DECOUPLED ESTIMATION OF NOMINAL DIRECTION AND ANGULAR SPREAD BASED ON ASYMPTOTIC MAXIMUM LIKELIHOOD APPROACH

Bamrung Täu Sieskul *and* *Somchai Jitapunkul*

Digital Signal Processing Laboratory, Department of Electrical Engineering, Chulalongkorn University
Bangkok, Thailand

ABSTRACT

The problem of estimating the nominal direction and its underlying angular spread is considered herein. As encountered in spatially distributed source localization, the computation of these directional parameters might be regarded as two consecutive tasks. In this paper, we propose an *a*symptotic *m*aximum *l*ikelihood (AML) approach to successively estimate both of them. The first advantage of estimation in this way is that it requires only two successive 1-dimensional searches rather than joint 2-dimensional optimization as utilized in the *a*symptotic *m*aximum *l*ikelihood (AML) estimator. Therefore, the decoupled estimation in this way provides more numerical flexibility. Since it belongs to a large-sample approximation of the exact ML method, numerical simulation is conducted in order to validate its asymptotic efficiency producible with respect to Cramér-Rao bound. Although its non-asymptotic performance is inferior to that provided by the joint AML approach, it appeared that, in the region of large number of temporal snapshot, the proposed AML estimator for decoupled estimation is the same as the AML criterion which employed the joint 2-dimensions.

1. INTRODUCTION

Most works involved direction finding problem were based on *m*aximum *l*ikelihood (ML) estimation due to its producible optimality [1]. To arrive at extremal quantity, an optimization search of the likelihood function seems to be inevitable in a complex model. In general, the physical model with well-described characterization would requires large number of model parameters. As a consequence, the larger the number of model parameters, the larger the dimension of optimization over parameter space. Unfortunately, this might allow the ML estimator to be unsuitable for being incorporated into real-word applications. This is because of suffering from implementation aspects, for instance, computational complexity and memory consumption.

In the presence of local scattering around the vicinity of source, most classical point source models will suffer from the lack of identifiability in the presence of large number of directions. To deviate from the given problem, it is reasonable to assume that a number of multipaths should be large enough so that their path gains can be characterized, under the central limit theorem, by a Gaussian random variable whose associated directions are also random [2]–[3]. With a priori knowledge of angle probability distribution, it appeared in general possible to govern all deviated angles into a parametric model as well. As a matter of course, an incoming source signal immediately consists itself of three individual arrival parameters, such as, nominal direction, angular spread and power observed by the sensor array. Since the exact likelihood function with all 4 parameters, which also include the spatially uncorrelated noise variance, can not be concentrated on explicitly [2], it is therefore conflictive to account for implementations as indicated before. Recently, a large-sample approximation of the exact ML is proposed in [4]. It requires joint 2-dimensional search and yields lower in error variances than the WLS (*w*eighted *l*east *s*quares) estimator in [2]. Furthermore, relying on these restrictions, the reducible computation might be admitted by two successive one-dimensional WLS searches [5] for estimating the nominal direction and angular spread.

Here we propose an *a*symptotic *m*aximum *l*ikelihood (AML) approach to successively estimate both of them. The first advantage of estimation in this way is that it requires only two successive 1-dimensional searches instead of joint 2-dimensional optimization as utilized in the AML estimator [4]. Since it belongs to a large-sample approximation of the exact ML method, numerical simulation is conducted in order to validate its asymptotic efficiency producible with respect to Cramér-Rao bound. Although its non-asymptotic performance is inferior to that provided by the AML approach, it appeared that the proposed AML estimator for decoupled estimation still keeps the asymptotic efficiency.

2. SPATIALLY DISTRIBUTED SOURCE MODEL

Restrict our attention to a signal transmitting through a channel and then impinging on the *u*niform *l*inear *a*rray (ULA). With phase reference at the first element, or *f*irst *e*lement *r*eference (FER), the array response vector $\boldsymbol{a}(\phi) : [-\frac{\pi}{2}, \frac{\pi}{2}] \mapsto$

$\mathbb{C}^{N_E\times 1}$ can be, in general, written ideally as

$$\boldsymbol{a}(\phi) \triangleq \begin{bmatrix}1 & e^{\imath k d_E \sin(\phi)} & \dots & e^{\imath k d_E (N_E-1)\sin(\phi)}\end{bmatrix}^{\mathsf{T}} \quad (1)$$

where $k = \frac{2\pi}{\lambda}$ designates the wave number with associating wavelength λ and N_E is the number of sensor elements. As previously developed, most local scattering models assume that the nominal angle ϕ is deterministic while angular deviation δ_ϕ and associating path gain γ are considered as stochastic quantities. According to linear regression analysis, the array output at time instant n_T can be characterized in a flat fading channel by the snapshot $\boldsymbol{x}[n_T] \in \mathbb{C}^{N_E\times 1}$. Mathematically speaking, it can be represented as [3, p. 25]

$$\boldsymbol{x}[n_T] = s[n_T] \sum_{n_P=1}^{N_P} \gamma_{n_P}[n_T]\boldsymbol{a}(\phi+\delta_{\phi_{n_P}}[n_T])+\boldsymbol{n}[n_T] \quad (2)$$

where N_P denotes the number of scattering paths and $\boldsymbol{n}[n_T] \in \mathbb{C}^{N_E\times 1}$ designates the additive noise at sensor array. For a large number of rays, the channel vector

$$\boldsymbol{h}[n_T] \triangleq \sum_{n_P=1}^{N_P} \gamma_{n_P}[n_T]\boldsymbol{a}(\phi + \delta_{\phi_{n_P}}[n_T]) \quad (3)$$

seemed, under the central limit theorem, plausible to hold a circularly-symmetric complex-valued Gaussian process, *i.e.*, $\boldsymbol{h}[n_T] \sim \mathcal{N}_c\left(\boldsymbol{0}; \boldsymbol{\Sigma}_{hh}, \boldsymbol{O}\right)$. This N_E-dimensional variate implicitly provides the statistic $\boldsymbol{\Sigma}_{hh} \triangleq \mathcal{E}\langle \tilde{\boldsymbol{h}}[n_T]\tilde{\boldsymbol{h}}^{\mathsf{H}}[n_T]\rangle \in \mathbb{C}_{\mathbb{H}}^{N_E\times N_E}$, where $\tilde{\boldsymbol{h}}[n_T] \triangleq \boldsymbol{h}[n_T] - \mathcal{E}\langle \boldsymbol{h}[n_T]\rangle = \boldsymbol{h}[n_T]$. For taking an incoherently distributed channel into account, the second-order statistic of a certain incoming ray yields [3]

$$\mathcal{E}\langle \gamma_{n_P}[n_T]\gamma^*_{\acute{n}_P}[\acute{n}_T]\rangle = \sigma_\gamma^2 \delta_{n_P,\acute{n}_P}\delta_{n_T,\acute{n}_T} \quad (4)$$

where $\delta_{\bullet,\bullet}$ signifies the Krönecker delta function and σ_γ^2 is the power due to any path. Over spatial continuum of incoming rays, it can be approximated as

$$\boldsymbol{\Sigma}_{hh}(\rho, \phi, \sigma_\phi) \approx \rho \int f(\delta_\phi|0; \sigma_\phi^2)\boldsymbol{a}(\phi + \delta_\phi)\boldsymbol{a}^{\mathsf{H}}(\phi + \delta_\phi)d\delta_\phi \quad (5)$$

where $\rho \triangleq N_P\sigma_\gamma^2$ signifies the cluster power due to all paths and $f(\delta_\phi|0; \sigma_\phi^2)$ denotes the conditional PDF for random deviation δ_ϕ given a priori knowledge of the angular spread σ_ϕ. In instead of such physical angles ϕ and σ_ϕ, the spatial frequency response is preferable due to the better accuracy of approximating the first-order Taylor series around the array broadside [3]. In general, the spatial frequency ω and its associating standard deviation σ_ω are provided by

$$\omega(\phi) = kd_E \sin(\phi) \quad (6a)$$

$$\sigma_\omega(\phi, \sigma_\phi) = kd_E \cos(\phi)\sigma_\phi. \quad (6b)$$

Accounting for small angular spread, the so-called *spatial frequency* approximation results in a separable form as

$$\boldsymbol{\Sigma}_{hh}(\rho, \omega, \sigma_\omega) \simeq \rho\, \boldsymbol{D}_a(\omega)\boldsymbol{B}(\sigma_\omega)\boldsymbol{D}_a^{\mathsf{H}}(\omega) \quad (7)$$

where $\boldsymbol{D}_a(\omega) : [-kd_E, kd_E] \mapsto \mathbb{C}_{\mathbb{D},\mathbb{U}}^{N_E\times N_E}$ is diagonal and unitary matrix parameterized by nominal angle and $\boldsymbol{B}(\sigma_\omega) : \mathbb{R}_+^{1\times 1} \mapsto \mathbb{R}_{\mathbb{S},\mathbb{T}}^{N_E\times N_E}$ is symmetric Toeplitz matrix parameterized by angular spread. Their $(n_E, \acute{n}_E)$-th elements can be expressed by [3, p. 22])

$$[\boldsymbol{D}_a(\omega)]_{[n_E,\acute{n}_E]} = e^{\imath(n_E-1)\omega}\delta_{n_E,\acute{n}_E} \quad (8a)$$

$$[\boldsymbol{B}(\sigma_\omega)]_{[n_E,\acute{n}_E]} = f_{\mathcal{F}}((n_E - \acute{n}_E)\sigma_\omega|0, 1) \quad (8b)$$

whence characteristic function $f_{\mathcal{F}}(t|, 0, 1) \triangleq \mathcal{F}(f(\delta_\omega|, 0, 1))$ is equivalent to the Fourier transform $\mathcal{F}(\cdot)$ of the associating random variable whose PDF holds zero-mean and unit variance. If additive noise assumed is spatially uncorrelated noise and absolutely uncorrelated from channels, it results in

$$\boldsymbol{\Sigma}_{xx}[n_T] = p[n_T]\boldsymbol{D}_a(\omega)\boldsymbol{B}(\sigma_\omega)\boldsymbol{D}_a^{\mathsf{H}}(\omega) + \sigma_{\mathsf{n}}^2\boldsymbol{I} \quad (9)$$

where $p[n_T] \triangleq \rho|s[n_T]|^2$ stands for the total power observed at the sensor array. In what follows, we shall consider only the deterministic signal with constant modulus so that $\boldsymbol{\Sigma}_{xx}[n_T] = \boldsymbol{\Sigma}_{xx}(\boldsymbol{\theta}_\circ)\ ; \forall n_T$, where $\boldsymbol{\theta}_\circ$ is the true value of model parameter. Now suppose that based on the second-order statistic $\boldsymbol{\Sigma}_{xx}(\boldsymbol{\theta}_\circ)$ our problem is to find the nominal direction of arrival, ϕ, given the collected data $\boldsymbol{x}[n_T]; \forall n_T$, where true-valued parameter vector $\boldsymbol{\theta}_\circ \in \mathbb{R}^{4\times 1}$ in the considered model can be defined by

$$\boldsymbol{\theta}_\phi \triangleq \begin{bmatrix}\phi & \sigma_\phi & p & \sigma_{\mathsf{n}}^2\end{bmatrix}^{\mathsf{T}} \quad (10a)$$

$$\boldsymbol{\theta}_\omega \triangleq \begin{bmatrix}\omega & \sigma_\omega & p & \sigma_{\mathsf{n}}^2\end{bmatrix}^{\mathsf{T}} \quad (10b)$$

for the physical and spatial frequency models, respectively. Let us introduce the matrix trace, derivative with respect to scalar and Krönecker product operator as $\lceil \mathbf{A} \rfloor$, $\dot{\mathbf{A}}(\mathrm{x}) \triangleq \frac{\partial}{\partial \mathrm{x}}\mathbf{A}(\mathrm{x})$ and $\otimes$. Under the central limit theorem, the snapshot data is also of Gaussianity with $\boldsymbol{x}[n_T] \sim \mathcal{N}_c\left(\boldsymbol{0}; \boldsymbol{\Sigma}_{xx}, \boldsymbol{O}\right)$. To estimate the exact $\boldsymbol{\Sigma}_{xx}$, the sample covariance matrix $\hat{\boldsymbol{\Sigma}}_{xx} \in \mathbb{C}_{\mathbb{H}}^{N_E\times N_E}$ is given by

$$\hat{\boldsymbol{\Sigma}}_{xx} = \frac{1}{N_T}\sum_{n_T=1}^{N_T} \boldsymbol{x}[n_T]\boldsymbol{x}^{\mathsf{H}}[n_T]. \quad (11)$$

3. SEPARABLE PARAMETERIZATIONS

In this section, the column-stacking vectorization operator $\boldsymbol{v}_{\mathsf{c}}(\cdot)$ is performed to represent $\boldsymbol{\xi}_x \triangleq \mathcal{E}\langle \boldsymbol{x}^*[n_T] \otimes \boldsymbol{x}[n_T]\rangle = \boldsymbol{v}_{\mathsf{c}}(\boldsymbol{\Sigma}_{xx}) \in \mathbb{C}^{N_E^2\times 1}$ in a certain parameterization.

3.1. Nominal Direction Parameterization

Let us define $\tilde{\boldsymbol{B}}(p,\sigma_\omega,\sigma_{\mathsf{n}}^2) \triangleq p\boldsymbol{B}(\sigma_\omega) + \sigma_{\mathsf{n}}^2 \boldsymbol{I} \in \mathbb{R}_{\mathbb{S},\mathbb{T}}^{N_E\times N_E}$. This exhibits a separable parameter $\boldsymbol{\vartheta}_\omega \in \mathbb{R}^{(N_E+1)\times 1}$ as

$$\boldsymbol{\vartheta} \triangleq \begin{bmatrix}\omega & \boldsymbol{\eta}_\omega^{\mathsf{T}}\end{bmatrix}^{\mathsf{T}} \tag{12}$$

where $\boldsymbol{\eta}_\omega \in \mathbb{R}^{N_E\times 1}$ is the first column vector in $\tilde{\boldsymbol{B}}(p,\sigma_\omega,\sigma_{\mathsf{n}}^2)$. Such a parameterization results in

$$\boldsymbol{\xi}_x(\boldsymbol{\vartheta}_\omega) = \boldsymbol{\Omega}_\omega(\omega)\boldsymbol{\eta}_\omega \tag{13}$$

where full-rank matrix $\boldsymbol{\Omega}_\omega(\omega) : [-kd_{\scriptscriptstyle E}, kd_{\scriptscriptstyle E}] \mapsto \mathbb{C}_{\mathbb{F}}^{N_E^2\times N_E}$ is $\boldsymbol{\Omega}_\omega(\omega) \triangleq \boldsymbol{\Phi}_a(\omega)\boldsymbol{\Xi}$ with full-rank binary selection matrix $\boldsymbol{\Xi} \in \mathbb{B}_{\mathbb{F}}^{N_E^2\times N_E}$ corresponding to the Toeplitz structure of $\tilde{\boldsymbol{B}}(p,\sigma_\omega,\sigma_{\mathsf{n}}^2)$ and nominal frequency parameterization matrix $\boldsymbol{\Phi}_a(\omega) \triangleq \boldsymbol{D}_a^{\mathsf{H}}(\omega)\otimes\boldsymbol{D}_a(\omega) : [-kd_{\scriptscriptstyle E}, kd_{\scriptscriptstyle E}] \mapsto \mathbb{C}_{\mathbb{D},\mathbb{U}}^{N_E^2\times N_E^2}$. It was mentioned in [5] that based on the extended invariance principle the reparameterization between $\boldsymbol{\theta}_\omega$ and $\boldsymbol{\vartheta}_\omega$ yields the same performance.

3.2. Joint Parameterization of Nominal Direction and Angular Spread

Assume that we wish to joint estimate both ω and σ_ω. We must define $\boldsymbol{\eta}_{\omega,\sigma_\omega} \in \mathbb{R}^{2\times 1}$ as $\boldsymbol{\eta}_{\omega,\sigma_\omega} \triangleq \begin{bmatrix}p & \sigma_{\mathsf{n}}^2\end{bmatrix}$ so that

$$\boldsymbol{\xi}_x(\boldsymbol{\theta}_\omega) = \boldsymbol{\Omega}_{\omega,\sigma_\omega}(\omega,\sigma_\omega)\boldsymbol{\eta}_{\omega,\sigma_\omega} \tag{14}$$

where $\boldsymbol{\Omega}_{\omega,\sigma_\omega}(\omega,\sigma_\omega) : [-kd_{\scriptscriptstyle E}, kd_{\scriptscriptstyle E}] \times \mathbb{R}_+^{1\times 1} \mapsto \mathbb{C}_{\mathbb{F}}^{N_E^2\times 2}$ is $\boldsymbol{\Omega}_{\omega,\sigma_\omega}(\omega,\sigma_\omega) \triangleq \begin{bmatrix}\boldsymbol{v}_{\mathsf{c}}\left(\boldsymbol{D}_a(\omega)\boldsymbol{B}(\sigma_\omega)\boldsymbol{D}_a^{\mathsf{H}}(\omega)\right) & \boldsymbol{v}_{\mathsf{c}}(\boldsymbol{I})\end{bmatrix}$.

4. DECOUPLED AML ESTIMATOR

If we designate the nonparametric estimate $\hat{\boldsymbol{\Psi}}_{xx} \in \mathbb{C}_{\mathbb{H}}^{N_E^2\times N_E^2}$ as $\hat{\boldsymbol{\Psi}}_{xx} \triangleq \hat{\boldsymbol{\Sigma}}_{xx}^{\mathsf{T}} \otimes \hat{\boldsymbol{\Sigma}}_{xx}$, then the AML nuisance estimate becomes [4]

$$\hat{\boldsymbol{\eta}}_{\mathsf{AML}}(\boldsymbol{\iota}) = \left(\boldsymbol{\Omega}^{\mathsf{H}}(\boldsymbol{\iota})\,\hat{\boldsymbol{\Psi}}_{xx}^{-1}\boldsymbol{\Omega}(\boldsymbol{\iota})\right)^{-1}\boldsymbol{\Omega}^{\mathsf{H}}(\boldsymbol{\iota})\,\hat{\boldsymbol{\Psi}}_{xx}^{-1}\hat{\boldsymbol{\xi}}_x. \tag{15}$$

Plugging the incomplete $\hat{\boldsymbol{\eta}}_{\mathsf{AML}}(\boldsymbol{\iota})$ into $\boldsymbol{\xi}_x(\boldsymbol{\vartheta}_\omega) = \boldsymbol{\Omega}(\boldsymbol{\iota})\boldsymbol{\eta}$, we obtain

$$\boldsymbol{v}_{\mathsf{c}}(\widehat{\boldsymbol{\Sigma}}_{xx}(\boldsymbol{\iota})) = \boldsymbol{\Omega}(\boldsymbol{\iota})\hat{\boldsymbol{\eta}}_{\mathsf{AML}}(\boldsymbol{\iota}) \tag{16}$$

where $\widehat{\boldsymbol{\Sigma}}_{xx}(\boldsymbol{\iota}) \triangleq \boldsymbol{\Sigma}_{xx}(\boldsymbol{\iota},\hat{\boldsymbol{\eta}}_{\mathsf{AML}}(\boldsymbol{\iota}))$ is the concentrated covariance for AML estimate. Then, the AML estimator of the parameter of interest can be written as

$$\hat{\boldsymbol{\iota}}_{\mathsf{AML}} = \arg\min_{\boldsymbol{\iota}} \ell_{\mathsf{AML}}^{[N_T]}(\boldsymbol{\iota}) \tag{17a}$$

$$\ell_{\mathsf{AML}}^{[N_T]}(\boldsymbol{\iota}) = \lceil \widehat{\boldsymbol{\Sigma}}_{xx}^{-1}(\boldsymbol{\iota})\hat{\boldsymbol{\Sigma}}_{xx}\rfloor + \ln|\widehat{\boldsymbol{\Sigma}}_{xx}(\boldsymbol{\iota})|. \tag{17b}$$

Now the question is implicitly imposed in what the parameter of interest, $\boldsymbol{\iota}$, should be. The following procedure enables us to an obvious answer for decoupled estimation of nominal direction ϕ and its underlying angular spread σ_ϕ.

$$\boldsymbol{\iota} = \omega \overset{(13)}{\Rightarrow} \hat{\omega}_{\mathsf{AML}} = \arg\min_{\omega} \ell_{\mathsf{AML}}^{[N_T]}(\omega) \tag{18}$$

$$\boldsymbol{\iota} = \begin{bmatrix}\omega \\ \sigma_\omega\end{bmatrix} \overset{(14)}{\Rightarrow} \{\hat{\sigma}_\omega\}_{\mathsf{AML}} = \arg\min_{\sigma_\omega} \ell_{\mathsf{AML}}^{[N_T]}(\hat{\omega}_{\mathsf{AML}},\sigma_\omega). \tag{19}$$

Searching the minimum solution for $\hat{\omega}_{\mathsf{AML}}$ and $\{\hat{\sigma}_\omega\}_{\mathsf{AML}}$ according to two successive one-dimensional searches, we immediately obtain physical angle estimates via (6).

5. NUMERICAL EXAMPLES

To demonstrate the impact of the proposed estimator, we commonly employ the ULA with half-wavelength separation to receive a QPSK (*q*uaternary *p*hase *s*hift *k*eying) signal whose strength are controllable with respect to noise variance by $\mathrm{SNR} \triangleq 10\log\left(\frac{\sigma_{\mathsf{s}}^2}{\sigma_{\mathsf{n}}^2}\right)$. All significant parameters are set up, unless otherwise a variation on the parameter of interest will be specified individually in each figure, as the following table:

$\phi_\circ$	$\sigma_{\phi_\circ}$	σ_γ^2	SNR	N_P	N_E	N_R
$0°, 10°$	$5°$	0.01	10	100	8	$1,000$

Practically, the pseudo random number satisfied the Laplacian PDF $f_{\mathsf{L}}(\delta_\phi|0,1)$ can be modified from $\delta_{\phi_{\mathsf{L}}} = \frac{1}{\sqrt{2}}\ln\left(\frac{\delta_{\phi_{\mathsf{U}}}}{\acute{\delta}_{\phi_{\mathsf{U}}}}\right)$ [6] with any two independent uniform distributions $\delta_{\phi_{\mathsf{U}}} \sim \mathcal{U}[0,1]$ and $\acute{\delta}_{\phi_{\mathsf{U}}} \sim \mathcal{U}[0,1]$. Our empirical standard deviation is to average RMSE from a large number of independent runs (N_R).

Recently, it is shown that the AML estimator outperforms the WLS in non-asymptotic region [4]. Therefore, we shall investigate only the effect of decoupled estimation based on AML approach.

In Fig. 1, the joint AML estimator slightly outperforms the decoupled AML for small number of temporal snapshot. As shown in asymptotic performance assessment, both estimators achieve the CRB as the number of temporal snapshots tends to infinity.

For estimating the angular spread in Fig. 2, the decoupled AML estimator more deviates from the CRB than that shown in Fig. 1. This is because the second step for estimating the angular spread has imposed the uncertainty in nominal direction estimation. However, this effect will be gradually vanished when the nominal direction estimate is more accurate. In Fig. 2, it can be observed that both joint and decoupled AML estimations yields the same RMSE performance from large number of temporal snapshots.

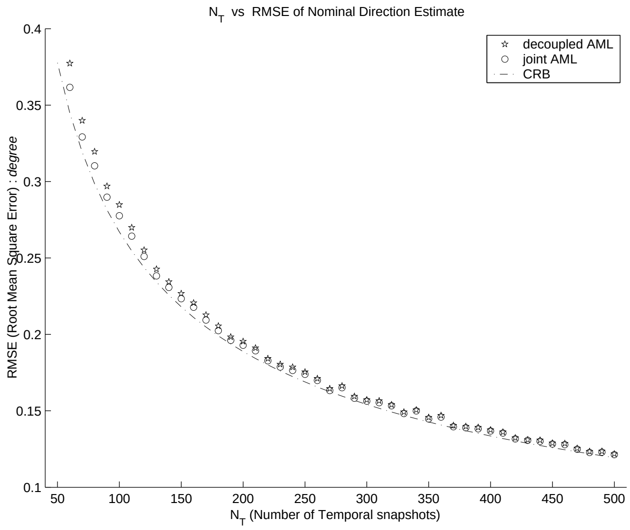

Fig. 1. Laplacian angle deviations : empirical and theoretical standard deviations of the error due to estimating the nominal direction ϕ as a function of number of snapshots N_T.

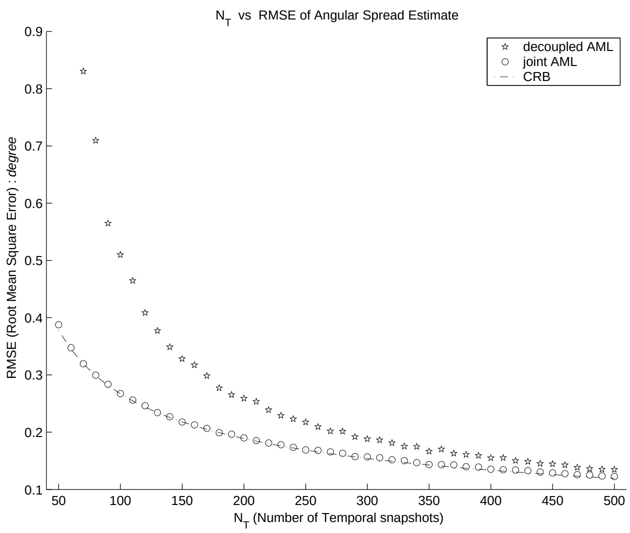

Fig. 2. Laplacian angle deviations : empirical and theoretical standard deviations of the error due to estimating the angular spread σ_ϕ as a function of number of snapshots N_T.

6. CONCLUSION

A decoupled approach with two steps has been proposed for estimating the nominal direction and its underlying angular spread. It is intended to provide more numerical flexibility than the joint estimation in a certain application, *e.g.*, the situation where the angular spread might be not of interest in a while. Numerical simulation was also conducted to validate the asymptotic efficiency with respect to the joint estimation and the CRB. The numerical results are verified that the decoupled estimation can attains the CRB as same as in the joint estimation when employing a large number of temporal sanpshots.

7. REFERENCES

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