Pythagorean Interval-Systems

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Abstract: The paper offers an alternative to the “Pythagorean homomorphism,” that is, the homomorphism \( P \) of \( \mathbb{Z} \times \mathbb{Z} \) into \( \mathbb{R} \) defined by
\[
P(s, g) = \log 1.5^n \cdot 2^m
\]
for any integer-pair \((s, g)\) satisfying, for some pair of integers \(m\) and \(n\),
\[
(s, g) = ((7, 4) \cdot m + (12, 7) \cdot n) = (7m + 12n, 4m + 7n).
\]
It is argued that the Pythagorean homomorphism is empirically invalid, and that the dichotomy of integer-pairs vs. real numbers is essential to the notion “music interval.”

Key-Words: Pythagorean homomorphism; integer-pairs; log frequency-ratios; musical intervals; music cognition.

1 Introduction
It is not difficult to see that musical intervals in conventional nomenclature correspond to integer-pairs. For example, “P1”=(0, 0), “ascending m3”=(3, 2), “ascending A2”=(3, 1), “ascending P5”=(7, 4), “descending P8”=(–12, –7), etc. The significance of this correspondence, however, is open to interpretation. If \( P \), for example, is the homomorphism of \( \mathbb{Z} \times \mathbb{Z} \) into \( \mathbb{R} \) defined by
\[
P(s, g) = \log 1.5^n \cdot 2^m
\]
for any integer-pair \((s, g)\) satisfying, for some pair of integers \(m\) and \(n\),
\[
(s, g) = ((7, 4) \cdot m + (12, 7) \cdot n) = (7m + 12n, 4m + 7n),
\]
then one may argue that integer-pairs are mere labels of “Pythagorean” log frequency-ratios, that is, logs of frequency-ratios of form \( \frac{3}{2}^r \cdot \frac{2}{3}^s \).

2 Problem Formulation
Despite its mathematical elegance and historical prestige, from the empirical point of view the homomorphism \( P \)—to which we shall refer henceforth as “Pythagorean”—is seriously flawed. For the homomorphism implies, contrary to fact, that human beings are capable of producing and perceiving frequency-ratios with infinite precision. There is no evidence that the Pythagorean homomorphism is empirically valid, while there is ample evidence to suggest that it is invalid. For example, there is no evidence to suggest that one necessarily hears, say, the Pythagorean log frequency-ratio \( \log \left( \frac{3}{2}^9 \cdot \frac{2}{3}^5 \right) \) (ca. 317.595 “cents”) as \( (7, 4) \cdot 9 + (12, 7) \cdot 5 \) = “A2,” rather than, say, \((3, 2)\) = “m3.” In fact, any log frequency-ratio of, say, ca. 300 cents—whether “Pythagorean” or not—would normally be heard as “m3”; in an appropriate context, the exact same log frequency-ratio could also be heard as “A2.” Although the Pythagorean homomorphism correctly prioritizes the integer-pairs \((12, 7)\) and \((7, 4)\), on the one hand, and the frequency-ratios \( \frac{3}{2}^9 \) and \( \frac{2}{3}^5 \), on the other, we need a theory that connects these important entities in an empirically defensible manner.

3 Problem Solution
3.1 The Cognitive Interval-System
DEFINITION 1: Let \( \text{NVLS} \) be a copy of the set \( \mathbb{Z} \times \mathbb{Z} \) of all integer-pairs. We shall refer to any element \((s, g)\) in \( \text{NVLS} \) as a note-interval.

DEFINITION 2: Let \( \text{NVLSS} \) be the module \( \{\text{NVLS}; \oplus\} \) over the ring of integers, where \((s, g) \oplus (s', g') = (s + s', g + g') \) for any two note-intervals \((s, g)\) and \((s', g')\) in \( \text{NVLS} \), “+” is the usual addition operation on the integers, and \( k \oplus (s, g) = (k \cdot s, k \cdot g) \) for any integer \( k \) and any note-interval \((s, g)\), “.” is the usual multiplication operation on the integers. We shall refer to \( \text{NVLSS} \) as the note-interval space.
DEFINITION 3: Fix any note-interval \((a, b)\) satisfying \(a > b > 1\). We shall refer to \((a, b)\) as a cognitive octave.

DEFINITION 4: Let \(O\) be an equivalence relation in NVLS satisfying \((s, g)O(s', g')\) for any two note-intervals \((s, g)\) and \((s', g')\) if, for some integer \(i\), \((s', g') = (s, g) + i \cdot (a, b)\). We shall refer to \(O\) as the octave relation, and to any two note-intervals \((s, g)\) and \((s', g')\) satisfying \((s, g)O(s', g')\) as octave related.

DEFINITION 5: Let \((s, g)\) be any note-interval in NVLS. We shall refer to any equivalence class of the form \{\((s, g) + n \cdot (a, b)\)\}, \(n\) runs over all integers, as a note-class interval. We shall use the square-bracket notation \([s, g]\) to denote the note-class interval \{\((s, g) + n \cdot (a, b)\)\}, \(n\) runs over all integers. We shall use the boldface notation \(\text{NVLS}^\text{class}\) to refer to the set \{\([s, g]\)\} of all note-class intervals.

DEFINITION 6: Let \(\text{NVLS}^\text{class}\) be the module \{\(\text{NVLS}; \otimes\)\} over the ring of integers, \([s, g] \otimes [s', g'] = [s + s', g + g']\) for any two note-class intervals \([s, g]\) and \([s', g']\) in NVLS, “+” is the usual addition operation on the integers, \(k = [k \cdot s, k \cdot g]\) for any note-class interval \([s, g]\) in NVLS and any integer \(k\), “-” is the usual multiplication operation on the integers. We shall refer to \(\text{NVLS}^\text{class}\) as the note-class interval space.

DEFINITION 7: Fix any cognitive octave \((a, b)\). Let NVLS be the note-interval space, let \(O\) be the octave relation, and let \(\text{NVLS}^\text{class}\) be the note-class interval space. We shall refer to the triple \(\Gamma = \{\text{NVLS}, O, \text{NVLS}^\text{class}\}\) as the cognitive interval-system relative to \((a, b)\).

### 3.2 The Psychoacoustical Interval-System

DEFINITION 8: Let \(\text{PVLS}\) be a copy of the set of all real numbers \(\mathbb{R}\). We shall refer to any element \(p\) in \(\text{PVLS}\) as a pitch-interval.

DEFINITION 9: Let \(\text{PVLSS}\) be the module \{\(\text{PVLS}; +\)\} over the ring of integers, “+” is the usual addition operation on the reals. We shall refer to \(\text{PVLSS}\) as the pitch-interval space.

DEFINITION 10: Let \(\text{FREQS}\) be a copy of the set \(\mathbb{R}^+\) of all strictly positive real numbers. We shall refer to any element \(f\) in \(\text{FREQS}\) as a frequency-ratio. Let \(\text{FREQSS} = \{\text{FREQS}; \cdot\}\) be the group \(\text{FREQS}\), “\(\cdot\)” is the usual multiplication operation on the reals. We shall refer to \(\text{FREQSS}\) as the frequency-ratio space.

DEFINITION 11: Let \(\text{PVLS}\) be the pitch-interval space, and let \(\text{FREQSS}\) be the frequency-ratio space. Finally, let \(\Psi\) be the function from \(\text{FREQS}\) into \(\text{PVLS}\) defined by \(\Psi(f) = \log(f)\) for any frequency-ratio \(f\) in \(\text{FREQS}\). We shall refer to \(\Psi\) as the psychoacoustical function, and to the triple \(\Psi = \{\text{PVLS}, \text{FREQSS}, \Psi\}\) as the psychoacoustical interval-system. Given the psychoacoustical interval-system \(\Psi\), we shall refer to the pitch-interval \(\log 2\) as the psychoacoustical octave, and to the pitch-interval \(\log 1.5\) as the psychoacoustical fifth.

### 3.3 Composite Interval-Systems

DEFINITION 12: Let \(\Gamma = \{\text{NVLS}, O, \text{NVLS}^\text{class}\}\) be the cognitive interval-system relative to \((a, b)\), and let \(\Psi = \{\text{PVLS}, \text{FREQSS}, \Psi\}\) be the psychoacoustical interval-system. Let \(\text{TUNE}\) be any function from NVLS×PVLS onto the open-closed (mathematical) interval \((0, 1]\), such that, for any fixed note-interval \((s, g)\), \(\text{TUNE}((s, g), p)\), \(p\) varies over all pitch-intervals, is any normal statistical distribution (see Fig. 1). We shall refer to \(\text{TUNE}\) as a tuning function. We shall write \(P^{\varepsilon}_{((s, g))}\), \(0 < \varepsilon \leq 1\), to refer to any pitch-interval \(p\) satisfying \(\text{TUNE}((s, g), p) = \varepsilon\).
DEFINITION 13: Let $\Gamma = \{\text{NVLSS}, \Omega, \text{NVLSS}\}$ be the cognitive interval-system relative to $(a, b)$, let $\Psi = \{\text{PVLSS}, \text{FREQSS}, \psi\}$ be the psycho-acoustical interval-system, and let $\text{TUNE}$ be any tuning function satisfying $\log_2 \left( \frac{p_{k(a, b)}}{p_{k(a, b)'}} \right) = 2^k$. We shall refer to any triple $\Gamma / \Psi = \{\Gamma, \Psi, \text{TUNE}\}$ as a composite interval-system.

3.4 Efficient Interval-Systems

DEFINITION 14: Let $\Gamma = \{\text{NVLSS}, \Omega, \text{NVLSS}\}$ be the cognitive interval-system relative to $(a, b)$. We shall refer to any note-interval $(q, r)$ in NVLS, $1 \leq q \leq a - 1$, as a cognitive fifth, if $(a, q) = 1$.

DEFINITION 15: Let $\Gamma = \{\text{NVLSS}, \Omega, \text{NVLSS}\}$ be the cognitive interval-system relative to $(a, b)$. Fix any cognitive fifth $(q, r)$, and let $\text{NVLS}^q_r$ be any subset of NVLS satisfying $\text{NVLS}^q_r = \{n \otimes [q, r]\}$, $n = 0, \pm 1, \ldots, \pm \left\lfloor \frac{q}{r} \right\rfloor$. We shall refer to any element $[u, v]$ of $\text{NVLS}^q_r$ as a primary note-class interval.

EXAMPLE 1: Let $(a, b) = (12, 7)$, $(q, r) = (7, 4)$. Then $\text{NVLS} = \{n \otimes [7, 4]\}$, $n = 0, \pm 1, \ldots, \pm 6$, satisfies $\text{NVLS} = \{[0, 0], [1, 1], [2, 1], [3, 2], [4, 2], [5, 3], [6, 3], [6, 4], [7, 4], [8, 5], [9, 5], [10, 6], [11, 6]\} = \{"P1," "m2," "M2," "m3," "M3," "P4," "A4," "d5," "P5," "m6," "M6," "m7," "M7"\}$.

DEFINITION 16: Let $\Gamma / \Psi = \{\Gamma, \Psi, \text{TUNE}\}$ be any composite interval-system. Fix any cognitive fifth $(q, r)$, and let $\text{NVLS}^q_r$ be the set of primary note-class intervals relative to $(q, r)$. We shall refer to any pair $\{\Gamma / \Psi, \text{NVLS}^q_r\}$ as an efficient interval-system, if (A): For any pair of note-intervals $(u, v)$ and $(u', v')$ such that $[u, v]$ and $[u', v']$ are primary note-class intervals, if $u > u'$, then $p^1_{(u, v)} > p^1_{(u', v')}$; (B): For any pair of note-intervals $(u, v)$ and $(u', v')$ such that $[u, v]$ and $[u', v']$ are primary note-class intervals, if $u - u' = 1$, then $|p^1_{(u, v)} - p^1_{(u', v')}|$ is maximal.

THEOREM 1: Let $\Gamma / \Psi, \text{NVLS}^q_r$ be any efficient interval-system. Then, for any pitch-interval $\log_2 u$, $u$ is any integer, there exists a note-interval

Fig. 1. A Sample Function TUNE, for Some Arbitrary but Fixed Note-Interval
(u, v), [u, v] ∈ NVLS for some v, satisfying
p^1_{(u, v)} = \log 2^u. Moreover, if u \not\equiv \frac{x}{2} (mod a), then v
is unique.

3.5 Coherent Interval-Systems

DEFINITION 17: Let \{Γ/Ψ, NVLS\} be any efficient
interval-system. We shall refer to \{Γ/Ψ, NVLS\}
as a consistent interval-system, if
\lim_{\varepsilon \to 0} p^r_{(u, v)} = \log 2^u, (u, v) is any note-interval satisfying
\[ u, v \in NVLS \] see Fig. 2.

DEFINITION 18: Let \{Γ/Ψ, NVLS\} be any
consistent interval-system, and assume that the
system satisfies \( b = \left\lfloor \frac{x}{2} \right\rfloor + 1 \) and \( (b, r) = 1 \). We shall
refer to \{Γ/Ψ, NVLS\} as a coherent interval-
system, if, for any pair of note-intervals \( (u, v) \) and
\( (u', v') \) such that \( [u, v] \) and \( [u', v'] \) are primary
note-class intervals, if \( u > u' \), then \( v \geq v' \).

PROPPOSITION 1: Let \{Γ/Ψ, NVLS\} be any
coherent interval-system. Set \( \alpha = 1 \)
TUNE((u, v), log 2^u = \frac{u}{2a}) \( (u, v) \) is any note-interval
satisfying \( [u, v] \in NVLS \) (see Fig. 2). Then, for
any pair of note-intervals \( (u, v) \) and \( (u', v') \) such
that \([u, v] \) and \([u', v'] \) are primary note-class
intervals, if \( p^e_{(u, v)} > p^e_{(u', v')} \), \( \alpha < \varepsilon, \varepsilon' \leq 1 \), then
\( u > u' \) and \( v \geq v' \).

THEOREM 2: Let \{Γ/Ψ, NVLS\} be any consistent
interval-system. Assume that the system satisfies

\[ b = \left\lfloor \frac{x}{2} \right\rfloor + 1 \] and \( (b, r) = 1 \). Then \{Γ/Ψ, NVLS\} is
a coherent interval-system if, and only if, either
(A): \( (a, b) = (3 + 2n, 2 + n) \), \( n \) is any non-negative
integer, and \( (q, r) = (2, 1) \) or \( (q, r) = (a - 2, b - 1) \); or
(B): \( (a, b) = (4 + 4n, 3 + 2n) \), \( n \) is any non-
negative integer, and \( (q, r) = (\frac{q}{2} - 1, \frac{b - 1}{2}) \), or
(\( (q, r) = (\frac{q}{2} + 1, \frac{b + 1}{2}) \).

PROPPOSITION 2: Let \{Γ/Ψ, NVLS\} be any
coherent interval-system. Then any note-interval
\( (s, g) \) satisfies \( (s, g) = m \cdot (q, r) + n \cdot (a, b) \), where
\( m = \pm s \cdot b \mp g \cdot a \), and \( n = \mp s \cdot r \pm g \cdot q \). Moreover,
p^1_{(s, g)} = \log 2^u.
3.6 Pythagorean Interval-Systems

**Definition 19**: Let \( \{\Gamma / \Psi, \text{NVLS}\} \) be any coherent interval-system. We shall refer to \( \{\Gamma / \Psi, \text{NVLS}\} \) as a Pythagorean interval-system, if \( p_{(q, r)} = \log 2^{\frac{3}{5}} \) is as close as possible to the psychoacoustical fifth log1.5.

**Theorem 3**: Let \( \{\Gamma / \Psi, \text{NVLS}\} \) be any Pythagorean interval-system. Then \((a, b) = (12, 7)\) and \((q, r) = (7, 4)\).

4 Conclusion

We offer the theory of Pythagorean interval-systems as an alternative to the Pythagorean homomorphism initially discussed. Both theories prioritize the conventional “P8” and “P5,” on the one hand, and the frequency-ratios \( \frac{5}{3} \) and \( \frac{3}{2} \), on the other. However, the Pythagorean homomorphism assumes a priori that “P8”=(12, 7) and “P5”=(7, 4), and proceeds to uniquely pair any note-interval \((s, g) = ((7, 4) \cdot m + (12, 7) \cdot n) \) with the pitch-interval \( \log 1.5^n \cdot 2^m \). By contrast, the theory of Pythagorean interval-systems assumes the existence of a generalized “cognitive octave”=(a, b) and a generalized “cognitive fifth”=(q, r). Assuming that, in general, a statistical normal distribution offers the only empirical defensible description of the relationship of any note-interval to the set of all pitch-intervals, the theory assumes that in the special case of the cognitive octave \((a, b)\), the distribution peaks at \( \log 2 \). On the basis of additional assumptions, in particular, “efficiency” (Definition 16) and “coherence” (Definition 18), and by assuming a relationship between the cognitive fifth \((q, r)\) and the psychoacoustical fifth \( \log \frac{3}{5} \) (Definition 19), the theory is ultimately able to fix \((a, b) = (12, 7)\) and \((q, r) = (7, 4)\) (Theorem 3). Note that in the case of a cognitive fifth \((q, r)\) the theory does not propose a distribution of pitch-intervals that peaks at \( \log \frac{3}{5} \). Rather, the distribution peaks at the “equal-tempered” pitch-interval \( \log 2^\frac{3}{5} \) (Theorem 1). Superficially, then, the term “Pythagorean” in “Pythagorean interval-system” may seem misleading. So-called “Pythagorean intonation” and equal temperament, after all, are traditionally viewed as lying on two opposite sides of a conceptual divide. We can only hope that the present paper will contribute to a reassessment of this position.

In closing, it should be useful to consider the assumptions of “efficiency” and “coherence” from an empirical vantage point.

**Efficiency**. Condition (B) of Definition 16, \( |u - u'| = 1 \) implies that \( \left| p_{(u, v)} - p_{(u', v')} \right| \) is maximal, addresses the finding that given two arbitrary yet distinct pitch-intervals, the human capacity to distinguish one from the other is not without limits. In particular, the closer to each other on the pitch-interval continuum the two pitch-intervals lie, the more a human listener will find it difficult to tell them apart.

Definition 16 leads directly to Theorem 1, the cognitive content of which is readily apparent. Indeed, the one-to-one relationship of certain pitch-intervals to certain note-intervals described in Theorem 1 is, arguably, what allows musical communication to take place in the first place. Note that this relationship involves a generalization of equal temperament.

**Coherence**. While Theorem 1 may allow the existence of musical communication in the first place, Definition 18, from which Proposition 1 directly follows, would seem to facilitate musical communication among human beings in particular, given their finite memories.

Let \( p \) be any pitch-interval satisfying \( p \neq \log 2^{\frac{3}{5}}, s \neq \frac{q}{r} \) (mod a) is any integer. Then there exists a unique note-interval \((u, v)\), \([u, v]\) is a primary note-class interval, such that \( p_{(u, v)} \) satisfies \( \alpha < \varepsilon \leq 1 \) (see Fig. 2). Given now another pitch-interval \( p' \) satisfying \( p' \neq \log 2^{\frac{3}{5}}, a \) human listener need not search among all “primary” note-intervals, in order to find the unique note-interval \((u', v')\) such that \( p_{(u', v')} \) satisfies \( \alpha < \varepsilon \leq 1 \). Suppose that \( p' > p \). Then one may search only among note-intervals \((u', v'), [u', v']\) is a primary note-class interval, satisfying both \( u' > u \) and \( v' > v \).

Although we are unable to prove any of the stated propositions and theorems within the limits of the present paper, we may tentatively conclude that integer-pairs are not reducible to log frequency-ratios, as the Pythagorean homomorphism implies. Integer-pairs and log frequency-ratios capture two essential—though related—attributes of the notion “music interval.”