Orthogonal Finite-Length Sequence Sets With Impulsive Autocorrelation Function

Yoshihiro TANADA
Department of Computer Science and Systems Engineering
Yamaguchi University
2-16-1 Tokiwadai, Ube, Yamaguchi, 755-8611
Japan

Abstract: - Self-orthogonal finite-length sequence has impulsive autocorrelation function with no sidelobes except at both shift ends. In this paper, sets of self-orthogonal finite-length sequences with and without zero-correlation zone are presented for the application to synchronous or quasi-synchronous CDMA communication system without interferences. Sequence set starting from a pair of sequences of length 5 as well as other sets are derived from general solution for a sequence which is given by the sequence spectrum composed of primitive sequence spectra. Derived sequences are applicable to fast convolution or correlation processing.

Key-words: - Self-orthogonal, finite-length, pseudonoise sequence, zero-correlation zone, sequence set, aperiodic correlation, convolution, sequence spectrum, quasi-synchronous CDMA, fast signal processing.

1 Introduction
A finite-length sequence with an impulsive autocorrelation function is useful for the radar, the sonar and the communication systems where a transmitted signal of long length is compressed to an impulse in a receiver. A finite-length sequence with zero-sidelobe autocorrelation function except at both shift ends is called a self-orthogonal or shift-orthogonal finite-length sequence since its shifted sequences are orthogonal [1]-[3]. For a code division multiple access (CDMA) communication system, a set of self-orthogonal finite-length sequences with different patterns is desired.

This paper presents orthogonal sets of the above sequences for the application to a synchronous or quasi-synchronous CDMA communication system without interferences. The sequence sets are derived from the general solution for a self-orthogonal finite-length sequence which is given by the sequence spectrum composed of primitive sequence spectra. Orthogonal sequences with and without zero correlation zone are derived, which have the formation applicable to fast convolution or correlation processing.

2 Solution of Self-Orthogonal Finite-Length Sequence
A finite-length sequence whose autocorrelation function has no sidelobes except at left and right shift ends is an ideal finite-length PN (pseudonoise) sequence, and can be called a self-orthogonal or shift-orthogonal finite-length sequence, since its shifted sequences are orthogonal within a limited shift range. An aperiodic autocorrelation function of a complex-valued self-orthogonal finite-length sequence \{a_{M,\ell, i}\} of length M, member \ell and ordinal \ i is expressed as

\[ \rho_{M,\ell, i, i'} = \frac{1}{M} \sum_{i=0}^{M-1} a_{M,\ell, i} a^{*}_{M,\ell, i-i'} \]

\[ = \begin{cases} 1 & i = 0, i' = M - 1 \\ \varepsilon_{M-1} & i = 0, i' = -(M - 1) \\ \varepsilon_{M-1}^{*} & i = M - 1, i' = 0 \\ 0 & \text{elsewhere} \end{cases} \]  

(1)

where \ a_{M,\ell, i} = 0 \ for \ i < 0 \ and \ i > M - 1, \ and \ i' \ is \ shift, \ and \ * \ denotes \ complex \ conjugate, \ and \ \varepsilon_{M-1} \ is \
where

\[ a_{M,\ell}(t) = \sum_{i=0}^{M-1} a_{M,\ell,i} \delta(t - iT_c) \]  

where \( \delta(t) \) is Dirac’s delta function of time \( t \). The Fourier spectrum of \( a_{M,\ell}(t) \) is given by

\[ A_{M,\ell}(f) = \int_{-\infty}^{\infty} a_{M,\ell}(t) e^{-j2\pi ft} \, dt \]

\[ = \sum_{i=0}^{M-1} a_{M,\ell,i} z^{-i} \]  

where \( z = e^{j2\pi fT_c} \) and \( f \) is frequency. The energy spectrum of \( a_{M,\ell}(t) \) is related to the aperiodic autocorrelation function \( \{ \rho_{M,\ell,\ell'} \} \) as follows:

\[ |A_{M,\ell}(f)|^2 = \sum_{i'=-(M-1)}^{M-1} \rho_{M,\ell,\ell'} z^{-i'} \]

\[ = M \{ e_{M-1} z^{M-1} + 1 + e_{M-2} z^{M-2} + \cdots + e_0 \} \]  

which is factorized to

\[ |A_{M,\ell}(f)|^2 = (1 + 1 - \frac{4}{\alpha_M^2})^{(M-1)/2} \frac{1}{\alpha_M \beta_M} \]  

The sequences of length 2 with the spectra \( \{ z^{-1} - \beta_M e^{-j\phi_M} e^{j(2m+1)\pi / M-1} \} \) and \( \{ z^{-1} + \beta_M e^{-j\phi_M} e^{j(2m+1)\pi / M-1} \} \) which we call prime sequences are reversed each other and have the same autocorrelation function, neglecting coefficients, since there exist the following relations:

\[ |z^{-1} - \beta_M e^{-j\phi_M} e^{j(2m+1)\pi / M-1}|^2 \]

\[ = \alpha_M \alpha_M + \beta M (e_{1,m} z + 1 + e_{1,m} z^{-1}) \]

and

\[ |z^{-1} + \beta_M e^{-j\phi_M} e^{j(2m+1)\pi / M-1}|^2 \]

\[ = \beta M (\alpha_M + \beta M) (e_{1,m} z + 1 + e_{1,m} z^{-1}) \]

Using Eqs. (9) to (12), the energy spectrum is rewritten by

\[ |A_{M,\ell}(f)|^2 = M \cdot (\alpha_M + \beta M) \frac{M-1}{\alpha_M + \beta M} \]

\[ \times \prod_{m=0}^{M-2} (e_{1,m} z + 1 + e_{1,m} z^{-1}) \]  

which implies that the autocorrelation function of the self-orthogonal finite-length sequence is given by the \( (M - 1) \)-multiple convolution of the autocorrelation function.
functions of the prime sequences. From the above
together with the prime-sequences. From the above
to the prime-sequences.

According to Eq. (15), we can obtain
with real coefficients in Eq. (15). The combina-
tion of the energy spectrum, the sequence spec-
tation for the signal transmission by themsel-
valeued sequences for a given length
real-valued sequences [1]. These sequences in-
through convolution or correlation processing [2], a set of orthog-
sequences for fast
orthogonal sequences etc.

to the targets of the energy spectrum, the sequence spec-
tation for the signal transmission by them-
ues for the signal transmission by themselves.

For every length \( M \) and \( \gamma_{M,\ell,0} = \gamma_{M,\ell,1} = \cdots = \gamma_{M,\ell,M-2}, \ell = \ell_0 \), we have the following sequence spectrum:

\[
A_{M,\ell,0}(f) = \sqrt{\frac{M}{\varepsilon_{M-1}}} e^{j\varphi_M} K_{M,\ell} \times \left\{ z^{-\ell(M-1)} - \gamma_{M,\ell,0,0} e^{-j\varphi_M} \right\}
\]

which gives the sequence with successive zero values when \( M > 2 \). For such \( M \) that \( M - 1 \) is a compound number, we have the other zero-valued sequences. For example, when \( M = 7 \), we have the following two sequence spectra with zero-valued coefficients:

\[
A_{7,\ell_1}(f) = \sqrt{7\varepsilon_6} e^{j\varphi_7} K_{7,\ell_1} \times (z^{-3} - \gamma_{7,\ell_1,0} e^{-j\varphi_7} e^{j\frac{2\pi}{7}}) \times (z^{-3} - \gamma_{7,\ell_1,1} e^{-j\varphi_7} e^{j\frac{3\pi}{7}})
\]

where \( \gamma_{7,\ell_1,0} = \gamma_{7,\ell_1,2} = \gamma_{7,\ell_1,4} \), \( \gamma_{7,\ell_1,1} = \gamma_{7,\ell_1,3} = \gamma_{7,\ell_1,5} \) and \( \gamma_{7,\ell_1,0} \neq \gamma_{7,\ell_1,1} \) and

\[
A_{7,\ell_2}(f) = \sqrt{7\varepsilon_6} e^{j\varphi_7} K_{7,\ell_2} \times (z^{-2} - \gamma_{7,\ell_2,1} e^{-j\varphi_7} e^{j\frac{2\pi}{7}}) \times (z^{-2} - \gamma_{7,\ell_2,2} e^{-j\varphi_7} e^{j\frac{3\pi}{7}})
\]

where \( \gamma_{7,\ell_2,0} = \gamma_{7,\ell_2,3}, \gamma_{7,\ell_2,1} = \gamma_{7,\ell_2,4}, \gamma_{7,\ell_2,2} = \gamma_{7,\ell_2,5} \) and \( \gamma_{7,\ell_2,0} \neq \gamma_{7,\ell_2,1} \). The sequences corresponding to Eqs. (19) and (20) take successive two zero-values and one zero-value, respectively, between both neighboring non-zero values. On the other hand, we have the sequence spectrum of non-zero-valued sequence for \( M = 3 \)

\[
A_{3,\ell_3}(f) = \sqrt{3\varepsilon_2} e^{j\varphi_3} K_{3,\ell_3} \times (z^{-1} - \gamma_{3,\ell_3,0} e^{-j\frac{2\pi}{3}} e^{j\frac{2\pi}{3}}) \times (z^{-1} - \gamma_{3,\ell_3,1} e^{-j\frac{2\pi}{3}} e^{j\frac{2\pi}{3}})
\]

where \( \gamma_{3,\ell_3,0} \neq \gamma_{3,\ell_3,1} \), and that for \( M = 4 \)

\[
A_{4,\ell_4}(f) = \sqrt{4\varepsilon_3} e^{j\varphi_4} K_{4,\ell_4} \times (z^{-1} - \gamma_{4,\ell_4,0} e^{-j\frac{2\pi}{4}} e^{j\frac{2\pi}{4}}) \times (z^{-1} - \gamma_{4,\ell_4,1} e^{-j\frac{2\pi}{4}} e^{j\frac{2\pi}{4}}) \times (z^{-1} - \gamma_{4,\ell_4,2} e^{-j\frac{2\pi}{4}} e^{j\frac{2\pi}{4}})
\]

where \( \gamma_{4,\ell_4,0} \neq \gamma_{4,\ell_4,1} \). Since replacing \( z^{-1} \) in Eq. (21) with \( z^{-3} \) and \( z^{-1} \) in Eq. (22) with \( z^{-2} \) leads to the spectra of Eqs. (19) and (20), respectively, the sequences corresponding to Eqs. (19) and (20) can be obtained by inserting successive two zero-values and one zero-value between both neighboring values in the sequences corresponding to Eqs. (21) and (22), respectively. Fig. 2 illustrates the sequences \( \{a_{7,\ell_1,i}\} \) and \( \{a_{7,\ell_2,i}\} \) from \( A_{7,\ell_1}(f) \) and \( A_{7,\ell_2}(f) \).
of the zero-value sequences, the binomial neighbors non-zero values. As seen in the spectra 4 Synthesis of Orthogonal Set of Sequences.

The crosscorrelation function between the sequences \( \{a_{M,\ell,i}\} \) and \( \{a_{M,\ell',i}\} \) is given by

\[
\rho_{M,\ell,\ell',i,i'} = \frac{1}{M} \sum_{i=0}^{M-1} a_{M,\ell,i} a_{M,\ell',i-i'}^* (23)
\]

which is related to the cross-energy spectrum

\[
A_{M,\ell}(f) \cdot A_{M,\ell'}^*(f) = M \sum_{i'=-(M-1)}^{M-1} \rho_{M,\ell,\ell',i-i'} z^{-i'} (24)
\]

If the cross-energy spectrum includes the factor of the lowest degree binomial \( z^{-\mu} + c, \mu = 2, 3, \ldots \) where \( c \) is constant, then the crosscorrelation function takes successive \( \mu - 1 \) zero-values between both neighboring non-zero values. As seen in the spectra of the zero-valued sequences, the binomial \( z^{-\mu} + c \) is obtained so that its factors \( \prod_{m=0}^{\mu-1} \left\{ z^{-1} - \frac{1}{c^n} e^{(2m+1)\pi n} \right\} \) may be shared to \( A_{M,\ell}(f) \) and \( A_{M,\ell'}^*(f) \). The crosscorrelation function with zero values is utilized for the synthesis of orthogonal sequence set in the next section.

4 Synthesis of Orthogonal Set of Sequences

The shifted sequences \( \{a_{M,\ell,i-i-n}\} \) from a self-orthogonal finite-length sequence \( \{a_{M,\ell,i}\} \) satisfy the following orthogonality from Eq. (1):

\[
\frac{1}{M} \sum_{i=0}^{M-1} a_{M,\ell,i} a_{M,\ell,i-n}^* = 0 \quad (25)
\]

which is effective for detecting the impulse response of the signal transmission path in communication system, control system, radar etc. The additional orthogonality of the sequences with different patterns is desired for a synchronous CDMA communication system without interchannel interferences. In this section, orthogonal sets of the different sequences are synthesized.

Every crosscorrelation function of an orthogonal sequence set is desired to take zero values at the specific shift. Hence, every sequence spectrum is desired to have the common factors composed of the product of plural primitive sequence spectra which is also the spectrum of the compound sequence with length \( \geq 3 \). We discuss on the synthesis of orthogonal sets of sequences as concrete examples.

A set of orthogonal sequences starting from a pair of sequences with length 5 is explained as follows. We select a pair of real-valued sequences of \( M = 5 \) and \( \varphi_5 = \pi \) of the spectra

\[
A_{5,0}(f) = -\alpha_5 \sqrt{5} \times (z^{-1} + \alpha_5) (z^{-1} + \beta_5) \times (z^{-2} + \beta_5^2) \quad (26)
\]

\[
A_{5,1}(f) = -\beta_5 \sqrt{5} \times (z^{-1} + \alpha_5) (z^{-1} + \beta_5) \times (z^{-2} + \alpha_5^2) \quad (27)
\]

The cross-energy spectrum of the spectra \( A_{5,0}(f) \) and \( A_{5,1}(f) \) is given by

\[
A_{5,0}(f) \cdot A_{5,1}^*(f) = -5 \times \alpha_5^2 \times (z^{-2} - \alpha_5^2) \times (z^{-2} - \beta_5^2)^2 \quad (28)
\]

which is expanded to the polynomial with the coefficients of zero values for the terms of the odd power of \( z^{-1} \). Then, the sequences \( \{a_{5,0,1}\} \) and \( \{a_{5,1,1}\} \) corresponding to \( A_{5,0}(f) \) and \( A_{5,1}(f) \), respectively, are orthogonal at the odd shift \( i' = \pm1, \pm3 \). Next, replacing \( z^{-1} \) and \( z^{-2} \) in Eqs. (26) and (27) with \( z^{-2} \) and \( z^{-4} \), respectively, and additionally replacing \( z^{-2} - \alpha_5 \) with \( (z^{-1} + \alpha_5)(z^{-1} + \beta_5) \) while replacing \( z^{-2} + \beta_5^2 \) with \( z^{-2} + \alpha_5^2 \) or \( z^{-2} + \beta_5^2 \) makes the four spectra of the sequences of length 9, as

\[
A_{9,0}(f) = -\alpha_5^2 \sqrt{9} \times (z^{-1} - \alpha_5)(z^{-1} + \beta_5) \times (z^{-2} + \beta_5^2)(z^{-4} + \beta_5^2) \quad (29)
\]
\[ A_{0,1}(f) = -\alpha_0 \sqrt{9|\varepsilon_8|} (z^{-1} - \alpha_0) (z^{-1} + \beta_0) \times (z^{-2} + \alpha_0^2) (z^{-4} + \beta_0^4) \]  
(30)

\[ A_{0,2}(f) = -\beta_0 \sqrt{9|\varepsilon_8|} (z^{-1} - \alpha_0) (z^{-1} + \beta_0) \times (z^{-2} + \beta_0^2) (z^{-4} + \alpha_0^4) \]  
(31)

\[ A_{0,3}(f) = -\beta_0^3 \sqrt{9|\varepsilon_8|} (z^{-1} - \alpha_0) (z^{-1} + \beta_0) \times (z^{-2} + \alpha_0^2) (z^{-4} + \alpha_0^4) \]  
(32)

The cross-energy spectra \( A_{0,0}(f) \cdot A_{0,2}^*(f) \) and \( A_{0,1}(f) \cdot A_{0,3}^*(f) \) are given by

\[ A_{0,0}(f) \cdot A_{0,2}^*(f) = A_{0,1}(f) \cdot A_{0,3}^*(f) = -9|\varepsilon_8| \alpha_0^4 z^8 (z^{-1} - \alpha_0^4) \times (z^{-4} - \beta_0^4) (z^{-4} + \beta_0^4)^2 \]  
(33)

which explains that the crosscorrelation functions between \( \{a_{0,0,0}\} \) and \( \{a_{0,2,0}\} \) and between \( \{a_{0,1,0}\} \) and \( \{a_{0,3,0}\} \) take zero values at the shift \( i' \neq 0 \) mod 4. The cross-energy spectra \( A_{0,0}(f) \cdot A_{0,1}^*(f) \) and \( A_{0,0}(f) \cdot A_{0,3}^*(f) \) are given by

\[ A_{0,0}(f) \cdot A_{0,1}^*(f) = -9|\varepsilon_8| \alpha_0^2 z^8 (z^{-2} - \alpha_0^2) \times (z^{-2} + \beta_0^2) (z^{-4} + \alpha_0^4) \times (z^{-8} - \beta_0^8) \]  
(34)

\[ A_{0,0}(f) \cdot A_{0,3}^*(f) = -9|\varepsilon_8| \alpha_0^2 z^8 (z^{-2} - \alpha_0^2) \times (z^{-2} + \beta_0^2) (z^{-4} + \beta_0^4) \times (z^{-8} - \beta_0^8) \]  
(35)

which explain that the crosscorrelation functions between \( \{a_{0,0,0}\} \) and \( \{a_{0,1,0}\} \) and between \( \{a_{0,2,0}\} \) and \( \{a_{0,3,0}\} \) also take zero values at the shift \( i' \neq 0 \) mod 2. The crosscorrelation functions between \( \{a_{0,2,0}\} \) and \( \{a_{0,3,0}\} \) and between \( \{a_{0,3,0}\} \) and \( \{a_{0,3,0}\} \) is also zero at the shift \( i' \neq 0 \) mod 2. From the above correlation functions, a set of the sequences \( \{a_{0,0,0}\}, \{a_{0,1,0}\}, \{a_{0,2,0}\} \) and \( \{a_{0,3,0}\} \) is orthogonal at the shift \( i = 0 \) mod 4 where \( \{a_{0,1,0}\}, \{a_{0,2,0}\} \) and \( \{a_{0,3,0}\} \) are the shifted sequences from \( \{a_{0,1,0}\}, \{a_{0,2,0}\} \) and \( \{a_{0,3,0}\} \), respectively. Fig. 3 shows correlation functions in the shifted sequences \( \{a_{0,0,0}\}, \{a_{0,1,0}\}, \{a_{0,2,0}\} \) and \( \{a_{0,3,0}\} \), where \( \ell, \ell' \) denotes correlation function between \( \ell \)th and \( \ell' \)th shifted sequences. Generally, we obtain a set of orthogonal sequences of length

Fig. 3 Correlation functions between shifted sequences of length 9.

\[ M = 2^{\nu+1} + 1, \text{ members } L = 2^\nu \text{ and zero-correlation shift } i' = 0 \mod 2^\nu, \nu = 1, 2, \ldots. \] By including the shifted sequences at \( i' = 2^\nu \), we can use totally \( 2^{\nu+1} \) orthogonal sequences.

A set of sequences with zero-correlation zone is derived from the above set. Replacing \( z^{-1}, z^{-2} \) and \( z^{-4} \) in Eqs. (29), (30), (31) and (32) with \( z^{-2}, z^{-1} \) and \( z^{-8} \) and additionally replacing \( z^{-2} - \alpha_0 \) with \( (z^{-1} - \alpha_17) (z^{-1} + \beta_17) \) gives the four spectra of the sequences of length 17, as

\[ A_{17,0}(f) = -\alpha_0^7 \sqrt{17|\varepsilon_{16}|} (z^{-1} - \alpha_17) \times (z^{-1} + \beta_17) (z^{-2} + \beta_17^2) \times (z^{-4} + \beta_17^4) (z^{-8} + \beta_17^8) \]  
(36)

\[ A_{17,1}(f) = -\alpha_0^3 \sqrt{17|\varepsilon_{16}|} (z^{-1} - \alpha_17) \times (z^{-1} + \beta_17) (z^{-2} + \beta_17^2) \times (z^{-4} + \alpha_17^4) (z^{-8} + \beta_17^4) \]  
(37)

\[ A_{17,2}(f) = -\beta_0 \sqrt{17|\varepsilon_{16}|} (z^{-1} - \alpha_17) \times (z^{-1} + \beta_17) (z^{-2} + \beta_17^2) \times (z^{-4} + \beta_17^4) (z^{-8} + \alpha_17^8) \]  
(38)

\[ A_{17,3}(f) = -\beta_0^3 \sqrt{17|\varepsilon_{16}|} (z^{-1} - \alpha_17) \times (z^{-1} + \beta_17) (z^{-2} + \beta_17^2) \times (z^{-4} + \alpha_17^4) (z^{-8} + \alpha_17^8) \]  
(39)

Hence, a set of the sequences \( \{a_{17,0,0}\}, \{a_{17,1,0}\}, \{a_{17,2,0}\}, \{a_{17,3,0}\} \) and \( \{a_{17,3,0}\} \) are orthogonal at the shift \( i = 0, \pm 1 \) mod 8. Generally, we obtain a set of orthogonal sequences of length \( M = 2^{\nu+2} + 1 \), numbers \( L = 2^\nu \), and zero-correlation shift \( i' = \)
0, ±1, $mod 2^{\nu+1}, \nu = 1, 2, \cdots$. If the values of the sequences are too large, we can use such method that the terms $z^{-8} + \alpha_1^{8}$ and $z^{-8} + \beta_1^{8}$ in Eqs. (36) to (39) are replaced by a common term $(z^{-1} + \sqrt{2} \beta_{1/2} z^{-2} + \alpha_{1/2})(z^{-1} - \sqrt{2} \beta_{1/2} z^{-2} + \beta_{1/2})$, while the members reduce by half and the total orthogonal sequences including shift sequences do not change.

Similarly, starting from a pair of sequences with length 7, we obtain a set of orthogonal sequences of length $M = 3 \cdot 2^\nu + 1$, members $L = 2^\nu$ and zero-correlation shift $i' = 0 \ mod \ 2^\nu, \nu = 1, 2, \cdots$, and that of length $M = 3 \cdot 2^\nu + 1 + 1$, members $L = 2^\nu$ and zero-correlation shift $i' = 0, \pm 1 \ mod \ 2^\nu+1, \nu = 1, 2, \cdots$. For the zero-correlation shift $i' = 0, \pm 1 \ mod \ 2^\nu+1, \nu = 1, 2, \cdots$. For the zero-correlation shift $i' = 0, \pm 1, \cdots \pm s$, replacing $z^{-1}, z^{-2}$ etc. in the original set of spectra with $z^{-s+(s+1)}, z^{-2(s+1)}$ etc. and additional replacing $z^{-(s+1)} - \alpha_M$ with $(z^{-1} - \alpha_r) \cdot \{z^{-s} + \beta_r \cdot z^{-(s+1)} + \cdots + \beta_s^r\}$ e.g. makes a set of orthogonal sequences of length $r, r = M(s+1) - s$. Using the similar method, the other sets of sequences such as complex-valued sequences and even-length sequences are obtained.

5 Conclusion
Orthogonal finite-length sequence sets with impulsive autocorrelation function are constructed for the application to a synchronous or quasi-synchronous CDMA communication system without interferences. A general solution is given by the sequence spectrum which is the product of primitive sequence spectra. The combination of the primitive sequences forms zero-valued crosscorrelation functions and generates a set of orthogonal sequences, which are shifted each other. Orthogonal sequence sets with and without zero correlation zone are derived. A set of orthogonal sequences starting from a pair of sequences of length 5 is explained as well as the other sets of sequences. The sequence sets have the formation applicable to fast convolution or correlation processing. The application of the sequence set to the synchronous or quasi-synchronous CDMA is to be reported hereafter.

References: