). As noted above, the reliability polynomial degree does not exceed the number of edges in the graph. Since the number of edges in an \( FG \) depends linearly on the number of vertices in the graph, then the size of the symbolic expression for \( \text{Rel}(s, t, p) \) is \( O(n) \) for an \( n \)-vertex \( FG \).

It is of interest to find the efficient way of generating the symbolic expression for \( \text{Rel}(s, t, p) \) of an \( FG \). The expression can be constructed directly based on (1) but the time of its generation will be exponential. Another way to generate is in use of the algorithm of Provan and Ball [7] for computing \( \text{Pr}[EP(s, t)] \). The algorithm determines two-terminal reliability in time that is polynomial in the number of mincuts of the graph. As shown in [4], the total number of mincuts in an \( FG \) is polynomial, and \( \text{Pr}[EP(s, t)] \) of a probabilistic \( FG \) can be computed in a polynomial time. However, the symbolic expression for \( \text{Pr}[EP(s, t)] \) of an \( FG \) generated by formulae of Provan and Ball [7] increases exponentially as the number of vertices in the graph grows. It is clear that the order of an expression size is a lower bound of the running time necessary for the expression generation.

The solution of the problem could be found by analysis of coefficients of \( \text{Rel}(s, t, p) \) calculated by one of the above mentioned ways. Consider the data presented in Tables 1 and 2.

Table 1 includes coefficients of \( \text{Rel}(s, t, p) \) for an \( n \)-vertex \( FG \) \((n = 2, 3, ..., 9)\). Here \( d \) is a degree of a polynomial item. The number found in the cell with the coordinates \( x \) and \( y \) is a coefficient near the \( x \) degree item of \( \text{Rel}(s, t, p) \) corresponding to an \( y \)-vertex \( FG \). For instance, \( \text{Rel}(s, t, p) \) of a 7-vertex \( FG \) is

\[
1 - 2p^2 - 6p^3 + 3p^4 + 28p^5 - 33p^6 - 4p^7 + 21p^8 - 7p^9 - 2p^{10} + p^{11}.
\]

Table 2 includes numbers of \( k \)-edge mincuts \((k = 1, 2, ..., 5)\) in an \( n \)-vertex \( FG \) \((n = 2, 3, ..., 9)\). Corresponding values are computed using the following theorem proved in [5] (we denote a set of all mincuts of the size \( k \) in an \( n \)-vertex \( FG \) as \( CF_k(n) \)).

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Table 1. Coefficients of a reliability polynomial for a Fibonacci graph.
Theorem 1. For \( n \geq 3 \), the mincuts of an \( n \)-vertex \( FG \) are characterized as follows:

1. A mincut of the maximum size in an \( n \)-vertex \( FG \) has \( \left\lceil \frac{n+1}{2} \right\rceil \) edges.
2. An \( n \)-vertex \( FG \) has mincuts of all sizes in the range from 2 to \( \left\lfloor \frac{n+1}{2} \right\rfloor \) and no other mincuts.
3. The number of mincuts of all sizes in an \( n \)-vertex \( FG \) is described by the following formulæ:

\[
|CF_3(3)| = 2,
|CF_3(4)| = 3, |CF_3(4)| = 1,
|CF_3(6)| = 2, |CF_3(6)| = 6, |CF_3(6)| = 1,
\]

for odd \( n \geq 5 \):

\[
|CF_3(n)| = 2,
|CF_3(n)| = n - 1 - 2(k - 3):
\]

\[
k = 3, 4, \ldots, \frac{n-1}{2}, \frac{n+1}{2},
\]

for even \( n \geq 8 \):

\[
|CF_3(n)| = 2,
|CF_3(n)| = n - 1 - 2(k - 3):
\]

\[
k = 3, 4, \ldots, \frac{n-2}{2}, \frac{n-2}{2},
\]

\[
|CF_{\frac{n+1}{2}}(n)| = 6,
|CF_{\frac{n+1}{2}}(n)| = 1.
\]

One can easily conjecture the following law, which we present as a theorem.

Theorem 2. A coefficient near \( p^k \) item in \( Rel(s, t, p) \) of an \( n \)-vertex \( FG \) \((n \geq 5)\) is equal to \(-1\) multiplied by the number of \( k \)-edge mincuts in the \( FG \) for \( k = 1, 2, 3 \). It holds for any \( n \) if we consider \( k = 1, 2 \) only.

Proof. We begin from \( k = 1 \). The 2-vertex \( FG \) has a single cut which is a single-edge mincut. \( Rel(s, t, p) = 1 - p \) for such an \( FG \). An \( n \)-vertex \( FG \) \((n \geq 2)\) has no single-edge cut (and no single-edge mincut correspondingly) and therefore, as follows from formula (1), a coefficient near the first degree item in its \( Rel(s, t, p) \) is equal to 0. Consider \( k = 2 \). As stated in Theorem 1 a mincut of the minimum size in an \( FG \) containing more than two vertices is a 2-edge mincut. It is clear that a mincut of the minimum size in an st-dag is a cut of the minimum size in this st-dag and vice versa. For this reason, the number of 2-edge mincuts in an \( n \)-vertex \( FG \) is equal to the number of 2-edge cuts in this \( FG \). Since a coefficient near \( p \) equals 0 in our case, then as follows from (1), a coefficient near \( p^2 \) in \( Rel(s, t, p) \) of an \( n \)-vertex \( FG \) is equal to \(-\sigma_2\), and hence, it is equal to \(-|CF_3(n)|\). Move on to \( k = 3 \). An \( n \)-vertex \( FG \) has \(|CF_3(n)|\) 3-edge mincuts. Besides, it has 3-edge cuts which are not mincuts. These cuts can be derived by augmentation of edges in corresponding 2-edge mincuts. Adding a supplementary edge to each of these mincuts generates a 3-edge cut. According to Theorem 1, there are two 2-edge mincuts in an \( n \)-vertex \( FG \) for \( n \geq 5 \). They are \( \{(1,2), (1,3)\} \) and \( \{(n-1,n), (n-2,n)\} \), and they are not intersected (have no common edge). Hence, such an \( m \)-edge \( FG \) has \( 2(m-2) \) 3-edge cuts which are not mincuts and the general number of 3-edge cuts in this graph is \( \sigma_3 = |CF_3(n)| + 2(m-2) \). In our case, the first nonzero member of the sum

\[
\sum_{k=0}^{m} \sigma_k p^k (1-p)^{m-k} \text{ in } (1) \text{ is } 2p^3(1-p)^{m-2} = 2p^3(1-(m-2)p + \ldots). \text{ The coefficient near } p^3 \text{ is } -2(m-2). \]

Therefore, the total coefficient near \( p^3 \) in \( Rel(s, t, p) \) of an \( FG \) is \(-(|CF_3(n)| + 2(m-2) - 2(m-2)) = -|CF_3(n)|\). When \( n < 5 \) we have intersected 2-edge mincuts and, besides, the 4-vertex \( FG \) has three 2-edge cuts (see Theorem 1). Thus, the proof of the theorem is complete. ■

Corollary 3. In \( Rel(s, t, p) \) of an \( n \)-vertex \( FG \), a coefficient near \( p^3 \) is equal to \(-2 \) for \( n = 3 \) or \( n \geq 5 \), and a coefficient near \( p^3 \) is equal to \(-n \) for \( n = 5 \) or \( n \geq 7 \).

Proof. Immediate from Theorems 1 and 2. ■
Hence, the laws for the coefficients near items of higher degrees are not shown here. An interesting open problem is to tally up all the coefficients of the reliability polynomial of a Fibonacci graph using information on the number of mincuts of a given size only. However, supposing the failure probability \( p \) is sufficiently small we may neglect higher degrees for estimation of numeric value of Rel\((s, t, p)\).

Further examination of Table 1 leads to the following conjectures concerning the high degree coefficients of Rel\((s, t, p)\) of a Fibonacci graph:

1. An absolute value of a coefficient near \( p^{2n-3} \) item (an item of the highest degree) in Rel\((s, t, p)\) of an \( n \)-vertex \( FG \) is equal to 1. In addition, the coefficient is negative for even \( n \) and is positive for odd \( n \).
2. An absolute value of a coefficient near \( p^{2n-4} \) item in Rel\((s, t, p)\) of an \( n \)-vertex \( FG \) \((n \geq 3)\) is equal to 2. In addition, the coefficient is positive for even \( n \) and is negative for odd \( n \).
3. An absolute value of a coefficient near \( p^{2n-5} \) item in Rel\((s, t, p)\) of an \( n \)-vertex \( FG \) \((n \geq 4)\) is equal to \( 2(n - 4) + 1 \). In addition, the coefficient is positive for even \( n \) and negative for odd \( n \).
4. An absolute value of a coefficient near \( p^{2n-6} \) item in Rel\((s, t, p)\) of an \( n \)-vertex \( FG \) \((n \geq 5)\) is equal to \( 6(n - 4) + 3 \). In addition, the coefficient is negative for even \( n \) and is positive for odd \( n \).

### 3 Generalization of Theorem 2 to Some Other St-dags

Theorem 2 can be generalized to some other st-dags as follows.

**Theorem 4.** Suppose \( G \) is an \( m \)-edge st-dag that has no mincut of the size less than \( l \) \((l \geq 1)\). In such a case, a coefficient near \( p^k \) item \((1 \leq k < l)\) in Rel\((s, t, p)\) of \( G \) is equal to 0, and a coefficient near \( p^l \) item in Rel\((s, t, p)\) of \( G \) is equal to \(-1\) multiplied by the number of \( l \)-edge mincuts in \( G \). If, besides, no two \( l \)-edge mincuts of \( G \) have more than \( l - 2 \) common edges then a coefficient near \( p^{l+1} \) item in Rel\((s, t, p)\) of \( G \) is equal to \(-1\) multiplied by the number of \( l+1 \)-edge mincuts in \( G \).

**Proof.** It is clear that \( G \) has no cut of the size less than \( l \). That is, \( \sigma_k = 0 \) for \( k < l \) in formula (1). Hence, all coefficients near \( p^k \) items \((1 \leq k < l)\) in Rel\((s, t, p)\) of \( G \) are equal to 0. As noted in the proof of Theorem 2, a mincut of the minimum size in an st-dag is a cut of the minimum size in this st-dag and vice versa. For this reason, the number of \( l \)-edge mincuts in \( G \) (denoted \( \mu_l \)) is equal to the number of \( l \)-edge cuts in \( G \). Since a coefficient near \( p^k \) \((1 \leq k < l)\) equals \( 0 \) in our case, then as follows from (1), a coefficient near \( p^l \) in Rel\((s, t, p)\) of \( G \) is equal to \(-\sigma_l \), and hence, it is equal to \(-\mu_l \). Suppose \( G \) has \( \mu_{l+1} \) \( l+1 \)-edge mincuts. Besides, \( G \) has \( l+1 \)-edge cuts which are not mincuts. These cuts can be derived by augmentation of edges in corresponding \( l \)-edge mincuts. If no two \( l \)-edge mincuts of \( G \) have more than \( l - 2 \) common edges, then adding a supplementary edge to each of \( l \)-edge mincuts generates a unique \( l+1 \)-edge cut. Hence, \( G \) has \( \mu_l(m - l) \) \( l \)-edge cuts which are not mincuts and the general number of \( l+1 \)-edge cuts in \( G \) is \( \sigma_{l+1} = \mu_{l+1} + \mu_l(m - l) \). In our case, the first nonzero member of the sum

\[
\sum_{k=0}^{n} \sigma_k p^k (1 - p)^{n-k} = (1 - \sigma_k p^k (1 - p)^{n-k}) = -\sigma_l p^l (1 - (m - l)p + \ldots). \]

The coefficient near \( p^{l+1} \) is \(-\sigma_l(m - l) \). On the other hand, \( \sigma_l = \mu_l \). Thus, the total coefficient near \( p^{l+1} \) in Rel\((s, t, p)\) of \( G \) is \(-\mu_{l+1} + \mu_l(m - l) - \sigma_l(m - l) = -\mu_{l+1} \). The proof of the theorem is complete.

For example, Rel\((s, t, p)\) of the st-dag from Fig. 2 is

\[
1 - p^2 - 4p^3 + 6p^4 + 2p^5 - 8p^6 + 5p^7 - p^8.
\]

This st-dag has no single-edge mincut, one 2-edge mincut \((\{g, h\})\), and four 3-edge mincuts \((\{a, c, h\}, \{b, c, h\}, \{d, f, h\}, \{e, f, h\})\). Hence, no 2-edge mincuts of this st-dag have common edges. Thus, coefficients near \( p, p^2, p^3 \) items of its Rel\((s, t, p)\) are equal to 0, \(-1, -4\), respectively.

![Fig. 2. An st-dag.](image)

**Corollary 5.** In any st-dag \( G \), a coefficient near \( p \) item in Rel\((s, t, p)\) of \( G \) is equal to \(-1\) multiplied by the number of single-edge mincuts in \( G \).

**Proof.** Immediate from Theorem 4.

**Corollary 6.** Suppose an st-dag \( G \) has only one single-edge mincut. Then a coefficient near \( p \) item in
Rel\((s, t, p)\) of \(G\) is equal to \(-1\) and a coefficient near \(p^2\) item in \(\text{Rel}(s, t, p)\) of \(G\) is equal to \(-1\) multiplied by the number of 2-edge mincuts in \(G\).

**Proof.** The statement about a coefficient near \(p\) follows immediate from Corollary 5. Since there is only one single-edge mincut, then there are no single-edge mincuts with common edges and, by Theorem 4, a coefficient near \(p^2\) item in \(\text{Rel}(s, t, p)\) of \(G\) is equal to \(-1\) multiplied by the number of 2-edge mincuts in \(G\). ■

For example, an st-dag from Fig. 3 has only one single-edge mincut (\(\{a\}\)) and four 2-edge mincuts (\(\{b, c\}, \{b, e\}, \{d, c\}, \{d, e\}\)). Its \(\text{Rel}(s, t, p)\) is

\[
1 - p - 4p^2 + 8p^3 - 5p^4 + p^5.
\]

According to Corollary 6, a coefficient near \(p\) equals \(-1\) and a coefficient near \(p^2\) equals \(-4\).

![Fig.3. An st-dag with only one single-edge mincut.](image)

Another st-dag pictured in Fig. 4 has two single-edge mincuts (\(\{a\}, \{f\}\)) and four 2-edge mincuts (\(\{b, c\}, \{b, e\}, \{d, c\}, \{d, e\}\)). Its \(\text{Rel}(s, t, p)\) is

\[
\]

By Corollary 5, a coefficient near \(p\) equals \(-2\). However, an absolute value of a coefficient near \(p^2\) is not equal to a number of 2-edge mincuts.

![Fig.4. An st-dag with two single-edge mincuts.](image)

### 4 Conclusion and Future Work

In this paper we investigated the reliability polynomial of an \(n\)-vertex Fibonacci graph. The coefficients near the first three items of the reliability polynomial have been computed. This result was generalized to some other two-terminal directed acyclic graphs. More work is needed to tally up all the coefficients of the reliability polynomial.

**References:**


