Alternative Approach to Continuous-Time Stochastic Systems

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ABSTRACT
This paper is based on an alternative approach to system theory and deals with an original definition of continuous-time models of discrete-time causal systems, namely with the definition of continuous-time stochastic systems. Despite the general system theory has been intensively studied since 1960’s, none of the developed theories has been fully accepted by professional community up to now. The alternative system theory is based on quite new system paradigms consequent upon attentive observations and results in an axiomatic system theory with correctly and unambiguously defined notions.

KEY WORDS
System theory, Causality law, Stochastic causal system, Diffusion system

1 Introduction
The presented paper deals in detail with originally reviewed definition of models of continuous-time stochastic systems in an alternative approach to system theory, which is based on quite new system paradigms consequent upon attentive observations and results in an axiomatic system theory with correctly and unambiguously defined notions [4], [5]. Due to the fact that we are able to obtain only a finite number of independent observations while studying system properties in the real world, continuous-time systems can be studied within the new approach to system theory only as limits of previously defined (suitable) sequences of discrete-time causal systems. The extension (continualization) of discrete-time systems to uncountable infinite sets, mostly given in terms of continuous real-number intervals, is considerable only if necessarily missing information about the system properties can be appropriately amended to these larger sets. Hence, the extension to infinite sets cannot be based on observations but has to be postulated by an appropriate limit process called the continualization procedure. The time continuity of either the real system model or its trajectory is thus only a hypothesis but not an experimen-

tally proved fact of the matter. The cybernetic continuous-time system is therefore to be derived from a sequence of discrete-time causal systems.

However, there is a possibility that the limit of a given discrete-time causal system sequence does not exists. Consequently, there is no continuous-time counterpart to the given discrete-time causal system. On the other hand, there is always a discrete-time causal system sequence converging to a given continuous-time system. Hence, we deduce that the set of discrete-time systems is (in particular sense) greater than the set of continuous-time systems.

The alternative system theory [3], [4], [5], [7] is based on the assumption that
• the system is an abstract concept purpose of which is to model such a part of the real world that is not influenced by its surroundings
• the system processes are driven by an internal system mechanism governed by a newly reviewed principle and law of causality
• the system behavior is generally stochastic and can be reduced to deterministic one only when stochastic properties can be really neglected
• the system contains only a finite number of variables taking on only a finite number of values. This assumption can be extended to infinite cases when some consistency conditions are fulfilled
• the system is composed of subsystems mutually interconnected only by directional connections, which can be canceled or restored by an external intervention.

These paradigms lead to such axioms that define directly the discrete-time systems only. Suppose there is a given continuous-time system. Then, in order to exhaustively describe its properties, it is necessary to find such a sequence of discrete-time causal systems that (formally) converges to the given continuous-time system. However, since the terms of causality principle and law, causal probability, causal function easily defined in the discrete-time domain are losing their meanings in the continuous-time domain, full attention has to be given to the process of the discrete-time system continualization.
2 Continualization of discrete systems

Note that a particular sequence of linear discrete-time stochastic causal systems converging to a linear continuous-time stochastic system must be constructed unambiguously. Therefore, it is necessary to place some constraint conditions on every newly constructed discrete-time causal stochastic system in the corresponding sequence. These conditions will be discussed later. Now, it is essential that we derive a useful tool to construct a continuous interval from a set of discrete time-point set.

2.1 Extension of the time-points set

The continualization method is based on the extension of the time-points set. Let us suppose that there is given a causal system \( CS_0 \) defined on the time-points set \( T_0 = \{0, h_0, 2h_0, 3h_0, \ldots, nh_0\} \) where \( h_0 = t_i - t_{i-1}, i = 1, 2, \ldots, n \) and \( nh_0 = \vartheta \), \( \vartheta \) is an arbitrary positive fixed time instant (multiple of \( h_0 \)). \( T_0 \) can be extended by inserting a new time point between each two ones of \( T_0 \), e.g., in the middle of their distance. In this way the equidistance of the time-points set is ensured also in a new set \( T_1 = \{0, h_1, 2h_1, 3h_1, \ldots, nh_1\} \) distance between elements of which is \( h_1 = \frac{h_0}{2} \) and \( n_1 h_1 = \vartheta \). A sequence of time points sets is clearly defined as \( T_k \) for any arbitrary \( k^{th} \) step of the extension of the original set \( T_0, T_k = \{0, h_k, 2h_k, 3h_k, \ldots, \vartheta\}, k = 1, 2, \ldots \), where \( h_k = \frac{h_{k-1}}{2} = \frac{h_0}{2^k} \) and again \( n_k h_k = \vartheta \). Such a sequence of sets is increasing for every choice of \( h_0 > 0 \), i.e. \( T_{k-1} \subset T_k, k = 1, 2, \ldots \). The limit of this sequence for \( k \to \infty \) exists if \( \bigcap_{k=1}^{\infty} T_k = \bigcup_{k=1}^{\infty} T_k \) whereupon the limit is equal to \( \lim_{k \to \infty} T_k = \bigcup_{k=1}^{\infty} T_k \).

From the construction of the sets \( T_k \) it is obvious that for \( \lim_{k \to \infty} T_k \) it holds \( \bigcup_{k=0}^{\infty} T_k \subset (0, \vartheta) \) and \( \bigcup_{k=0}^{\infty} T_k \) is a dense set in \( (0, \vartheta) \). The (topological) closure of the set \( \bigcup_{k=0}^{\infty} T_k \) is the interval \( (0, \vartheta) \). A dense subset is "large" enough to ensure correctness of the following abstract thoughts. Let \( J \) is an interval in \( \mathbb{R} \) and \( H \) is a dense subset of \( J \). If \( f, g \) are two continuous functions on \( J \) then \( f(x) = g(x) \) for \( x \in H \) implies \( f = g \) on \( J \).

2.2 Causal system continualization

The system trajectory is according to real observations generated in a sequence of certain segments, which are determined by an ordered decomposition \( D \) of the definition domain \( D \) of the system trajectory \( s \), [12]. Each fundamental segment \( s[D^{(k, l)}] \) is according to the principle and the law of causality generated (not necessarily in the deterministic way) by its comprehensive immediate cause \( s[C^{(k, l)}] \), whereas for each of these relations the causality law is required to hold. Such an approach assures an unambiguous system trajectory description. The causal system \( CS \) is then defined as an ordered triplet \( CS = (T, V, C) \), where the set of all causal relations \( C \) was added to the general abstract system \( S = (T, V) \), see e.g. [3], [4], [11], [12].

Consider a causal system \( CS_0 \),

\[
CS_0 = (T_0, V_0, C_0),
\]

where \( T_0 = \{0, h_0, 2h_0, 3h_0, \ldots, nh_0\} \) is a set of time-points, \( V_0 \) is a set of all system variables and \( C_0 \) is the set of all system causal relations. Our aim is to propose such a sequence of causal systems

\[
CS_k = (T_k, V_k, C_k), \quad k = 0, 1, 2, \ldots,
\]

that (formally) converges for \( k \to \infty \) to a continuous-time system. In every step \( k \), it is (according to the causal system definition [3]) necessary to define the whole set of system attributes \( A_k \), state variables \( s_k(t), t \in T_k \) together with their domain \( V_k \) as well as the set \( \Omega_k \) of all system trajectories and the set \( S_k \) of all system events. From natural and consistency reasons we choose \( A_k = A_{k-1} \), \( s_k(t) = s_{k-1}(t) \) and \( V_k = V_{k-1} \). The system attributes \( A_k \) and the system state \( s_k(t) \) take on their values for all \( t \in T_k \cap T_{k-1} \); therefore it is necessary to define the domain \( D_k \) of the system trajectory \( s_k \), [3]. The system \( CS_k \) can now be defined as an ordered triplet from the equation (2). For \( k \to \infty \) we obtain a continuous-time system defined on a dense subset of \( (0, \vartheta) \). However, with the extension of time-points set we cannot define all real valued time instants, e.g. instants represented by some irrational numbers. Such instants denoted \( \tau \) can be defined by \( \tau = \lim_{k \to \infty} t_k \), where \( t_k \) is a sequence of time points from the dense set \( \bigcup_{k=0}^{\infty} T_k \). This produces the (topological) closure of \( \bigcup_{k=0}^{\infty} T_k \). Similarly, we can define the continuous-time system trajectory for all \( t \in (0, \vartheta) \). The continualization method proposes a clarification of problems with determination of causal (cause-effect) relations within continuous-time systems because there is no such time instant (real number) \( t' \), \( t' \neq t > 0 \) that is immediately preceding to a time instant (real number) \( t, t \in (0, \vartheta) \). Therefore, a high attention should be payed to the properties of the dynamic system variables during the process of continualization. On the other hand, there is no difficulty with the static state variables and, consequently, there is no difficulty with the structural terms definition of continuous-time systems [11].

3 Linear stochastic causal system continualization

The problem of linear stochastic causal systems continualization is connected with a choice of the random processes convergence. For some choices it was solved in the past [3], [8]. This paper deals with a new approach based on the extension of the transition probability density functions of a discrete-time stochastic causal system (convergence in distribution). It will be shown that the continualization method leads to diffusion systems on specific conditions.
3.1 Linear stochastic causal system

If we admit an axiom that each fundamental segment $s|D_{k,l}$ of the system trajectory $s \in \Omega$ ($\Omega$ is a set of all system trajectories) is generated by its comprehensive immediate cause $s|\mathcal{C}_{[k,l]}$ with a particular probability, we can extend the causal system with a set $\mathcal{P}$ of all probabilistic mappings $P^{(k,l)}$ and define the stochastic causal system as an ordered quadruplet

$$\mathcal{PCS} = (T, \mathcal{V}, \mathcal{C}, \mathcal{P}),$$

where the set $\mathcal{P}$ consists of parametric probabilities

$$P^{(k,l)}(s|D^{(k,l)} : s|\mathcal{C}^{(k,l)}),$$

$k = 0, 1, 2, \ldots, e$, $l = 1, 2, \ldots, m$. $P^{(k,l)}$ is called the causal probability of the system $\mathcal{PCS}$, see [12] for more details. Thus, properties of linear stochastic causal systems are comprehensively described by the set $\mathcal{P}$ of all causal probabilistic mappings, which are in the state-space unambiguously represented by conditional probability density function of the Gaussian (normal) distribution

$$f(s(t+h)|s(t)) = \frac{1}{(2\pi)^{\frac{N}{2}} \cdot \text{det}Q} \cdot e^{-\frac{1}{2} \cdot (s(t+h)-A \cdot s(t))^{T} \cdot Q^{-1} \cdot (s(t+h)-A \cdot s(t))},$$

for all $t, t + h \in T$, $s(t)$ is a state vector at time $t$. From now on, we shall write the last equation in brief as $f(s(t+h)|s(t)) \sim \mathcal{N}\{A \cdot s(t), Q(t)\}$. A probabilistic initial condition of the system is then given by $f(s(0)) \sim \mathcal{N}\{m, R\}$.

3.2 Problem formulation

Consider a linear time-invariant stochastic causal system $\mathcal{PCS}_0$ properties of which are described by the transition probability density function

$$f_0(s_0(t+h_0)|s_0(t)) \sim \mathcal{N}\{A_0 \cdot s(t), Q_0(t)\}$$

$t, t + h_0 \in T_0$ with a probabilistic initial condition

$$f_0(s_0(0)) \sim \mathcal{N}\{m_0, R_0\}$$

where $s_0(t) \in \mathbb{R}^N$ is a state vector, $A_0(t) = A_0$ is for all $t \in T_0$ a nonsingular constant real square matrix of dimension $N$ whose eigenvalues are either real positive or complex conjugate, $Q_0(t) = Q_0$ is for all $t \in T_0$ a constant real conditional covariance matrix of the system state $s_0(t + h_0)$ conditioned by the state vector $s_0(t)$, $m_0$ is a real vector of the mean and $R_0$ is a real covariance matrix of the generally stochastic initial condition $s_0(0)$.

Remark: The demand placed on eigenvalues of the matrix $A_0$ follows from previous research [8].

3.3 Problem solution

The aim of the continualization problem of discrete-time linear stochastic causal systems is to derive a class of all continuous-time linear stochastic systems. In the process, it suffices that each continuous-time linear stochastic system is defined only once. Our goal is to continualize (i.e. derive a continuous-time system) the system given in equations (6) and (7), which represents a class of discrete-time linear stochastic causal systems. First, a sequence of discrete-time linear stochastic causal systems is to be proposed with using the extension of the time-points set (see paragraph 2.1). The process of definition of the sets $T_k$, $\mathcal{V}_k$ and $\mathcal{C}_k$ was discussed in paragraph 2.2. It remains to define the sequence of the sets $\mathcal{P}_k$ represented by transition probability density functions $f_k(s_k(t + h_k)|s_k(t))$. The determination of the systems $\mathcal{PCS}_k$ thus consists in finding $f_k(s_k(t + h_k)|s_k(t))$ from given $f_{k-1}(s_{k-1}(t + h_{k-1})|s_{k-1}(t))$, $k = 1, 2, \ldots$, including initial conditions. Subsequently, the continuous-time linear stochastic systems can be derived with a proper limity procedure.

3.3.1 Sequence of linear stochastic causal system

From philosophical background of the continualization problem we demand that the states $s_k(t)$, $t \in T_k$ of the system $\mathcal{PCS}_k$ of the order $N$ are determined by the same type of transition probability density functions as the states $s_0(t)$, $t \in T_0$ of the system $\mathcal{PCS}_0$ (see (6),

$$f_k(s_k(t + h_k)|s_k(t)) \sim \mathcal{N}\{A_k \cdot s_k(t), Q_k\}$$

$t, t + h_k \in T_k$ with probabilistic initial conditions

$$f_k(s_k(0)) \sim \mathcal{N}\{m_k, R_k\}.$$ 

Furthermore, in every step $k$ we demand the properties of each newly constructed system $\mathcal{PCS}_k$ correspond to the properties of previously derived system $\mathcal{PCS}_{k-1}$ in terms of the following equality

$$f_k(s_k(t + 2h_k)|s_k(t)) = f_{k-1}(s_{k-1}(t + h_{k-1})|s_{k-1}(t)),$$

where $t, t + h_{k-1} \in T_{k-1}$, $h_k = \frac{h_{k-1}}{2}$, $k = 1, 2, \ldots$ including their initial conditions

$$f_k(s_k(0)) = f_{k-1}(s_{k-1}(0)).$$

The probabilistic determination $f_k(s_k(t + 2h_k)|s_k(t))$ of the state $s_k(t + 2h_k)$ conditioned by the state $s_k(t)$ can be found by a routine computation (see (8)),

$$f_k(s_k(t + 2h_k)|s_k(t)) \sim \mathcal{N}\{A_k^2 \cdot s_k(t), A_k Q_k A_k^T + Q_k\}.$$ 

The relation (9) expresses the equality of the transition probability density functions $f_k(s_k(t + 2h_k)|s_k(t))$ and
f_{k-1}(s_{k-1}(t+h_k)|s_{k-1}(t)) for all \( t+h_k = t+2h_k \in T_{k-1} \subset T_k \) provided the value of any arbitrary states \( s_k(t) = s_{k-1}(t) \) (12) at time \( t \in T_{k-1} \) is given. Similarly, the probabilistic determinations of initial conditions are according to (10) equal. Such demands are apparently natural. It can be easily shown that if the above stated conditions are fulfilled, all derived transition probability density functions of systems PCS_{k-1} and PCS_k are also equal for all \( t \in T_{k-1} \subset T_k \). Moreover, from principal reasons stated in [8] we demand that the expected length of the trajectory of each newly constructed system PCS_k is minimal from all systems satisfying (9) and (10), which is formulated by

\[
A_k = \arg \min_{A_k^T = A_{k-1}} \left\{ \mathbb{E}\left\{ d(PCS_k) \right\} \right\}. \tag{13}
\]

The parameters \( m_k \) and \( R_k \) of the system sequence PCS_k can be found from (10) by comparing the corresponding transition probability density functions of systems PCS_k and PCS_{k-1}.

\[
m_k = m_{k-1}, \quad R_k = R_{k-1}. \tag{14}
\]

Direct equations for any arbitrary step \( k, k = 1, 2, \cdots \) can easily be derived from these recursive equations as follows

\[
m_k = m_0, \quad R_k = R_0. \tag{15}
\]

The parameters \( A_k, Q_k \) can be found from (9) by comparing the corresponding transition probability density functions \( f_k(s_k(t+2h_k)|s_k(t)) \) and \( f_{k-1}(s_{k-1}(t+h_k)|s_{k-1}(t)) \) wherefrom the following equations were derived,

\[
A_k^2 s_k(t) = A_{k-1} s_{k-1}(t), \tag{16}
\]

\[
A_k Q_k A_k^T + Q_k = Q_{k-1}. \tag{17}
\]

With respect to (12), the equation (16) is for nonsingular matrices \( A_k^2 \) and \( A_{k-1} \) satisfied by

\[
A_k^2 = A_{k-1}. \tag{18}
\]

From this recursive equation it is possible to derive the direct formula in any arbitrary step \( k \)

\[
A_k^2 = A_0. \tag{19}
\]

solution of which gives the parameter \( A_k \). Before finding a solution to (19) it is necessary to introduce the matrix equation \( X^m = A \). Note that there exists a discrete as well as a continuous character of multivalence in the general solution to \( X^m = A \), which is given by

\[
X = UX \left[ \sqrt{J_A^{(1)}}, \sqrt{J_A^{(2)}}, \cdots, \sqrt{J_A^{(u)}} \right] X_{J(A)}^{-1} U^{-1}, \tag{20}
\]

where the Jordan canonical form of the matrix \( A \) with eigenvalues \( \lambda_1, \lambda_2, \cdots, \lambda_u \) is given by formula \( A = U \cdot J \cdot U^{-1} = U \cdot [J_{A}^{(1)}, J_{A}^{(2)}, \cdots, J_{A}^{(u)}] \cdot U^{-1} \). The discrete (in this case finite) character of multivalence of the right-hand side of (20) arises from the choice of the distinct branches of the function \( \sqrt{\lambda} \) in the various blocks of the quasi-diagonal matrix \( J \) (for \( \lambda_j = \lambda_k \) the branches of \( \sqrt{\lambda} \) in the \( j \)th and \( k \)th diagonal blocks may even be distinct). The continuous character arises from arbitrary parameters contained in \( X_{J} \), which is determined by the solution to the special case of the matrix equation \( A \cdot X = X \cdot B \) for \( A = B \).

**Theorem 1** Let eigenvalues of a nonsingular square matrix \( A = U \cdot [J_A^{(1)}, J_A^{(2)}, \cdots, J_A^{(u)}] \cdot U^{-1} \) be given in the form

\[
\lambda_i^{(1)} = \max_{\lambda_i^{(u)}} \left\{ \cos \varphi(\lambda_i^{(1)}) + j \cdot \sin \varphi(\lambda_i^{(1)}) \right\},
\]

\( i = 1, 2, \cdots, u \). If every block \( \sqrt{J_A^{(1)}} \) in the general solution

\[
X = UX \left[ \sqrt{J_A^{(1)}}, \sqrt{J_A^{(2)}}, \cdots, \sqrt{J_A^{(u)}} \right] X_{J(A)}^{-1} U^{-1} \tag{21}
\]

to a matrix equation \( X^m = A \) is for \( m > 1 \) determined by the branch

\[
\lambda_i^{(1)} = \sqrt{\lambda_i^{(1)}(X)} = \max_{\lambda_i^{(u)}} \left\{ \cos \varphi(\lambda_i^{(1)}(X)) + j \cdot \sin \varphi(\lambda_i^{(1)}(X)) \right\},
\]

of the function \( \sqrt{\lambda_i^{(1)}(X)} \) where

\[
\varphi(\lambda_i^{(1)}(X)) = \min_{\lambda_i^{(u)}} \left\{ \arctan \left( \frac{\varphi(\lambda_i^{(1)}(X)) + 2\pi}{m} \right) \right\}, \tag{23}
\]

then there exists only one solution to the equation \( X^m = A \).

**Proof 1** The discrete character of multivalence of the right-hand side of (21) is eliminated by the specification (23) of one particular branch (22) of the function \( \sqrt{\lambda_i^{(1)}(X)} \). Consequently, every block \( \sqrt{J_A^{(1)}} \) containing the same eigenvalue \( \lambda_i \) must be determined by the same branch of \( \sqrt{\lambda_i^{(1)}(X)} \), which holds for all distinct eigenvalues of \( A \). Thus the specification (23) eliminates also the continuous character of the right-hand side of (21), i.e. \( X_{J} \) and \( J_A^{(1)}, J_A^{(2)}, \cdots, J_A^{(u)} \) commute. The solution (21) with respect to (22) and (23) to \( X^m = A \) is then unique.

The general solution to the equation (19) is represented by formula

\[
A_k = U_{(A_0)} X_{J(A_0)} \cdot \sqrt{J_A^{(1)}}, \sqrt{J_A^{(2)}}, \cdots, \sqrt{J_A^{(u)}} \cdot X_{J(A_0)}^{-1} U_{(A_{k-1})}^{-1}, \tag{24}
\]

According to the theorem 1, it is necessary to determine every block \( \sqrt{J_A^{(i)}} \) \( i = 1, 2, \cdots, u \) by the branch

\[
\lambda_i^{(1)}(A_k) = \max_{\lambda_i^{(u)}} \left\{ \cos \varphi(\lambda_i^{(1)}(A_k)) + j \cdot \sin \varphi(\lambda_i^{(1)}(A_k)) \right\}, \tag{25}
\]
of the function $\sqrt[2]{\frac{\lambda_i^{(1)}}{(\lambda_{\lambda})}}$, where

$$\varphi(\lambda_{\lambda}) = \arg \min_{\lambda_i} \left| \arctan \left( \frac{\lambda_i^{(1)}}{2\pi} \right) \right|. \quad (26)$$

The last formula correspond to the demand (13) placed on $A_\lambda$. With respect to the properties of eigenvalues of $A_0$ (paragraph 3.2) we can introduce the following theorem.

**Theorem 2** The eigenvalues of a non-singular real square matrix $X$ are either real positive or complex conjugate in the quadrant 1 or 4 of the complex plane if the general solution to the matrix equation $X^m = A$ for $m > 1$ is obtained according to the theorem 1, where eigenvalues of $A$ are either real positive or complex conjugate.

**Proof 2** A non-singular real square matrix $X$ has eigenvalues either real or there exists a complex conjugate eigenvalue to each complex eigenvalue of $X$ because it holds: if coefficients $b_0, b_1, \ldots, b_N$ of the characteristic polynomial

$$p(\lambda_{\lambda}) = \det(\lambda E - X) = \lambda^N + b_N \lambda^{N-1} + \cdots + b_1 \lambda + b_0 = 0$$

of the matrix $X$ are real valued then also

$$p(\lambda_{\lambda}(A)) = \lambda^N + b_N \lambda^{N-1} + \cdots + b_1 \lambda + b_0 = 0,$$

where $\lambda_{\lambda}(A)$ is complex conjugate to $\lambda_{\lambda}(X)$. If $\lambda_{\lambda}(A)$ is complex conjugate to the eigenvalue $\lambda_{\lambda}(A)$ of $A$ then

$$\varphi(\lambda_{\lambda}(A)) = -\varphi(\lambda_{\lambda}(A)), \quad |\lambda_{\lambda}(A)| = |\lambda_{\lambda}(X)|.$$  

From the equation (22) it holds

$$|\lambda_{\lambda}(A)| = |\lambda_{\lambda}(X)|$$

and from (23) it follows

$$\varphi(\lambda_{\lambda}(X)) = \frac{\varphi(\lambda_{\lambda}(A))}{m} = -\frac{\varphi(\lambda_{\lambda}(A))}{m} = -\varphi(\lambda_{\lambda}(X)),$$

For any real-valued eigenvalue $\lambda_{\lambda}(X)$ of $X$ it holds $|\lambda_{\lambda}(X)| = \frac{n}{\lambda_{\lambda}(A)}$, $\varphi(\lambda_{\lambda}(A)) = \frac{\varphi(\lambda_{\lambda}(A))}{m} = 0$. From (23) it directly follows $\varphi(\lambda_{\lambda}(X)) \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$ for all $\lambda_{\lambda}(X), i = 1, 2, \ldots, u$.

The solution (24) with (25) and (26) is according to the theorem 1 unique and $A_k$ is according to the theorem 2 real square matrix for each $k = 1, 2, \ldots$.

**Remark:** The eigenvalues of the matrix $A_k$ converge for $k \rightarrow \infty$ to continuous modes, $\lim_{k \rightarrow \infty} \varphi(\lambda_{\lambda}(A_k)) = 1$ and $\lim_{k \rightarrow \infty} \varphi(\lambda_{\lambda}(A_k)) = 0$.

Further, it is necessary to compute the matrix parameter $Q_k$ of the transition probability density function $f_k(s_k(t + 2h_k)|s_k(t))$. The formula (17) is given in the form of Lyapunov’s equation. Let us define a vector $v_Q$, $i = 0, 1, 2, \ldots$

$$v_Q = \begin{bmatrix} Q_i^{(1,1)} & Q_i^{(2,1)} & \cdots & Q_i^{(2,1)} & Q_i^{(2,2)} & \cdots & Q_i^{(N,N)} \end{bmatrix}^T.$$  

(27)

where $Q_i^{(p,q)}$ is an element of $Q_i$ in the $p$-th row and in the $q$-th column. We will also use the Kronecker’s product [2] denoted by the symbol "$\otimes$".

**Remark:** In the following, the symbol "$\left[ vM \right]$, $\dim vM = n \times 1$ according to (27), formally stands in the place of a square matrix $M$, $\dim M = n \times n$.

Using the Lyapunov’s lemma [10], the equation (17) can be rewritten in an equivalent formula

$$\left( (A_k \otimes A_k + I) \cdot vQ_k \right) = \left[ vQ_{k-1} \right].$$  

(28)

The analysis of the last equation was accomplished in [8], which leads to the solution

$$vQ_k = \left( (A_k \otimes A_k + I) \cdot vQ_{k-1} \right),$$

(29)

provided the inversion $(A_k \otimes A_k + I)^{-1}$ exists, i.e. if for any arbitrary eigenvalues $\lambda_1(A_k), \lambda_2(A_k)$ of the matrix $A_k$ it holds $\lambda_1(A_k) \cdot \lambda_2(A_k) \neq -1$. This condition is implicitly satisfied by (25) and (26). The proof can be found in [8]. The formula (28) presents an implicit recursive equation for computing the covariance matrix $Q_k$. Moreover, the direct equation can be found. The solution to the recursive equation (28) in any arbitrary integer step $k > 0$ with initial condition $Q_0$ is

$$\left( \prod_{i=1}^{k} (A_i \otimes A_i + I) \right) \cdot vQ_k = \left[ vQ_0 \right].$$  

(30)

With respect to $A_k^i \otimes A_k^j = (A_k \otimes A_k)^i$ (using properties of Kronecker’s product: $A_1(A_2 \otimes B_1)B_2 = (A_1B_1) \otimes (A_2 \otimes B_2)$) and thus $A_k^1A_k^{p-1} - A_kA_k^{p-1} = A_k(A_k \otimes A_k)(A_k^{p-2} \otimes A_kA_k^{p-2}) = \cdots = (A_k \otimes A_k)^0$ and with respect to (19), the multiplication on the left-hand side of (30) can be transformed into a sum,

$$\prod_{i=1}^{k} (A_i \otimes A_i + I) = \sum_{i=0}^{2^{k-1}} (A_k \otimes A_k)^i.$$  

(31)

After substituting $A_k^{(KR)} k = A_k \otimes A_k$, the equation (30) assumes the following form

$$\left( \sum_{i=0}^{2^{k-1}} A_k^{(KR)} k \right) \cdot vQ_k = \left[ vQ_0 \right].$$  

(32)

Now, we can compute the sum $\sum_{i=0}^{2^{k-1}} A_k^{(KR)} k$ on condition that $(A_k^{(KR)} k - I), k = 1, 2, \ldots$ is non-singular

$$\sum_{i=0}^{2^{k-1}} A_k^{(KR)} k = (A_k^{(KR)} k - I) \cdot (A_k^{(KR)} k - I)^{-1},$$

(33)
where \((A_{(KR)}^{k} k - I)\) and \((A_{(KR)} k - I)^{-1}\) commute.

**Remark:** The sum \(\sum_{i=0}^{2^{k}-1} A_{i}^{k}(K)\) can be computed separately for each element of \(A_{k}(K)\) if \((A_{(KR)} k - I)\) is singular.

If we substitute (33) into (32) we obtain

\[
\left( A_{(KR)}^{k} k - I \right) (A_{(KR)} k - I)^{-1} \cdot vQ_{k} = \left[ vQ_{0} \right].
\]  
(34)

Hence the direct equation for computing the matrix parameter \(Q_{k}\) assumes the following form

\[
vQ_{k} = \left[ (A_{(KR)} k - I) (A_{(KR)} 0 - I)^{-1} \cdot vQ_{0} \right],
\]  
(35)

where \(A_{(KR)}^{k} = A_{(KR)} 0\), which follows from properties of Kronecker’s product with respect to (19).

Finally, the sequence of systems \(\mathcal{PCS}_{k}, k = 1, 2, \ldots\) is described by transition probability density functions \(f_{k}(s_{k}(t) + h_{k})|s_{k}(t))\) and initial conditions \(f_{k}(s_{k}(0))\) parameters of which were given in (15), (24) and (35). Now, we can proceed to the limit case of the system sequence \(\mathcal{PCS}_{k}\) for \(k \to \infty\).

### 3.3.2 Continuous-time stochastic system

In order to emphasize the dependency of the transition probability density function \(f_{k}(s_{k}(t)|s_{k}(\tau)), t > \tau, k = 0, 1, 2, \ldots\) on four variables \(x, t, y, \tau\), where \(x = s_{k}(t)\) denotes the value of the random process \(\{s_{k}(t), t > 0\}\) at time \(t \in T_{k}\) and \(y = s_{k}(\tau)\) is a given arbitrary value of the random process \(\{s_{k}(\tau), \tau > 0\}\) at time \(\tau \in T_{k}\), \(T_{k}\) is the corresponding set of time instants (see paragraph 2.1), we shall write

\[
f_{k}(s_{k}(t)|s_{k}(\tau), \tau < t) = f_{k}(x, t; y, \tau).
\]  
(36)

Using the discrete forward Kolmogorov’s equation we can derive the following formula [8]

\[
\frac{\partial f(x, t; y, \tau)}{\partial t} = -\sum_{i=1}^{N} \sum_{j=1}^{N} \left[ \alpha^{(i)}(x, t) \cdot f(x, t; y, \tau) \right] + \frac{1}{2} \sum_{i, j=1}^{N} \left[ \beta^{(i,j)}(x, t) \cdot f(x, t; y, \tau) \right],
\]  
(37)

where

\[
\alpha^{(i)}(x, t) = \lim_{h_{t} \to 0} \frac{1}{h_{t}} \int_{(z)} (z^{(i)} - x^{(i)}) f_{k}(z, t + h_{t}; x, t) dz,
\]  
(38)

\[
\beta^{(i,j)}(x, t) = \lim_{h_{t} \to 0} \frac{1}{h_{t}} \int_{(z)} (z^{(i)} - x^{(i)})(z^{(j)} - x^{(j)}) f_{k}(z, t + h_{t}; x, t) dz.
\]  
(39)

\[
\lim_{h_{t} \to 0} \frac{1}{h_{t}} \int_{(z)} (z^{(i)} - x^{(i)})^{p} f_{k}(z, t + h_{t}; x, t) dz = 0,
\]  
(40)

\(p > 2\). With respect to the derived parameters \(A_{k}, Q_{k}\) we obtain

\[
\alpha(x, t) = \lim_{h_{t} \to 0} \frac{1}{h_{t}} \int_{(z)} (z - x) f_{k}(z, t + h_{t}; x, t) dz = \lim_{h_{t} \to 0} \frac{1}{h_{t}} \left( A_{(KR)}^{k} k - I \right) x = \frac{1}{h_{t}} \ln A_{0} x,
\]  
(41)

\[
\beta(x, t) = \lim_{h_{t} \to 0} \frac{1}{h_{t}} \int_{(z)} (z - x)^{2} f_{k}(z, t + h_{t}; x, t) dz = \lim_{h_{t} \to 0} \frac{1}{h_{t}} \left( A_{(KR)}^{k} k - I \right) \left( A_{(KR)} 0 - I \right)^{-1} vQ_{0} + \left( A_{0}^{k} - I \right) x \cdot x^{T} \left( A_{0}^{k} - I \right)^{T} = \frac{1}{h_{t}} \left( \ln A_{(KR)} 0 \cdot \left( A_{(KR)} 0 - I \right)^{-1} vQ_{0} \right).
\]  
(42)

The necessary and sufficient condition of the existence of \(\ln A_{0}\) and \(\ln A_{(KR)} 0\) is \(\det A_{0} \neq 0\), which is implicitly satisfied by the proper problem formulation. Note, a discrete as well as a continuous character of multivalence generally exist in \(\ln A_{0}\) again. The matrix logarithm is, however, defined in the equations (41), (42) unambiguously by a limit of the corresponding formulas. The proof of the limit existence can be found in [8]. The initial conditions of (37) are \(f(x^{(i)}; y^{(i)}, \tau) = \delta(x^{(i)} - y^{(i)}), i = 1, 2, \ldots., N, \delta\) denotes the Dirac delta function and the boundary conditions are given by \(f(\infty, t; y, \tau) = f(-\infty, t; y, \tau) = 0\).

The limit case of the system sequence \(\mathcal{PCS}_{k}\) for \(k \to \infty\) is described by the partial differential equation (37) of the parabolic type which is for general coefficients \(\alpha(s(t), t)\) and \(\beta(s(t), t)\) called the vector “forward Kolmogorov’s equation” (also known as the “Fokker-Planck equation”). As this equation describes the transition probability density function of a vector random process from a given (arbitrary) initial state (values of the process \(s(\tau), \tau > 0\) at time \(\tau\) to the state \(s(t), t > 0\) at time \(t\), the process \(s(t), t > 0\) is called the Markov process. If the Fokker-Planck equation (37) holds for a transition probability density function of a Markov process \(s(t), t \geq 0\), i.e. if its infinitesimal moment characteristics are given by (41), (42), then such a Markov process is called the “diffusion process”. Similarly, a derived continuous-time system that generates its state as a Markov random process \(s(t), t \geq 0\) will be called the diffusion system. The column parameter \(\alpha(s(t), t)\) is called the drift coefficient and the matrix parameter \(\beta(s(t), t)\) is called the diffusion coefficient of a diffusion system.

Note that the general solution to the Fokker-Planck equation (37) is very hard to find (it is practically impossible). However, it is often acceptable to find only the moment characteristics of the transition probability density function \(f_{k}(x, t; y, \tau)\). The solution to (37) with respect to (41), (42) can be found on specific conditions (note that \(\beta(s(t), t) = \beta\) is in (42) a constant matrix independent of the state \(x\) in the form of a transition probability density
function of Gaussian distribution
\[ f(x, t; y, \tau) = \frac{1}{(2\pi)^{\frac{d}{2}} \cdot \sqrt{\det P(t)}} \cdot e^{-\frac{1}{2} \left( x - \mu(t) \right)^\top (P(t))^{-1} \left( x - \mu(t) \right)}, \]
for \( t > \tau \); \( t, \tau \in T = (0, \vartheta) \), where
\[ \frac{\partial \mu(t)}{\partial t} = F \cdot \mu(t), \]
\[ \frac{\partial P(t)}{\partial t} = F \cdot P(t) + P(t) \cdot F^\top + \beta \]
for \( t > \tau \) and fixed \( \tau \). The initial conditions are represented by \( \mu(\tau) = y \) and \( P(\tau) = 0 \). The dependence of (43), (44), (45) on \( y \) and \( \tau \) was left out in order to keep the formulas transparent.

This approach to the continuous-time stochastic systems definition can be also used to prove the existence of the Wiener-Lévy stochastic process which is considered as a perfect source of uncertainty and it is described in an exact mathematical way [6], [8]. With respect to the solution to the Fokker-Planck diffusion equation, it is also possible to derive an equivalent description of diffusion systems given by Itô stochastic differential equation
\[ ds(t) = F(s(t), t)dt + G(s(t), t)dw(t), \]
where \( F(s(t), t) \) is given in (44), \( G(s(t), t) \cdot G^\top(s(t), t) = \beta(s(t), t) \) and \( w(t) \) denotes the Wiener-Lévy stochastic process, \( E\{dw(t)\} = 0 \), \( E\{dw(t)dw^\top(t)\} = Idt \) (Itô isometry). The stochastic differential equation (46) is written with differentials instead of derivatives because the derivative \( \frac{dw(t)}{dt} \) of the Wiener-Lévy stochastic process \( \{w(t); t \geq t_0\} \) does not exist in a strict mathematical way. The variable \( dw(t) \) is in (46) fictive and does not represent any real attribute of the system. It is used to model random errors within the system description (46), which is equivalent to the original system description given by the Fokker-Planck diffusion equation (37).

4 Conclusions

An original approach to the process of continualization was presented in this paper in terms of an alternative approach to system theory, which is based on quite new system paradigms. It is assumed that the properties of the internal system mechanism are governed by newly reviewed principle and law of causality. Due to the fact that we are able to obtain only a finite number of independent observations while studying system properties in the real world, all the system variables were directly defined on finite sets. Subsequently, a class of continuous-time stochastic systems was derived under carefully chosen continuity hypothesis. The systems from this class are described by the Fokker-Planck diffusion equation on exact conditions. The presented continualization method is straightforward and based on natural demands. It also proposes a clarification of problems with determination of causal (cause-effect) relations within continuous-time systems. Their existence was approved by appropriate mathematic theory. Finally, the presented method thus brings important results featuring in adequate description of real phenomena.

References