Attenuation of Stress Wave Propagation in Layered Elastic Media

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Abstract: - It is well known that there is an analogy between wave propagation in a periodically layered (nonhomogeneous) medium consisting of two elastic materials and that in a homogeneous viscoelastic medium. This interesting phenomenon was investigated in this paper. To show the analogy, an attenuation factor of wave was introduced. The attenuation effect of a layered medium was investigated using an exact analogous relaxation function for the layered medium. The effects of thickness ratio of the two constituent layers, their mechanical properties, and the layer thickness on the attenuation of stress wave propagation were examined.

Key-Words: - attenuation, wave propagation, layered medium, viscoelasticity, relaxation function.

1 Introduction
The behavior of wave propagation in layered media has been studied for decades. In his study, Barker [1] firstly reported that there was an analogy between layered elastic media and homogeneous viscoelastic media in terms of stress wave propagation, but he did not establish a direct analogy between the two. Starting from general viscoelastic layered media in which elastic layered media was a special case, Ting and Mukunoki [2-3] analytically verified Barker’s finding. However, Ting and Mukunoki [2-3] did not illustrate the wave attenuation effect on wave propagation in a layered medium with the viscoelasticity analogy. Later, Christensen [4-5] studied the same problem by a perturbation method and dielectric theory. He derived an approximate attenuation factor that indicated the effect of viscoelastic analogy for randomly layered media. The result from Christensen’s study was only good for long wave length time harmonic waves in layered elastic media.

In this paper, we were intending to demonstrate the attenuation effect because of the viscoelastic analogy of a layered elastic medium. Layered media constituted by two distinct elastic materials were studied. For simplicity, the layered media were assumed semi-infinite. Different thickness ratio and material combination for the layers were investigated.

2 Analogy and Attenuation

2.1 Viscoelastic Analogy
Consider a periodic layered medium as shown in Fig. 1. A typical cell of thickness $h$ consists of two layers with distinct mechanical properties. Layer 1, 3, 5, … are of material “1” with layer thickness $h_1$, density $\rho_1$, and two relaxation functions $\lambda_1(t)$ and $\mu_1(t)$; and Layer 2, 4, 6, … are of material “2” with layer thickness $h_2$, density $\rho_2$, and two relaxation functions $\lambda_2(t)$ and $\mu_2(t)$. Here the materials were assumed elastic, so that the two relaxation function $\lambda_i$ and $\mu_i$ ($i=1,2$) are independent of time $t$ and are identical to Lamé constants. For any stress wave propagating normal to the layering medium, an analogy between such a layered medium and a homogeneous viscoelastic medium could be established by a relaxation function $G(t)$ expressed in Laplace transform as $\mathcal{G}(s)$ [2]:
\[
\overline{G}(s) = \frac{\rho sh^2}{\{\cosh^{-1}[\cosh(k_1) - (\theta - 1)\cosh(k_2)]\}^2},
\]

where \(s\) is Laplace transform parameter,

\[
k_1 = h_1 \left[ \frac{\rho_1}{\lambda_1 + 2\mu_1} + h_2 s \frac{\rho_2}{\lambda_2 + 2\mu_2} \right],
\]

\[
k_2 = h_1 s \left[ \frac{\rho_1}{\lambda_1 + 2\mu_1} - h_2 s \frac{\rho_2}{\lambda_2 + 2\mu_2} \right],
\]

\[
\theta = \left( 1 + \frac{\rho_1(\lambda_1 + 2\mu_1)}{\rho_2(\lambda_2 + 2\mu_2)} \right)^2,
\]

\[
\rho = \frac{h_1}{h} \rho_1 + \frac{h_2}{h} \rho_2.
\]

The analogy given by (1) is that the stress response of a homogeneous viscoelastic medium is same as the stress response of a layered medium at the correspondent positions located at the mid-planes of the layers of material 1. The stress solution based upon the analogous homogeneous viscoelastic medium is

\[
\overline{\Sigma}(s) = \overline{p}(s)e^{-\theta s},
\]

where \(\overline{\Sigma}(s)\) is the Laplace transform of stress in the homogeneous viscoelastic medium; \(\overline{p}(s)\) is the Laplace transform of initial stress condition at \((x = 0)\). To make the analogy valid, \(x\) in (3) should be taken as \(n \times h\) \((n=0,1,2,\ldots)\), so that the simple form (3) could be used to find a stress response in a layered elastic medium. Solutions for arbitrary positions were also given by Ting and Mukunoki [2-3] based upon the solution found by (1), which were basically the same form as the later in terms of stress responses. Since our interest was the attenuation effect of wave propagation in a layered medium, the solution for the mid-planes of a layered medium from (1) - (3) would be enough for this study.

### 2.2 Attenuation Factor \(\eta\)

In general, expression of time harmonic wave propagation in a homogeneous viscoelastic medium can be expressed in the form [6]

\[
u(x,t) = \hat{u}e^{-\eta t}e^{i(k_1 x - \omega t)},
\]

where \(u(x,t)\) could be stress, or strain, or displacement; \(\hat{u}\) is the constant amplitude; \(\eta\) is the attenuation factor, \(\eta^*\) is real; and \(\omega\) is the frequency. Christensen [6] showed that the attenuation factor could be given by

\[
\eta = \frac{\omega \rho \frac{1}{2} G^V}{|G|},
\]

where \(G^* = G_1 + iG_2\) is the complex modulus and \(G^V\) is given by

\[
G^V = \text{Im}(\sqrt{G^*}).
\]

To complete the analogy, we need to find \(G^*\) and \(G^V\) for the layered medium.

### 3 Wave Attenuation in Layered Elastic Media

The attenuation factor discussed in section 2.2 was used to examine the attenuation effect of a layered medium.

#### 3.1 Calculation of \(\eta\)

The relaxation function in time domain \(G(t)\) could be obtained from (1) numerically by the method given by either [7] or [8]. Given \(G(t)\), complex modulus \(G^*\) can be readily computed via the Kronig-Kramers relation with which \(G_1\) and \(G_2\) are given by:

\[
G_1 = G(\infty) + \omega \int_0^\infty [G(t) - G(\infty)] \sin(\omega t) dt,
\]

\[
G_2 = G(\infty) - \omega \int_0^\infty [G(t) - G(\infty)] \cos(\omega t) dt.
\]
Because the integrands in (7) and (8) are only available numerically, \( G_1 \) and \( G_2 \) given by (7) and (8) can only be integrated numerically. Use the relation between \( G^* \) and \( G_1 \) and \( G_2 \), subsequently solve
\[
|G^*| = \sqrt{G_1^2 + G_2^2},
\]
and then substitute \( G_1 \) and \( G_2 \) into (6) to compute \( G^{IV} \).

With above manipulation, the attenuation factor \( \eta \) given by (5) can be numerically worked out if not analytically.

It should be pointed out that the analogous relaxation function \( G(t) \) defined by (1) does not monotonic decrease with time as a conventional relaxation function of viscoelasticity does [2-3]. Because of such a characteristic, we can only make the above computation of attenuation factor meaningful within a limited time; otherwise, the computation shows that the attenuation vanishes. In other words, the computed attenuation factor of the following section (section 3.2) is valid for a limited time after the wave front passes over.

### 3.2 Effect of Thickness and Impedance Ratios

Two sets of material systems were studied, i.e., steel/PMMA and ceramic/rubber, just for illustration purposes. They represent low and high impedance mismatches, respectively. Their mechanical properties are listed in Table 1.

<table>
<thead>
<tr>
<th>Material #</th>
<th>Material</th>
<th>Density (kg/m^3)</th>
<th>( \mu ) (GPa)</th>
<th>( \lambda ) (GPa)</th>
<th>Impedance Ratio (#1/#2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Steel</td>
<td>7.89</td>
<td>7.9</td>
<td>110</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>PMMA</td>
<td>1.15</td>
<td>0.</td>
<td>8.9</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Ceramic</td>
<td>3.98</td>
<td>136</td>
<td>136</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>Rubber</td>
<td>1.</td>
<td>.0007</td>
<td>.006</td>
<td>460</td>
</tr>
<tr>
<td>1</td>
<td>Ceramic</td>
<td>3.98</td>
<td>136</td>
<td>136</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Al</td>
<td>2.7</td>
<td>27</td>
<td>40</td>
<td>2.5</td>
</tr>
</tbody>
</table>

Fig. 2 and Fig. 3 were computed for the first two materials system listed in Table 1. The typical cell thickness \( h \) was taken as 0.517 mm. The frequency \( \omega \) was chosen as 1 for the computation. Fig. 2 shows the attenuation factor \( \eta \) varies with the thickness ratio of the thickness of material “1”, \( h_1 \) to the typical cell thickness, \( h \) given \( h \) fixed. The attenuation factor curves in Fig. 2 are normalized by a maximum attenuation factor, \( \eta_{\text{max}} \).

It is clearly shown by Fig. 2 that there is a thickness ratio that gives the maximum attenuation. The thickness ratio of maximum attenuation, \((h_1/h)_{\text{opt}}\) depends on the mismatch of the two materials in impedance. For instance, \((h_1/h)_{\text{opt}} = 0.55\) for steel/PMMA medium. According to Fig. 2, the layer with high impedance needs to be thicker in order to achieve the maximum attenuation if the impedance mismatch of the two materials becomes higher.

Fig. 3 shows the effect of impedance mismatch. It is seen that the attenuation increases as impedance mismatch increases.

By Table 1: Material Properties

![Fig. 2 Effect of thickness ratio.](image)

![Fig. 3 Effect of impedance mismatch.](image)
Fig. 2 does not vary for the given typical cell thickness, i.e., the \((h_1/h)_{\text{opt}}\) is almost unchanged.

### 3.3 Verification of Attenuation

The following discussion is to verify the maximum attenuation achieved in section 3.2 by illustrations of stress wave propagation in a layered medium made of steel/PMMA as an example.

Wave propagation implementation was made by solving (2) based upon the exact relaxation function (1), which should give an exact stress response in the layered medium corresponding the mid-planes of the layers of material “1”. The layered medium constituted by same materials with same typical cell thickness \(h=0.517\) mm but different thickness ratio was examined. Unit step force (initial stress condition) was applied on the mid-plane of the first layer of material “1” at time \(t=0\). For demonstration purpose, three representative thickness ratios were taken. They were thickness equal to \((h_1/h)_{\text{opt}}=0.55\) corresponding to the maximum attenuation (Case 2), smaller than \((h_1/h)_{\text{opt}}\) (Case 1), and larger than \((h_1/h)_{\text{opt}}\) (Case 3). Refer to Fig. 2 for their corresponding positions. In order to compare the attenuation effect of wave in the layered media properly, the comparison should be made at the equivalent time and equivalent spatial position.

Fig. 4 shows that the wave propagation in a layered medium corresponding to Case 1, 2 and 3 respectively. It should be noted that only the marked points in these figures represent the exact values of stress wave at the corresponding positions. The linking lines between markers are drawn for outlining purpose. In each figure, four approximate stress wave envelopes are illustrated, which correspond to different time with a...
time interval $\delta t$ equal to the time a wave propagates through a given typical cell. The front stress amplitudes in Fig. 4 were the stress responses at about $2\delta t$ after the real wave fronts pass over. (The stress amplitudes of real wave fronts does not change with the thickness ratio.)

If only the stress amplitude of the front (not the real wave front) in Fig. 4 at each corresponding time among the three cases is compared, Case 2 always gives the smallest. In addition, by checking the attenuation rate of the wave front through spatial dimension for the three cases, Case 2 also offers the highest rate.

The following could be summarized in accordance with Fig. 4:

- Different thickness ratio results in different attenuation of stress wave propagating in a layered medium constituted by same materials and with same typical cell thickness.
- The highest attenuation happens to the thickness ratio of Case 2, which agrees to the results calculated by the attenuation factor formulated by (5).

As it can be seen from Fig. 4, the propagation and attenuation of wave in a layered medium is, somehow, different from the propagation and attenuation of wave in a real viscoelastic medium. A step function stress wave propagating in a layered medium would build up its amplitude or even exceed the unity in amplitude, while the same wave in a homogeneous viscoelastic medium does not [9].

3.4 Effect of Layer Thickness

Another interesting question answered is the effect of typical cell thickness on the attenuation. The third material system listed in Table 1 was used for this purpose.

Assume the two constituent materials are ceramic and aluminum, respectively. The typical thicknesses are $h=h_0/2$, and $h=h_0$, where $h_0$ was taken as 3 mm. The thickness ratio $h_1/h$ was assumed 0.2 for both layers. As described in the proceeding section, we propagated a unit step stress wave at $x=0$ into the layered medium and checked the stress response at a certain position.

Figure 5 shows the stress responses of the wave in the two materials. The curves in Figs. 5 (a) and (b) had been normalized to the same scale so that the two stress responses could be compared directly. Different from those in Fig. 4, the wave fronts in Fig. 5 are real wave fronts.

The results indicate that the medium with a thinner cell ($h=h_0/2$) provides a higher attenuation than the medium with a thicker cell ($h=h_0$) in terms of stress amplitude at the corresponding wave fronts. On the other hand, the time needed for the stress amplitude to rise to unity after the passage of the wave front is shorter for the thinner cell than the thicker cell. This can be discerned by the difference in $\Delta x$ as indicated in Fig. 5. Obviously, $\Delta x$ in Fig. 5 (a) is smaller than that in Fig. 5 (b).

Based upon these results, we note that there are two features of the layer thickness effect, i.e., the attenuation at the wave front is greater and the rise time of wave amplitude is shorter in the medium with
thinner layers. In other words, if we keep reducing the typical cell thickness, eventually there will be another step wave front built up right behind the original wave front that diminishes.

4 Conclusion
Based upon this study, the following can be concluded:
1. An Analogy between elastic layered media and viscoelastic media exists in the attenuation behavior at the wave front.
2. Given other conditions the same, the thickness ratio of the two constituent layers can affect the wave attenuation in the layered medium and can be optimized to achieve the maximum attenuation.
3. A higher impedance mismatch between the two materials leads to a higher attenuation.
4. Thinner layers result in a greater attenuation at the wave front but a shorter rise time to attain the original input stress amplitude.

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References: