Traffic regulation in the urban transportation network

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Abstract: In this paper, we consider a real-time regulation problem of the traffic within a disturbed urban transportation network. From a given timetable, we aim to find a new schedule of various buses after detection of a disturbance at a given time. The main objective is to find a solution maximizing the level of service for all passengers by minimizing their waiting times at stops and their transit times in connecting nodes. This problem was already studied with evolutionary approaches and multi-agents techniques. In this paper, we prove that the problem has an optimal solution in the case of illimited buses capacities and we propose a formulation which allows its resolution without requiring the use of heuristics.

Keywords: transportation systems, non-linear optimisation.

1 Problem formulation

1.1 Description
Let us consider an urban transportation network which consists of several lines. Each line is represented by a set of successive stations. On each line, a set of buses pass by stations according to schedules fixed in the timetable. At every time, passengers arrive at the different stations to board these buses. The arrival statistical distributions of passengers at a given station are known.

1.2 Problem
At a given moment, a disturbance occurs in the network and affects a bus at its arrival into a station of a certain line. The problem is to correct the timetable (schedule of the buses) such that the quality of service for passengers boarding buses is maximized. The service quality can in this case be reduced to an aggregation of several criteria like the minimization of waiting time of the passengers caused by the delays, the minimization of the increase of the total travel time and the minimization of the total transit time spent in the connecting nodes [1, 2, 5].

2 Mathematical Formulation
In this section, we consider the regulation problem as it has been treated in [1, 2, 5]. We present the different characteristics of a such problem, its constraints as well as the different criteria.

2.1 Initial data
- The transportation network consists of $N$ lines. Each line $l$ ($1 \leq l \leq N$) contains $n_l$ stations $S_k^l$ ($1 \leq k \leq n_l$).
- For each line $l$ ($1 \leq l \leq N$), there are $m_l$ buses $B_i^l$ ($1 \leq i \leq m_l$). Each bus can simply represent a journey ($B_i^l$ - a journey from $S_k^l$ to $S_{k+1}^l$).
- $\forall 1 \leq l \leq N$, $\forall 1 \leq i \leq m_l$, $\forall 1 \leq k \leq n_l$, the departure time of $B_i^l$ on $S_k^l$ is $d(B_i^l, S_k^l) = d_k^l$.
- $\forall 1 \leq l \leq N$, $\forall 1 \leq i \leq m_l$, $\forall 1 \leq k \leq n_l$, the charge of a vehicle at a station is $C(B_i^l, S_k^l) = c_i^l$.

- Such departure times previously calculated by taking account of several factors like the passenger’s arrival statistical distributions in order to maximize the service quality. These distributions are supposed to be known and precisely evaluated.

Thus, we denote by $n_k^l(t)$ the number of passengers per unit of time which arrive at the station $S_k^l$ at the time $t$.

Departure times must respect two constraints: the minimal duration constraint and the transit minimal duration constraint.

- The minimal duration is the duration that a vehicle puts to reach $S_{k+1}$ from $S_k$.
∀1 ≤ l ≤ N, ∀1 ≤ i ≤ m, ∀1 ≤ k ≤ n - 1, d'_{i,k+1} - d'_{i,k} ≥ \delta m'_{i,k}. The data m'_{i,k} are calculated according to the distance between \( S'_{i} \) and \( S'_{k+1} \) and to the traffic state in the periods corresponding to the departure of the \( i^{th} \) bus.

- The transit minimal duration. If we suppose that the line \( l_i \) and the line \( l_i \) are crossed in a node which corresponds to the station \( S'_{i} = S'_{k} \). For each bus \( B'_{i} \) of the line \( l_i \) there is a bus \( B'_{i,k} \) which arrives after \( B'_{i} \) and take a proportion of passengers equal to \( \chi'_{i,k} \) who change from \( B'_{i} \) to \( B'_{i,k} \). (\( \chi(i) \) \( \in \{1,\ldots,m\} \)) . The constraint of the transit minimal duration is given by:

\[
d'_{i,k+1} - d'_{i,k} \geq \tau_{i,k}^{\delta i+k}. \]

This constraint is taken into account in order to ensure a maximum number of transits realized. Different nodes are presented in a list \( S \) and each node is represented by the quadruplet \( (l_i, l_k, k_i, k_k) \). (note that \( (l_i, l_k, k_i, k_k) \neq (l_i, l_k, k_k, k_i) \)).

### 2.2 Perturbation information

- At a given time \( d'_{i,k} \), a disturbance occurs in the network and affects the bus \( B'_{i} \) which will arrive at the station \( S'_{i} \) with a delay a duration \( \delta \). Thus, we can write:

\[d'_{i,k} = d'_{i,k} + \delta,\]

- The problem is to find the new transit times \( d'_{i,k} \) of the buses at the different stations such that \( d'_{i,k} > d_{i,k} \). It is obvious that \( d'_{i,k} = d'_{i,k} \) for all \( d'_{i,k} \leq d_{i,k} \).

- Two possibilities can be considered. In the first one, we do nothing. In this case, the delay of the disturbed bus will be propagated through the stations according to the equation:

\[d'_{i,k} = d'_{i,k} + \delta, \quad \forall k \geq k_i.\]

This solution is of course naive because it does not guarantee the performance. The second consists in updating all the departure times of the buses by following the regulation strategy. The constraints of the minimum duration necessary to go from a station to the next one and of the minimum duration to carry out a transit must be respected, i.e.

\[\forall 1 \leq l \leq N, \forall 1 \leq i \leq m, \forall 1 \leq k \leq n - 1, \quad \text{we have:} \quad d'_{i,k+1} - d'_{i,k} \geq \delta m'_{i,k}. \]

And for each node \( (l_i, l_k, k_i, k_k) \in S \) we have:

\[d'_{i,k+1} - d'_{i,k} \geq \tau_{i,k}^{\delta i+k}. \]

- According to the adopted strategy, delaying or advancing the buses, we can have constraints like:

\[d'_{i,k} - d'_{i,k} \leq \pi_{i,k} \] (maximum delay constraint) and \( d'_{i,k} \geq \nu_{i,k} \) (in case of regulation by delay, we have \( d'_{i,k} = \nu_{i,k} \)).

### 2.3 Criteria

In this section, we also consider the same criteria as those evoked in [1][2][3]. The service quality can be reduced to an aggregation of the following criteria:

- The minimization of the sum of waiting times of the passengers caused by the buses' delays: the figure 1 describes an example of arrival distribution \( \mu_{i}^{\prime}(l) \) at a station \( S'_{i} \) between two successive departure moments \( d'_{i,k} \) and \( d'_{i,k+1} \) of two buses from the same line. The waiting time \( A \) of passengers for the different stations and for all buses can be described in the following equation:

\[A = \sum_{i=1}^{N} \sum_{l=1}^{m} \int_{d_{i,k}}^{d_{i,k+1}} \mu_{i}(l) (d'_{i,k+1} - d_{i,k} - l) \mathrm{d}l. \]

We suppose that the arrival rate of passengers at a stop is constant and equal to \( \mu_{i}^{\prime} \). We obtain:

\[A = \sum_{i=1}^{N} \sum_{l=1}^{m} \mu_{i}(l) (d'_{i,k+1} - d'_{i,k}) = \sum_{i=1}^{N} \sum_{l=1}^{m} \int_{d_{i,k}}^{d_{i,k+1}} \mu_{i}(l) (d_{i,k+1} - d_{i,k} - l) \mathrm{d}l. \]

In [1, 2, 5], the authors have defined the gain on the total waiting time of passengers at stations noted by \( E(\Delta A) \). If the arrival rate of passengers at stops is constant, we have:

\[E(\Delta A) = \sum_{i=1}^{N} \sum_{l=1}^{m} \int_{d_{i,k}}^{d_{i,k+1}} \frac{1}{2} (d'_{i,k+1} - d_{i,k}) \text{ where } I_{i,k} \text{ and } I_{i,k+1} \text{ is the delay imposed by the regulation of the bus } B'_{i}\text{ at the station } S'_{i} \text{ in the different stations } S'_{k} \text{. Indeed, } E(\Delta T) = \sum_{i=1}^{N} \sum_{l=1}^{m} \sum_{k=1}^{n} r_{i,k}' C_{i,k}^{\prime} \text{ where } r_{i,k}' \text{ is the delay caused by the regulation of the bus } B'_{i}\text{ at the station } S'_{i} \text{ and } C_{i,k} = \mu_{i}(l) (d'_{i,k} - d_{i,k}) - \mu_{i}(l) (d'_{i,k+1} - d_{i,k+1}). \]

To minimize the transit duration of passengers in a node from the line \( l_i \) to the line \( l_k \) or the opposite, a quality indicator which measures the gain, induced by the regulation, on the total duration of the transit was proposed in [1][2][3]. It is supposed that the number of passengers carrying out the correspondence from \( B'_{i} \) to \( B'_{i,k} \) is proportional to the charge of \( B'_{i} \) at its arrival at the
node \((i_t, i_j, h_i, k_i)\) with a rate equal to \(\psi_{(i_t, i_j)}^{i_k}\). We calculate the number of passengers in transit in each node \(i\) in the following way: 

\[
n_pl(r_{i_{(i_t, i_j)}}, r_{i_{(i_t, i_j)}}^{i_k}) = \psi_{(i_t, i_j)}^{i_k} C_{i\rightarrow i_k}^{i_k} \quad \text{with} \quad C_{i\rightarrow i_k}^{i_k},
\]

the charge of \(r_{i_{(i_t, i_j)}}\) on its arrival at the node, i.e. at its departure from the station \(S_{i_{(i_t, i_j)}}\). We assume that the rates \(\psi_{(i_t, i_j)}^{i_k}\) can be considered constant for all buses [4]. The total waiting time of passengers can be given by:

\[
A_{\text{waiting}} = \sum_{(i_t, i_j, h_i, k_i) \in S} \text{waiting}(r_{i_{(i_t, i_j)}}, r_{i_{(i_t, i_j)}}^{i_k}, S_{i_{(i_t, i_j)}}) = \sum_{(i_t, i_j, h_i, k_i) \in S} \psi_{(i_t, i_j)}^{i_k} C_{i\rightarrow i_k}^{i_k} (d_{i_{(i_t, i_j)}}^{i_k} - d_{i_{(i_t, i_j)}}^{i_k}).
\]

Thus, the quality indicator can be deduced by comparing the values of the transit durations without and with regulation. The gain on the total transfer time is then equal to \(E(A_{\text{waiting}}) - A_{\text{waiting}}\) (without regulation) - \(A_{\text{waiting}}\) (with regulation). We aim, therefore, to maximize this gain to reduce the durations of transit at the nodes.

2.4 Global evaluation function
In order to aggregate the three quality indicators previously presented, \(E(\Delta A), E(\Delta A_{\text{waiting}})\) and \(E(\Delta T)\), in a one global function, the authors of [1][2][3][4] have defined weights for the different criteria. In fact, an importance degree could be fixed to each criterion, according to the different constraints which intervene. The cost function to be maximized can be reduced to the following one:

\[
\alpha E(\Delta A) + \beta E(\Delta A_{\text{waiting}}) - \gamma E(\Delta T)
\]

with \(\alpha, \beta\) and \(\gamma\) are positive parameters fixed by the regulator. Such parameters give the weight of each criterion.

![Fig 1. Arrival distribution of \(\mu_k(t)\) between \(d_{i_{(i_t, i_j)}}^{i_k}\) and \(d_{i_{(i_t, i_j)}}^{i_k}\)](image)

3 Transportation systems with limited capacities
As we can notice, the problem can easily be reduced to an optimisation problem with several variables. If we assimilate the set of \(d_{i_{(i_t, i_j)}}^{i_k}\) to a vector \(z = (z_1, z_2, \ldots, z_p, \ldots, z_r)\) from \((\Re_+)^p\) with \(r = \sum_{i=1}^{\infty} n_i m_i\), the regulation problem is reduced to the following problem (II):

\[
(I): \text{Maximize } f(z)
\]

such that \(g_h(z) \leq 0\)

with \(h \in \{1,2,\ldots,p\}\)

\(f(.)\) is a second degree polynomial function in \(z\), \(p\) is the total number of constraints on the \(d_{i_{(i_t, i_j)}}^{i_k}\) and \(g_h(.)\) are linear forms in \(z\). The problem (II) is then elementary. We introduce the Lagrange multiplier \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots, \lambda_p)\) and the parameters \(\psi(y, \lambda, y) = f(z) + \sum_{i=1}^{p} \lambda_i (g_i(z) + y_i)\) where all the conditions of partial derivability are satisfied for all variables in the first order as well as in the second order. Therefore, the optimal solution \(z^*\) exists and it is calculable.

4 Example and Results
We consider for the sake of illustration a bus network composed of \(N=3\) lines, these three lines are crossed in a node. Each line contains \(n_i = 10\) stops and \(m_i = 5\) buses. Let us study a disturbance that affects the third bus \(B_3^l\) of the line 1. This disturbance is detected at 16h00 and it is caused by a technical problem which obliges the bus \(B_3^l\) to have a standstill at the stop \(S_3^l\). All lines have a passage frequency of 45 minutes. The maximal delay is \(\pi_{i_k} = 30\) minutes, the minimal duration is \(d_{i_{(i_t, i_j)}}^{i_k} = 12\) minutes and the transit minimal duration is \(\tau_{i_{(i_t, i_j)}}^{i_k} = 9\) minutes

We assume that the studied horizon is included in homogeneous period of the day, then the arrival distributions of passengers at stops are constant and equal to \(\mu = 2\) passengers per minute. We suppose also that the number of passengers in a bus from a line arriving at a node and willing to take a bus of another line is proportional to the load of the bus with a rate of 10%. These rates between the concerned lines are supposed constant. Hence, the non-linear system to optimize is:
\[ \Psi (\mathbf{d}_i^{10}, \mathbf{d}_i^{11}, \ldots, \mathbf{d}_i^{19}) = \]
\[ \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{J}} \left( d_{i,j}^{10} - d_{i,j}^{11} + d_{i,j}^{11} - d_{i,j}^{10} \right) + \]
\[ \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{J}} \left( d_{i,j}^{11} - d_{i,j}^{12} \right) \left( d_{i,j}^{10} - d_{i,j}^{11} - d_{i,j}^{11} - d_{i,j}^{10} \right) + \]
\[ \sum_{i \in \mathcal{N}} \left( d_{i,j}^{12} - d_{i,j}^{13} \right) \left( d_{i,j}^{11} - d_{i,j}^{10} - d_{i,j}^{10} - d_{i,j}^{11} \right) \]
subject to:

- \( \forall 1 \leq i \leq N, \forall 1 \leq j \leq m_i, \forall 1 \leq k \leq n_i - 1, d_{i,j,k+1} - d_{i,j,k} \geq d_{i,j,k} \)
- \( \forall (i_1, i_2, k_1, k_2) \in \mathcal{I}, d_{i_1,j_1,k_1} - d_{i_2,j_2,k_2} \geq 0 \)
- \( \forall 1 \leq i \leq N, \forall 1 \leq j \leq m_i, \forall 1 \leq k \leq n_i, d_{i,j,k}^{10} - d_{i,j,k}^{11} \leq \tau_{i,j} \)
- \( \forall 1 \leq i \leq N, \forall 1 \leq j \leq m_i, \forall 1 \leq k \leq n_i, d_{i,j,k}^{11} \leq d_{i,j,k}^{12} \)

The optimal solution given by the resolution of the non-linear system above is obtained instantaneously. It is illustrated on the following figures (Fig 2, Fig 3 and Fig 4):

![Fig 2. vehicles rescheduling for the line 1](image1)

![Fig 3. vehicles rescheduling for the line 2](image2)

![Fig 4. vehicles rescheduling for the line 3](image3)

The thin lines represent the planned schedules of the buses whether the bold ones represent the disturbed ones and the new schedules resulting from the application of the optimal solution correspond to the dashed lines. We notice that the third vehicle of the line 1 has made a delay of 21 minutes this perturbation has generated a lacuna between B3 (the disturbed vehicle) and its predecessor B2. In order to equilibrate between the buses arrival times, the vehicles are delayed as shown and the objective function is optimized.

Finally, the application of the resolution approach explained above provides regulators of traffic with an immediate decision that are based on a projection in the future of the network state.

5 Conclusion

In conclusion, the problem considered in this article is a very elementary optimisation problem. We also note the absence of any preliminary analysis of complexity in [1-5]. It would have been more judicious for the authors of these references to study the complexity of the problem before any resolution by genetic algorithms. We note clearly that the problem does not represent any combinatorial aspect. All the variables are real, the constraints are linear and the criteria are polynomials functions with several variables. So it can easily be solved using many softwares like for example the Maximal software “MPL for Windows with the CPLEX solver”.

6. Références
