Design of optimal controller for discrete plant with static dependencies

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Abstract: - A new approach to general system theory enables to correctly design an optimal controlling subsystem for a controlled subsystem with a possible use of non-dynamic interactions between some input and output variables. The design of the optimal control system is divided into 2 steps. The first step is to determine a set of admissible structures of the controlling subsystem for a given structure of the controlled subsystem. The second step is to determine an optimal behavior of the discrete stochastic control system. The determination of the set of all admissible controller structures is solved with using the graph theory where graphs present the structures of the controlling and controlled subsystems. The optimal behavior of the control system is given according to the Bellman principle.

Key-Words: - Control system, Optimal control, System theory, Graph theory, Bellman principle

1 Introduction

The theory of optimal systems [1,2,3] etc. deals with systems which embody optimal behavior. The term optimal means that the behavior of control systems (CS) minimizes some given quality index. This paper is based on a theoretic background given by the system theory formulated in [4], which enables to solve the problem of the optimal behavior of the stochastic CS in general because the controlled subsystem can be possibly described with non-dynamic interactions between some input and output variables.

The new approach to the system theory [4] shows that the design of the optimal CS has to be generally divided into two steps. The first is to define the set ***P*** of all admissible controller structures for the structure of a given plant. Only the knowledge of the complete set ***P*** guarantees finding of the optimal CS. The second step is the calculation of the optimal behavior of the control system *S* with the knowledge of the set ***P***. The term *optimal controller structure* denotes controller structure which produce the optimal behavior of the CS (this structure can be time variable).

The determination of the optimal controller structure is evident if the plant includes only output variables whose values are generated (directly or indirectly) by no values of input variables at the same time instant (static dependence). This leads to the well known controller structure with static dependencies between each input and each output variables.

If the plant is a MIMO subsystem with some static dependencies, the determination of the optimal controller structure is more complicated. This problem is mentioned in [3] and it will be shown here that the proposed optimal controller structure may include some static dependencies in this case too. The controller structure without acceptable static dependencies and the given MIMO plant do not necessary constitute the optimal CS.

The aim of this paper is to design the discrete optimal CS for a general plant, i.e. for a plant that includes some static dependencies.

This paper is organized as follows. A discrete control system is described in the section 2. The problem of the standard design of the optimal control strategy is formulated in the section 3 and the generalized problem of the optimal control is formulated in the section 4. The section 5 presents the applications of graph theory in system theory. Rules delimitative the set of admissible "static input-output structures" (this term is explained in the definition 2) of the controller are formulated in the section 6 and a design of the optimal control is solved in the section 7. The illustrative example is given in the section 8 followed by the conclusion.

2 Discrete control system

The presented problem of the optimal control system design is based on a new approach to system theory [4] because this approach enables a correct design of an optimal controlling subsystem. The CS is composed of a given controlled subsystem S_1 (plant) and a controlling subsystem S_2 (controller) which is to be proposed. Σ_1 and Σ_2 are interconnected only by informational interconnections; precisely defined in [4]. This situation is illustrated in the Fig. 1 where $\boldsymbol{x}_k$ is vector of non-measurable $\boldsymbol{v}_k$ of inner and $\boldsymbol{y}_k$ and $\boldsymbol{u}_k$ of measurable variables of the CS *S* at the time *k*.

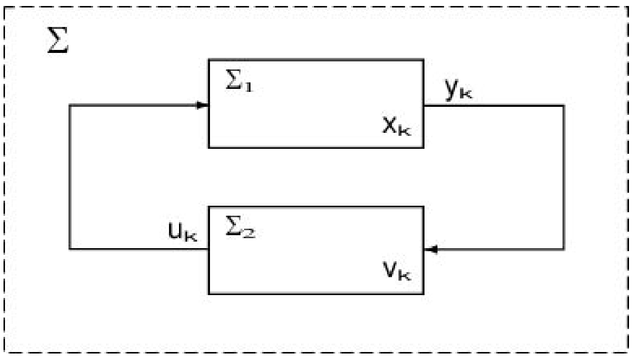

Figure 1: Stochastic control system

The CS S will be studied on a finite time set

$$T = \{0,\ 1,\ \ldots,\ F\}, \tag{1}$$

where F is a control horizon.

3 Standard problem of the optimal control

The controlled subsystem is usually described by so-called causal probability density functions (pdf)

$$f\left(\mathbf{x}(k),\mathbf{y}(k)\middle|\mathbf{x}(k-1),\mathbf{y}(k-1),\mathbf{u}(k-1)\right) \quad k \in T, \tag{2}$$

where the parametric part of $f_k(\cdot)$ represents a complete immediate cause of variables ***x(k)*** and ***y(k)*** of the plant at the time ***k***.

The aim of the optimal control design is to choose such an admissible controller that minimizes the mean value *J* of the loss function *L*

$$J = E\left\{L\left(\mathbf{x}_0^F,\mathbf{y}_0^F,\mathbf{u}_0^F\right)\right\}, \tag{3}$$

where $\mathbf{u}_0^F$ denotes the sequence $\left(\mathbf{u}(0),\ldots,\mathbf{u}(F)\right)$ and accordingly $\mathbf{x}_0^F,\mathbf{y}_0^F$.

It is possible to prove that the controller (or controllers) minimizing the formula (3) is generally described by (deterministic) functions

$$\begin{aligned} \mathbf{v}(k) &= g_k^v\left(\mathbf{v}(k-1),\mathbf{y}(k-1)\right) \\ \mathbf{u}(k) &= g_k^u\left(\mathbf{v}(k),\mathbf{y}(k)\right) \quad k \in T. \end{aligned} \tag{4}$$

The design of the discrete optimal CS can be solved e.g. with using *mathematical programming* [5,6] or with the *Bellman principle* [1,3]. General basses of this principle are mentioned in sections 7 and 8.

The application of the Bellman principle is nontrivial and leads to solutions that are not analytically solvable in a many cases. In a special case (linearity of the CS, additive quadratic loss function, Gaussian causal pdf) the problem of the minimization of the expression (3) leads to the LQG problem solvable with using the *Riccati equation* [1]. Another special case is the LnQ problem (Linear non-quadratic problem). This problem is solved for example in [7,8].

4 Generalized problem of the optimal control

Thanks to appropriate definition of the system in the system theory [4], it is possible to precisely specify the sets of input variables *$U(S_q)=\{u_1,\ldots,u_m\}$* and of output variables *$Y(S_q)=\{y_1,\ldots,y_n\}$* of the subsystem S_q at the time instant ***k***.

In the general, the controller and the plant can include some static dependencies. This property of the subsystem leads to the introduction of a relation *(A:B)*. This relation means that each variable from the set A at the time k is determined by each variable from the set B at the same time k. With this relation, it is possible to describe static dependencies of the subsystems and thus, to avoid causal (algebraic) loops in the discrete CS.

Suppose the plant S_1 with some static dependencies. That the sets $\hat{U}_i \subset U\left(\Sigma_1\right),\ i = 1,\ldots,k$, and $\hat{Y}_j \subset Y\left(\Sigma_1\right)$, $j = 1,\ldots,l$, exist with the property $(\hat{Y}_j : \hat{U}_i)$ for some i and j. The problem is that the equation (4) makes it possible to create the controller with property *(U:Y)*. It holds

$$\begin{aligned} ((\hat{Y}_j : \hat{U}_i) \wedge (((U:Y) \wedge (\hat{Y}_j \subset Y) \wedge (\hat{U}_i \subset U)) \\ \Rightarrow (\hat{U}_i : \hat{Y}_j))) \Rightarrow (\hat{Y}_j : \hat{Y}_j), \end{aligned}$$

and at least one *causal (algebraic) loop* exists within the CS and it is in a contradiction to the *causality law* because the given plant and the controller described by the equation (4) can form unrealistic CS.

Therefore, we need to find another description of the controlling subsystem. The simplest way to avoid the creation of the causal (algebraic) loop in the CS is to design the controller without any static dependencies. However, it will be shown that such controller does not necessary lead to the optimal CS because the optimal controller structure can generally contain some static dependencies.

5 The graph theory in the system theory

Definition 1 Consider a subsystem S_1 with m input variables *$\{u_i;\ i = 1,\ldots,m\}$* and n output variables *$\{y_j;\ j = 1,\ldots,n\}$*. Suppose that the output variable y_j at the time k depends (directly or indirectly) on the input variable u_i at the same time instant k. This dependence is called a "static input-output dependence" (SI-OD) of the subsystem.

Definition 2 A set of all static input-output dependencies of the subsystem S_1 is called a "static input-output structure" (SI-OS) of the subsystem.

Only the SI-ODs of subsystems S_1 and S_2 must be watched to prevent causal (algebraic) loops in the discrete CS. This is why the set *P* can be found by the determination of the set *Q* of all admissible SI-OSs of the controller S_2. Each controller structure with the SI-OS beeing an element of the set *Q* is an element of the set *P*.

The delimitation of the complete set *Q* is not a trivial problem and it seems that the graph theory is an appropriate instrument for the solving of this puzzler. A disadvantage of the graph theory is a multivalent terminology. This paper uses the terminology defined in [9].

The SI-OS of the subsystem S_l can be represented by a bigraph $\overrightarrow{B_l}(\Sigma_l) = (V, E)$ where the set $V(\overrightarrow{B_l})$ is the set of vertices and the set $E(\overrightarrow{B_l})$ is the set of edges.

Remark 1 Bigraph is a bipartite graph with directed edges only and its mathematical specification is $\overrightarrow{B_l} = (V, E)$ where $V(\overrightarrow{B_l}) = U(\overrightarrow{B_l}) \cup Y(\overrightarrow{B_l})$; $U(\overrightarrow{B_l}) \cap Y(\overrightarrow{B_l}) = \{\}$; $E(\overrightarrow{B_l}) = U(\overrightarrow{B_l}) \times Y(\overrightarrow{B_l})$.

The set $U(\overrightarrow{B_l})$ of the bigraph $\overrightarrow{B_l}$ represents the set $U(S_l)$ of the input variables of the subsystem S_l and the set $Y(\overrightarrow{B_l})$ constitutes the set $Y(S_l)$ of the output variables of the subsystem S_l. The edge $e_{ij} = (u_i, y_j) \in E(\overrightarrow{B_l})$ represents the SI-OD between variables u_i and y_j and the set $E(\overrightarrow{B_l})$ represents SI-OS of the subsystem Σ_l.

It is reasonable to determine that the number of the input values of the S_2 is identical as the number of the output values of the S_1 and the number of the input values of the S_1 is identical as the number of the output values of the S_2, hence $U(S_2)=Y(S_1)$ and $U(S_1)=Y(S_2)$. This fact means in the graph theory, that $V(\overrightarrow{B_1}(\Sigma_1)) = V(\overrightarrow{B_2}(\Sigma_2)) \equiv V$ and all SI-ODs of the controller and the plant can be represented by one directed graph (digraph) $\overrightarrow{G}(\Sigma) = \overrightarrow{B_1} \hat{\cup} \overrightarrow{B_2} \equiv (V, E(\overrightarrow{B_1}) \cup E(\overrightarrow{B_2}))$ (illustrated in the Fig. 2).

The SI-OS of the plant is given and the SI-OS of the controller is to be found. When the SI-OS of the controller is being proposed, the *causality law* must be respected. It means that the CS *S* cannot contain causal (algebraic) loops [4], hence, the graph $\overrightarrow{G}(\Sigma)$ must be a digraph without directed cycles (directed acyclic graph (DAG)).

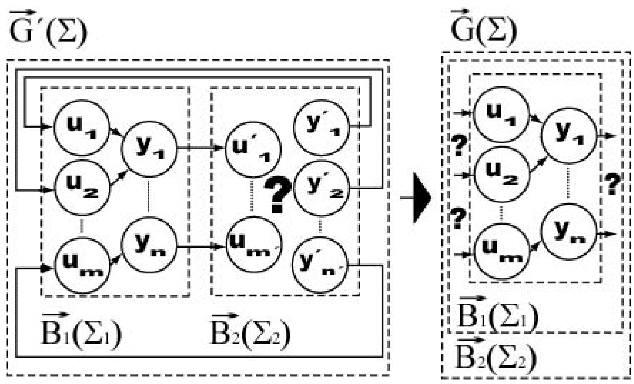

Figure 2: Description of the CS Σ by the DAG

6 Set of all admissible SI-OSs of the controller

Consider the controller S_2 with *n* input and *m* output variables. Suppose a set *Q* with elements $\overrightarrow{Q_i}$ where each $\overrightarrow{Q_i}$ is a bigraph describing the SI-OS of the controller S_2 admissiable to control the plant S_1. The set *Q* is finite and its cardinality is

$$|Q| \le 2^{m \cdot n}.$$

Some elements of the set *Q* are major than the others. A set of these major elements is denoted by S $(S \subset Q)$ with elements $\overrightarrow{S_i}$ $(i = 1,\ldots,l;\ l \le |Q|)$. Elements $\overrightarrow{S_i}$ are bigraphs and each bigraph $\overrightarrow{Q_i}$ $(\overrightarrow{Q_i} \in Q)$ is a subgraph of some bigraph $\overrightarrow{S_j}$ $(\overrightarrow{S_j} \in S)$. The bigraphs from the set *S* are major because SI-OSs of the subsystem S_2 represented by bigraphs from the set $Q \setminus S$ do not use all available information at the time *k* which can be used.

The SI-OS of the optimal controller structure at the time instant *k* is an element of the set *S* but it is not generally simple to determine which particular element from the set *S* it is. This is a reason why all elements of the set *S* must be discovered. Each element $\overrightarrow{S_i} = (V(\overrightarrow{S_i}), E(\overrightarrow{S_i}))$ of the set *S* has to satisfy both of the rules:

1. $\overrightarrow{G} = \overrightarrow{B_1} \hat{\cup} \overrightarrow{S_i}$ is the DAG.
2. The graph $\overrightarrow{G^{rs}} = \overrightarrow{B_1} \hat{\cup} \overrightarrow{S_i^{rs}}$ is the directed cyclic graph for each admissible *r*, *s*. $\overrightarrow{S_i^{rs}}$ is a bigraph, $\overrightarrow{S_i^{rs}} = (V(\overline{S_i}), E(\overline{S_i}) \cup (u_r, y_s))$, $u_r \in U(\overrightarrow{S_i^{rs}}) = U(\overrightarrow{S_i})$, $y_s \in Y(\overrightarrow{S_i^{rs}}) = Y(\overrightarrow{S_i})$, $(u_r, y_s) \notin E(\overrightarrow{S_i})$.

The Elements $\overrightarrow{S_i}$ can be denoted as maximal elements of the set *Q* (the rule 2 associates this title).

The finding of one element of the set S is a polynomial problem with complexity $\leq O((m+n)^2(m+n)^3)$ (common using of a "hungry" algorithm [10] and the *Floyd's algorithm* [11]) but it seams that the finding of all elements of the set S is an problem with complexity $m!$ (m is the number of the input variables of the plant and "*!*" denotes factorial).

7 Bellman principle

The Bellman principle is used for the design of the optimal behavior of the CS. In a standard case (plant without SI-ODs) the Bellman principle has the well known form [3]

$$\begin{aligned} W_{F+1} &= E\{L(\cdot)|\mathbf{y}_0^F,\mathbf{u}_0^F\}, \\ W_k(\mathbf{y}_0^k,\mathbf{u}_0^{k-1}) &= \min_{\mathbf{u}_k\in U_k} E\{W_{k+1}(\mathbf{y}_0^{k+1},\mathbf{u}_0^k)|\mathbf{y}_0^k,\mathbf{u}_0^k\}, \\ \mathbf{u}_k^* &= \arg\min_{\mathbf{u}_k\in U_k} E\{W_{k+1}(\cdot)|\mathbf{y}_0^k,\mathbf{u}_0^k\}, \\ k &= F,...,0, \qquad J = E\{W_0(\cdot)\}. \end{aligned} \tag{5}$$

The function $W_k(\cdot)$ is called the *generalized Bellman function* and it is derived from the equation

$$\begin{aligned} J = \int L(\mathbf{x}_0^F,\mathbf{y}_0^F,\mathbf{u}_0^F)p(\mathbf{x}_0^F,\mathbf{y}_0^F,\mathbf{u}_0^F)\mathrm{d}\mathbf{x}_0^F\mathrm{d}\mathbf{y}_0^F\mathrm{d}\mathbf{u}_0^F = \\ \int ... \int L(\mathbf{x}_0^F,\mathbf{y}_0^F,\mathbf{u}_0^F)p(\mathbf{x}_0^F|\mathbf{y}_0^F,\mathbf{u}_0^F)\mathrm{d}\mathbf{x}_0^F \\ p(\mathbf{u}_F|\mathbf{y}_0^F,\mathbf{u}_0^{F-1})\mathrm{d}\mathbf{u}_F\, p(\mathbf{y}_F|\mathbf{y}_0^{F-1},\mathbf{u}_0^{F-1})\mathrm{d}\mathbf{y}_F \\ p(\mathbf{u}_{F-1}|\mathbf{y}_0^{F-1},\mathbf{u}_0^{F-2})\mathrm{d}\mathbf{u}_{F-1}...p(\mathbf{u}_0|\mathbf{y}_0)\mathrm{d}\mathbf{u}_0 \\ p(\mathbf{y}_0)\mathrm{d}\mathbf{y}_0. \end{aligned} \tag{6}$$

If the loss function is additive,

$$L(\mathbf{x}_0^F,\mathbf{u}_0^F,\mathbf{y}_0^F) = \sum_{i=0}^{F} L_i(\mathbf{x}_0^i,\mathbf{u}_0^i,\mathbf{y}_0^i), \tag{7}$$

the optimizing recursion (5) has the well known form [1,3]

$$\begin{aligned} V_{F+1} &= 0 \\ V_k(\mathbf{y}_0^k,\mathbf{u}_0^{k-1}) &= \min_{\mathbf{u}_k\in U_k} E\{L_k(\cdot)+V_{k+1}(\cdot)|\mathbf{y}_0^k,\mathbf{u}_0^k\}, \\ \mathbf{u}_k^* &= \arg\min_{\mathbf{u}_k\in U_k} E\{L_k(\cdot)+V_{k+1}(\cdot)|\mathbf{y}_0^k,\mathbf{u}_0^k\}, \\ k &= F,...0, \qquad J = E\{V_0(\cdot)\}. \end{aligned}$$

The function $V_k(\cdot)$ is called the *Bellman function.*

In general, when the plant includes some SI-ODs the application of the Bellman principle is a little different because the equation (6) must be specified precisely. It is necessary to determine an order of a computation of the S_2 output variables at time k (at first, the SI-OS of the controller S_2 must be determined).

This order is given by the DAG $\vec{G} = (\vec{B}_1 \hat{\cup} \vec{B}_1)$ where $\vec{B}_1$ represents the SI-OS of the plant S_1 and $\vec{B}_2$ represents some element from the set S. It is possible to use the following algorithm to find this order:

1. Consider DAG $\vec{G} = (\vec{B}_1 \hat{\cup} \vec{B}_1)$ with m nodes from the set U and n nodes from the set Y; $V(\vec{G}) = U \cup Y$; set $i=1$ and $\vec{G}_i \equiv \vec{G}$.
2. Find a node with no input edge in the graph $\vec{G}_i$. Remove this node from the graph $\vec{G}_i$ with all output edges of this node. Denote the obtained graph by $\vec{G}_{i+1}$ and the removed node by b_i. Set $i=i+1$.
3. *if i>(m+n) then* end the algorithm *else* go to the step 2

Remark 2 The edge (a_i, a_j) is called the input edge of the node a_j and the output edge of the node a_i.

Now, the set $V(\vec{G})$ can be divided into subsets A_j, $j=1,\dots,J\leq m+n$ by the algorithm:

1. Set $i=1$, $j=1$, $A_1=\{\}$. *If* $b_1 \in Y$ *then* $J=Y$ *else* $J=U$.
2. if $b_i \in J$ *then* $A_j=\{A_j, b_i\}$, set $i=i+1$ *else begin if* J is Y *then* $J=U$ *else* $J=Y$. set $j=j+1$, $A_j=\{b_i\}$, $i=i+1$, *end.*
3. *if i>(m+n) then* end the algorithm *else* go to the step 2.

Now, it is possible to derive more detailed equation then (6)

$$\begin{aligned} J = \int ... \int L(\mathbf{x}_0^F,\mathbf{y}_0^F,\mathbf{u}_0^F)p(\mathbf{x}_0^F|\mathbf{y}_0^F,\mathbf{u}_0^F)\mathrm{d}\mathbf{x}_0^F \\ p(\mathbf{a}_{J,F}|\mathbf{a}_{J-1,F},...,\mathbf{a}_{1,F},\mathbf{y}_0^{F-1},\mathbf{u}_0^{F-1})\mathrm{d}\mathbf{a}_{J,F} \\ p(\mathbf{a}_{J-1,F}|\mathbf{a}_{J-2,F},...,\mathbf{a}_{1,F},\mathbf{y}_0^{F-1},\mathbf{u}_0^{F-1})\mathrm{d}\mathbf{a}_{J-1,F} \\ ...p(\mathbf{a}_{1,F}|\mathbf{y}_0^{F-1},\mathbf{u}_0^{F-1})\mathrm{d}\mathbf{a}_{1,F} \\ p(\mathbf{a}_{J,F-1}|\mathbf{a}_{J-1,F-1},...,\mathbf{a}_{1,F-1},\mathbf{y}_0^{F-2},\mathbf{u}_0^{F-2})\mathrm{d}\mathbf{a}_{J,F-1} \\ ...p(\mathbf{a}_{2,0}|\mathbf{a}_{1,0})\mathrm{d}\mathbf{a}_{2,0}\,p(\mathbf{a}_{1,0})\mathrm{d}\mathbf{a}_{1,0} \end{aligned} \tag{8}$$

where $\mathbf{a}_{j,k}$ is the vector of elements of the set A_j at the time instant k and J is the number of sets A_j, $j=1..,J$.

Now, the slightly modified *Bellman principle* is used to minimize the function (8). The modification consists in partitioning of the Bellman function V_k into its parts $V_{i,k}(\cdot)$ (k constitutes the time instant and i constitutes the set A_i with input elements of the plant). This problem is illustrated in the section 8.

8 Illustrative example

Consider a plant S_1 described by:

$$\Sigma_1 : y_1(k) = u_1(k) + 2x(k)$$
$$y_2(k) = u_2(k) + x(k)$$
$$x(k+1) = a\,x(k) + x(k)$$

where x is a noise of a Gaussian distribution $N(0, s^2)$. The given quality index is

$$J = E\{\sum_{i=0}^{F}(y_1(i)^2 + y_2(i)^2)\}. \qquad (9)$$

If the controller structure does not include direct dependencies (mentioned in the section 4) the relevant description of the behavior of the controller which minimizes function (9) is

$$\Sigma_2^a : \quad u_1^*(k) = \quad -a\,(y_1(k-1) - u_1(k-1))$$
$$u_2^*(k) = \quad -a\,(y_2(k-1) - u_2(k-1))$$

and the value of the function (9) is $J(S^a)=5Fs^2$.

Now the behavior of the CS S will be computed with using the method presented in this contribution.

A bigraph $\overrightarrow{B}_1(\Sigma_1)$ of the SI-OS of the plant S_1 is

$$\overrightarrow{B}_1(\Sigma_1) = (V, \{(u_1, y_1), (u_2, y_2)\}),$$

where $V=\{u_1, u_2, y_1, y_2\}$ is the set of nodes. The set S has 2 elements

$$\overrightarrow{S}_1 : (V, \{y_2, u_1\}), \qquad \overrightarrow{S}_2 : (V, \{y_1, u_2\}).$$

Let $\overrightarrow{B}_2(\Sigma_2) = \overrightarrow{S}_1$.

$$b_1 = u_2, \quad b_2 = y_2, \quad b_3 = u_1, \quad b_4 = y_1$$

and

$$A_1 = \{b_1\}, \quad A_2 = \{b_2\}, \quad A_3 = \{b_3\}, \quad A_4 = \{b_4\}.$$

The equation (8) is in this case

$$J = \int ... \int L(\cdot) p(\mathbf{x}_0^F | \mathbf{y}_0^F, \mathbf{u}_0^F) p(y_1(F) | \mathbf{u}_0^F, y_2(F), \mathbf{y}_0^{F-1})$$
$$p(u_1(F) | u_2(F), y_2(F), \mathbf{u}_0^{F-1}, \mathbf{y}_0^{F-1}) p(y_2(F) | u_2(F),$$
$$\mathbf{u}_0^{F-1}, \mathbf{y}_0^{F-1}) p(u_2(F) | \mathbf{u}_0^{F-1}, \mathbf{y}_0^{F-1}) p(y_1(F-1) | \mathbf{u}_0^{F-1},$$
$$y_2(F-1), \mathbf{y}_0^{F-2}) p(u_1(F-1) | u_2(F-1), y_2(F-1), \mathbf{u}_0^{F-2},$$
$$\mathbf{y}_0^{F-2}) ... p(u_2(0)) \mathrm{d}y_1(F) ... \mathrm{d}u_2(0).$$

The estimation of the value $x(k)$ is

$$E\{x(k) | \mathbf{u}_0^{k-1}, \mathbf{y}_0^{k-1}\} = a\,(y_2(k-1) - u_2(k-1))$$

and the partitioned Bellman functions are

$$V_{F+1} = 0,$$
$$V_{3,F} = \min_{u_1(F)} E\{(u_1(F) + 2x(F))^2 | y_2(F), \mathbf{u}_0^F, \mathbf{y}_0^{F-1}\}$$
$$= \min_{u_1(F)} \{(u_1(F) + 2(y_2(F) - u_2(F)))^2\} = 0,$$
$$u_1^*(F) = 2(u_2(F) - y_2(F)),$$
$$V_{1,F} = \min_{u_2(F)} E\{y_2(F)^2 + 0 | \mathbf{u}_0^{F-1}, \mathbf{y}_0^{F-1}\} =$$
$$= \min_{u_2(F)} E\{u_2(F)^2 + 2u_2(F)x(F) + x(F)^2 | ...\},$$
$$u_2^*(F) = -a\,(y_2(F-1) - u_2(F-1)), \qquad V_{1,F} = s^2,$$
$$V_{3,F-1} = \min_{u_1(F-1)} E\{y_1(F-1)^2 + s^2 | y_2(F-1), ...\},$$
$$u_1^*(F-1) = 2(u_2(F-1) - y_2(F-1)), \qquad V_{3,F-1} = s^2,$$
$$u_2^*(F-1) = -a\,(y_2(F-2) - u_2(F-2)), \quad V_{1,F-1} = 2s^2,$$
$$\vdots$$

The resulting behavior of the controller which minimizes function (9) is

$$\Sigma_2^b : \quad u_1^*(k) = \qquad 2(u_2(k) - y_2(k))$$
$$u_2^*(k) = \quad -a\,(y_2(k-1) - u_2(k-1))$$

and the value of the function (9) is $J(S^b)=Fs^2$.

Let $\overrightarrow{B}_2(\Sigma_2) = \overrightarrow{S}_2$

$$b_1 = u_1, \quad b_2 = y_1, \quad b_3 = u_2, \quad b_4 = y_2$$

and

$$A_1 = \{b_1\}, \quad A_2 = \{b_2\}, \quad A_3 = \{b_3\}, \quad A_4 = \{b_4\}.$$

The principle of the calculation of the partitioned Bellman functions is analogous as in previous case of $\overrightarrow{B}_2(\Sigma_2) = \overrightarrow{S}_1$ and the resulting behavior of the controller which minimizes the function (9) is

$$\Sigma_2^c : \quad u_1^*(k) = \quad -a\,(y_1(k-1) - u_1(k-1))$$
$$u_2^*(k) = \qquad 0{,}5(u_1(k) - y_1(k)).$$

The value of the function (9) is $J(S^c)=4Fs^2$ in this case.

It is shown that

$$J(\Sigma^a) > J(\Sigma^c) > J(\Sigma^b)$$

and $J(\cdot)$ depends only on s^2 hence the optimal structure is $\overrightarrow{S}_1$ and the optimal controller is the subsystem Σ_2^b.

9 Conclusion

The problem of the discrete optimal CS design when the plant include SI-ODs was presented in this contribution.

A determination of the set of all admissible controller structures for the given structure of the plant is necessary for a design of the optimal CS. This determination can be transformed into the determination of the set of all admissible maximal SI-OSs of controller S_2 (set S). The problem is solved by using the graph theory.

A problem of the determination of the set S is that it is necessary to find all elements of this set for a design of the optimal CS because it is not generally possible to determine which element of the set S will produce the optimal control strategy.

The behavior which minimizes given quality index for each SI-OS from the set S must be computed whereas more detailed Bellman principle which was presented in sections 7 and 8 is used for the determination of these behavior. Now, it is possible to choose the *optimal behavior* of the CS *S* from these behavior by using Bellman principle.

It seams that the determination of the set *S* is complicated algorithmic problem but it was shown that finding of one element of the set *S* is polynomial problem only and the controller with the SI-OS from the set *S* produces better or equal behavior of the CS *S* that controller without acceptable SI-ODs.

The problem with time variable structure of the controller Σ_2 is mentioned in the section 1. This problem is very complicated and it is solved at present.

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