Radon Measure Model For Edge Detection Using Rotational Wavelets *

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Abstract

Based on a mathematical analysis involving Radon measure explicit formulars of continuous (integral) wavelet transformations are given and further used to direct how to use rotational wavelets in edge detection.

Key Words: Edge detection, Wavelet, Dirac function, Radon measure

1. Introduction

Continuous wavelets with a rotational parameter are proposed in pattern recognition in the literature ([1],[2],[4],[5],[6]). According to the authors knowledge, however, no actual applications and algorithms were given, and no mathematical analysis based on explicit computations were conducted in order to lay a mathematical foundation for the effectiveness of the rotational continuous wavelet transformation method. This work is to present a convincing mathematical model for the method. The theory gives the insight on how the method works, and the principles of choosing the parameters involved in practice. The formulation in below is with the 2-dimensional (2D) case, while the whole theory is valid for all general nD-cases as well (see [QZ1]).

In this note we will only deal with continuous (integral) wavelets that are the square integrable functions (denoted by $L^2$) in Lebesgue integration sense satisfying the admissibility condition

$$C_\phi = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|\hat{\phi}(\xi)|^2}{|\xi|^2} d\xi < \infty,$$

(1)

where $\hat{\phi}$ denoted the Fourier transform of $\phi$. In the case $\phi$ is also integrable itself, ie. $f \in L^1$,
the admissibility condition implies $\hat{\phi}(0) = 0$, equivalent to
\[
\int_{\mathbb{R}^n} \phi(x) dx = 0. \tag{2}
\]

If $\phi(x) = \phi(y)$ whenever $|x| = |y|$, then $\phi$ is said to be radial. If $\phi$ is not radial, then for some rotation $\rho$ of $\mathbb{R}^2$ the functions $\phi(x)$ and $\phi(\rho x)$ are different. This kind of wavelets (functions) will be called isotropy, or rotational wavelets (or functions). We will use the $L^2$-normalized dilation in $\mathbb{R}^2$ together with rotation of the wavelet under consideration:

$$\phi_{a,\rho,b}(x) = a^{-1} \phi(a^{-1} \rho(x - b)), \tag{3}$$

where $\rho$ is a rotation of $\mathbb{R}^2$. We accordingly form the associated continuous wavelet transform (CWT) (CWTs with respect to different normalizations in $L^p$ defer by constant multiples.)

$$ (W_\phi f)(a, \rho, b) = a^{-1} \int_{\mathbb{R}^n} f(x) \phi(a^{-1} \rho(x - b)) dx. \tag{4} $$

In the convolution notation, it is

$$ (W_\phi f)(a, \rho, b) = f \ast \hat{\phi}_{a,\rho,0}(b), $$

where $\hat{f}$ denotes the reflection: $\hat{f}(x) = f(-x)$.

The admissibility condition is indispensible in the sense that only with that condition the original signals (test functions) may be recovered through the inversion formula

$$ f(x) = \frac{1}{C_\phi} \int_0^\infty \int_{SO(n)} \int_{\mathbb{R}^n} W_\phi f(a, \rho, b) \phi_{a,\rho,b}(x) \frac{da}{a^3} db d\rho. $$

In the following theory, however, only the CWT part will be used, and the inversion formula will not be concerned. In below, when a rotational wavelet (or function) $\phi$ is given, for some rotation $\rho$, the wavelet (or function) $\phi(\rho x)$ is called a rotated wavelet (function).

The 2D isotropy Morlet wavelet, or simply 2D Morlet wavelet is given by

$$ \phi(x) = e^{i<k,x>} e^{-(1/2)|x|^2} e^{- (1/2)|k|^2} e^{-(1/2)|x|^2}, \tag{5} $$

where $k$ is a fixed vector. The wavelet can be decomposed into

$$ \phi(x) = \phi_M(x) + \phi_E(x), $$

where

$$ \phi_M(x) = e^{i<k,x>} e^{-(1/2)|x|^2} \tag{6} $$

and

$$ \phi_E(x) = e^{-(1/2)|k|^2} e^{-(1/2)|x|^2}. \tag{7} $$

It is easily verify that neither $\phi_M$ or $\phi_E$ is a wavelet. The function $\phi_M$ is incorporated with a rotation-sensitive-factor $e^{i<k,x>}$ and there rotational. When the vector $k$ is chosen to have a large norm, then the “CWTs” corresponding to the rotated $\phi_M$’s can be effectivel used to detect edge, as established in the following section.

## 2 Radon Measure Method

A Radon measure (on $\mathbb{R}^2$) is a countably additive non-negative set function on all the Borel
sets (σ-algebra generated by all open sets) that is of finite value on all compact sets ([3]). In the practical language, all the sets, such as a piece of straight line segment or a curve, or any kinds of area you can imagine, are Borel sets. The point of this concept is that under a Radon measure, a set of Lebesgue measure zero is no necessarily to have zero measure. The usual wavelet theory uses Lebesgue measure and integration: the signal functions \( f \in L^2 \) are assumed to be Lebesgue square-integrable. In that case, for any wavelet \( \phi \) and any rotated wavelet \( \phi(\rho(\cdot)) \) (or non-rotated case for the rotation \( \rho = \text{identity} \)), the corresponding CWTs satisfy

\[
\lim_{a \to 0} (a^{-1}W_{\phi}f)(a, \rho, b) = f(x) \int_{\mathbb{R}^2} \phi(x)dx = 0,
\]

for all \( x \in \mathbb{R}^2 \) except, possibly, a set of Lebesgue measure zero (Theorem 1.25, Chapter 1, [7]). In particular, if the edge of a pattern is of Lebesgue measure zero, then in the above limit procedure all its WCTs become zero rapidly, and any piece of the edge cannot be recognized by the CWT values of any wavelet. This suggests that if we want to keep the intuition that “edge” happens on a set of Lebesgue measure zero, then one has to give up Lebesgue measure. Or, alternatively, also practically, is that the signal function is in \( L^2 \), then the procedure \( a \to 0 \) should not be used. This latter case will be discussed in a separate paper. In the present one we exercise the intuition that the edge is a set of Lebesgue measure zero, and show, through an explicit formula, how rotational wavelets are used in edge detection.

The well known 1-dimensional Dirac function \( \delta \) can be used to introduce Radon measures in \( \mathbb{R}^2 \) everagely supported on curves. For instance, the line \( a_1x_1 + a_2x_2 = 0, a = (a_1, a_2), |a| = \sqrt{a_1^2 + a_2^2} = 1 \), corresponds to

\[
d\mu(x) = \delta(<a, x>)dx = \delta(a_1x_1 + a_2x_2)dx_1dx_2.
\]

We can show, for a continuous function \( f \), that

\[
\int_{\mathbb{R}^2} f(x)d\mu(x) = \int_{a_1x_1 + a_2x_2 = 0} f(x_1, x_2)ds,
\]

where \( ds \) is the arc-length measure on the line \( a_1x_1 + a_2x_2 = 0 \). Using \( d\mu \) as the signal to be detected, we can show

**Theorem 2.1** Let \( a \) be a unit vector. Denote by \( l(x) = \delta(<a, x>) \) the Dirac function supported on the line \( \{a_1x_1 + a_2x_2 = 0\} \). Let \( \phi \) be the Morlet wavelet and \( \phi_M \) and \( \phi_E \) the components in the decomposition (5), given by (6) and (7), respectively. Then we have

(i) For any \( a > 0 \) and \( b \in \mathbb{R}^2 \), in the pointwise sense,

\[
(W_{\phi_E}l)(a, \rho, b) = (2\pi)^{1/2}e^{-(1/2)|k|^2}e^{-d^2/(2a^2)},
\]

where \( d \) is the distance from \( b \) to the hyperplaneline \( <a, x> = 0 \).
(ii) For any \( a > 0 \) and \( b \in \mathbb{R}^2 \), in the pointwise sense,
\[
(W_{\phi_M})(a, \rho, b) = (2\pi)^{1/2}e^{i\cos \theta}d/a e^{-d^2/(2a^2)}e^{-(1/2)\sin^2 \theta |k|^2},
\]
where \( \theta \) denotes the angle between the vector \( \rho^{-1}k \) and the normal of the line, the rotation \( \rho_0 \) is the one that rotates the normal of the line \( <a, x> = 0 \) to the vector \( e_2 = <0, 1> \).

To prove the theorem we perform change of variable \( x = \rho_0 y \), where \( \rho_0 \) rotates the \( x_2 \)-axis to the direction of \( a \). Then we have \( \delta( <a, x> ) = \delta(y_2) \) that enables us to use the characteristic property of the Dirac function in the \( y_2 \)-integration in the iterated integration. Finally we invoke the Fourier transform formula of the function \( e^{-tx^2}, t > 0, x \in \mathbb{R} \), to conclude (i) and (ii) of the theorem. In a long journal paper we provide detailed proofs of this and related results, including the \( nD \) cases, and for a class of the Morlet type wavelets.

3 Explanation and Conclusion
we now see what the theorem tells. Owing to the characteristic property of \( l(x_1, x_2) \), for the points \( b \) not on the line \( l = <a, x> = 0 \), the assertions (ii) and (iii) for \( d \neq 0 \) show that both the CWTs of \( a^{-1/2}W_{\phi_M}h \) and that of \( W_{\phi_E}h \) tend to zero in the procedure \( a \to 0 \). In fact, whenever \( d > 0 \) the factor \( e^{-d^2/(2a^2)} \) tend to zero rapidly as \( a \to 0 \). If the point \( b \) is on the line, corresponding to \( d = 0 \) in (ii) and (iii), then the CWT values are not sensitive to the parameter \( a \), and, as a matter of fact, they are independent of \( a \). The CWT values for \( \phi_E \) are independent of the rotation, while the norm of the CWT values for \( \phi_M \) significantly depend on the rotation: when \( \rho^{-1}k \) is orthogonal to the line \( l \), that is \( \theta = 0 \), then the norms of the corresponding CWT values take the maximal possible value \( \sqrt{2\pi} \). In general, for arbitrary angle \( \theta \) the norm of the CWT value for \( \phi_M \) is equal to \( \sqrt{2\pi}e^{-(1/4)(\sin^2 \theta |k|^2)} \), that is when \( \theta \) is the closer to zero the greater.

Suppose the edge is a straight line \{\( a_1x_1 + a_2x_2 + c = 0 \)\} (For \( c \neq 0 \) we can use \( d\mu(x) = \delta( <a, x> + c)dx \) and perform change of variable by translation \( x = x' + b' \), where \( b' \in \{a_1x_1 + a_2x_2 + c = 0\} \) to obtain the same conclusions as in the theorem.). For any arbitrary chosen vector \( k \), the norm \( |k| \) the larger the better, under the procedure \( a \to 0 \) the CWT values \( a^{-1/2}(W_{\phi_M}h)(a, \rho, b) \) will rapidly decrease to zero when \( b \) is away from the line, and when \( b \) the further away the decaying more rapid. On the line, however, the norms of the CWT values do not depend on the parameter \( a \), and they are in fact of a constant value. If we adjust the rotation \( \rho \) so
that the vector $\rho^{-1}\mathbf{k}$ becomes orthogonal with the line $\{a_1x_1 + a_2x_2 + c = 0\}$, then the norms of the CWTs on the line will exhibit the greatest value. This suggests how one should do edge detection using rotational Morlet wavelet $\phi_M$. Suppose now we want to detect a face boundary (edge) that is not a straight line but a close oval-shaped curve. For an arbitrary chosen rotation $\rho$, as analyzed above, in the procedure $a \rightarrow 0$ we extract out from the magnitudes of the CTWs those boundary portions that are orthogonal or nearly orthogonal with the vector $\rho^{-1}\mathbf{k}$. Then by altering the rotation $\rho$, for instance choosing four or six or even eight evenly distributed rotations, we obtain boundary portions in almost all directions. By connecting them we obtain the whole face boundary.

For a half-white and half-black photo to detect the edge in this model one uses $h(<\mathbf{a}, \mathbf{x}>)$ in place of $\delta(<\mathbf{a}, \mathbf{x}>)$ in the CWT formula, where $h$ is the Heaviside function $h(x) = 1, x > 0$; and $h(x) = 0, x \leq 0$ and $\mathbf{a}$ the unit normal of the edge. Then we take the directional derivative to the CWT along the direction $\mathbf{p} = (p_1, p_2)$ orthogonal with the direction $\rho^{-1}\mathbf{k}$. The derivative will be passed on to $h(<\mathbf{a}, \mathbf{x}>)$. Invoking the result $h' = \delta$ in the generalized function theory we have

$$(\partial/\partial \mathbf{p})h(<\mathbf{a}, \mathbf{x}>) = <\mathbf{a}, \mathbf{p} > \delta(\mathbf{a}, \mathbf{x}).$$

The directional derivative will take the maximal value when the selected $\rho$ makes $\mathbf{a} = \mathbf{p}$ that means that $\rho^{-1}\mathbf{k}$ is orthogonal with $\mathbf{a}$, consistent with the conclusion on the estimates of the CWTs for the Dirac function in Theorem 2.1.

Note that we are now dealing with the ideal situation and the condition $a \rightarrow 0$ corresponds to the assumption that the edge is of Lebesgue measure zero. In the real life situation the parameter $a$ should not tend to zero but be restricted to a certain band comparable to the “width” of the edge. We will discuss this situation in another paper. Directed by this methodology experiments and algorithms have been developed in a series of work by Liming Zhang et al in [8-10].

**References**


