Zero-Term Rank Preservers over Fields and Rings

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Abstract: - The zero-term rank of a matrix is the maximum number of zeros in any generalized diagonal. This article characterizes the linear operators that preserve zero-term rank of $m\times n$ matrices when the matrices have entries either in a field with at least $mn+1$ elements or in a ring whose characteristic is not 2.

Key-Words: - Zero-term rank, term rank, linear operator, preserver, (P,Q,B)-operator, cell.

1 Introduction

Let $\mathrm{M}_{m,n}(F)$ denote the set of $m\times n$ matrices over F, where F is an algebraic set, usually a field. Let $A\in \mathrm{M}_{m,n}(F)$ and let $\#A$ denote the number of nonzero entries of A. Let $\mathbf{B}$ be the two element Boolean algebra, and $\overline{A}$ denote the $m\times n$ matrix with entries in $\mathbf{B}$ such that $\overline{a}_{i,j}=0$ if and only if $a_{i,j}=0$. Let $E_{i,j}$ be the matrix in $\mathrm{M}_{m,n}(F)$ which has a "1"in the (i,j) entry and is zero elsewhere. We call $E_{i,j}$ a *cell*. A matrix A is said to *dominate* matrix B if $a_{i,j}=0$ implies that $b_{i,j}=0$, and we write $A\geq B$.

The *term rank* of $A\in \mathrm{M}_{m,n}(F)$ is the maxumum number of nonzero elements on any generalized diagonal. Equivalently, the term rank of A is the smallest k such that for some permutation matrices P and Q, PAQ has an $(m-r)\times(n-s)$ submatrix of zeros and $r+s=k$. We denote the term rank of A by $t(A)$. Let J denote the matrix of all 1's. If B is a $(0,1)$ matrix such that $B=\overline{A}$ then the *zero-term rank* of A, $z(A)$, is the term rank of $J-B$, that is, $z(A)=t(J-B)$. In other words, the zero-term rank of a matrix is the maximum number of zeros in any generalized diagonal.

If $T\colon M_{m,n}(F)\to M_{m,n}(F)$ is a linear operator, define $\overline{T}\colon M_{m,n}(\mathbf{B})\to M_{m,n}(\mathbf{B})$ by

$$\overline{T}(\overline{A}) = \sum_{i=1}^{m}\sum_{j=1}^{n}\overline{T(a_{i,j}E_{i,j})} .$$

A linear operator T *preserves* a set X if $T(\mathrm{X})\subset X$. A linear operator T *strongly preserves* a set X if T preserves the set and T preserves the complement of the set in $\mathrm{M}_{m,n}(F)$. T preserves a function $f:\mathrm{M}_{m,n}(F)\to F$ if

$$f(T(X))=f(X)$$

for every $X\in \mathrm{M}_{m,n}(F)$.

In [1] and [2], Beasley and Pullman characterized the term rank preservers and term rank-1 preservers. In [3] Beasley, Song and Lee have characterized the zero-term rank preservers, as well as zero-term rank 1 preservers with additional conditions. Those works were over antinegative semirings. Our results below require that the entries of the matrices come from a field with at least $mn+1$ entries.

A linear operator $T\colon M_{m,n}(F)\to M_{m,n}(F)$ is called a (P,Q,B)-*operator* if there exist permutation matrices P and Q, and a matrix B all of whose entries are nonzero such that

$T(X) = P(X \circ B)Q$ for all $X \in M_{m,n}(F)$ or, if $m = n$, $T(X) = P(X \circ B)^t Q$ for all $X \in M_{m,n}(F)$, where $X \circ B$ denotes the Hadamard(or Schur) product of X and B, i.e., $X \circ B = (x_{i,j} b_{i,j})$. In [3], the linear operators which preserve zero-term rank were shown to be (P,Q,B)-operators. We now state that result for later reference.

Theorem 1.1 [3] If S is any antinegative semiring, and T is a linear operator on $M_{m,n}(S)$, then the following are equivalent:

) T is a (P,Q,B)-operator;

) T preserves zero-term rank;

) T preserves zero-term rank 1 and $\overline{T(J)} = J$.

Theorem 1.2 [2] If S is any semiring, and T is a linear operator on $M_{m,n}(S)$, then the following are equivalent:

) T is a (P,Q,B)-operator;

) T preserves term rank;

) T preserves term ranks 1 and 2.

2 Zero-term rank preservers over fields

We begin with two lemmas upon which the main theorems will rely.

Lemma 2.1 If F is a field with at least $mn+1$ elements and $T: M_{m,n}(F) \to M_{m,n}(F)$ preserves zero-term rank 1, then there exists $X \in M_{m,n}(F)$ such that $\overline{T(X)} = J$. That is $\#T(X) = mn$.

Proof. Choose $X \in M_{m,n}(F)$ such that $\#T(X) \geq \#T(A)$ for all $A \in M_{m,n}(F)$. Since F has at least $mn+1$ elements, we can find X such that $T(X) \geq T(A)$ for all $A \in M_{m,n}(F)$.

Suppose that $\overline{T(J)} \neq J$. Then, for some (i,j), $T(A) \circ E_{i,j} = 0$, for all $A \in M_{m,n}(F)$. Further by permuting rows and columns, we may assume that $(i,j) = (1,1)$. Also, since F has at least $mn+1$ elements, we may assume that $\overline{X} = J$, so that $z(X) = 0$. Let $E_{k,l}$ be a cell such that $T(E_{k,l})$ has a nonzero (s,t) entry with $s,t \geq 2$. If no such cell existed we should have that $z(X - x_{i,j}E_{i,j}) = 1$ for every cell $E_{i,j}$ and necessarily,

$$z(T(X - x_{i,j}E_{i,j})) = \min\{m,n\},$$

a contradiction. Now, for $T(E_{k,l}) = R = (r_{i,j})$, we have that $z(T(X - \frac{x_{s,t}}{r_{s,t}}E_{k,l})) \geq 2$,

and

$$z(X - \frac{x_{s,t}}{r_{s,t}}E_{k,l}) \leq 1.$$

Thus, we must have that $z(X - \frac{x_{s,t}}{r_{s,t}}E_{k,l}) = 0$. Let $Y = X - \frac{x_{s,t}}{r_{s,t}}E_{k,l}$. If $E_{c,d}$ is a cell whose image under T has an (s,t) entry which is zero, then $z(Y - y_{c,d}E_{c,d}) = 1$, while $z(T(Y - y_{c,d}E_{c,d})) \geq 2$, a contradiction. Thus the image of every cell has a nonzero (s,t) entry. Let $T(Y) = W$. If

$$T(E_{1,1}) = U \text{ and } T(E_{1,2}) = V,$$

then

$$T\left(Y - y_{1,1}E_{1,1} + \left(\frac{y_{1,1}u_{s,t} - w_{s,t}}{v_{s,t}}\right)E_{1,2}\right)$$

has zeros in the $(1,1)$ and (s,t) entries, and hence has zero term rank at least 2, while

$$z\left(Y - y_{1,1}E_{1,1} + \left(\frac{y_{1,1}u_{s,t} - y_{s,t}}{v_{s,t}}\right)E_{1,2}\right) = 1,$$

a contradiction. Thus $\overline{T(X)} = J$. □

The hypothesis in the above lemma requiring the field to have at least $mn+1$ elements can be seen to be necessary by the following example.

Example 2.1 Let $F = Z_2$. Consider

$$T: M_{3,1}(Z_2) \to M_{3,1}(Z_2)$$

defined by

$$T(E_{1,1}) = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad T(E_{2,1}) = \begin{bmatrix}1\\0\\1\end{bmatrix} \text{ and } T(E_{3,1}) = 0 .$$

Then T preserves zero-term rank 1, but $\overline{T(J)} \neq J$ since for any $A \in M_{3,1}(Z_2)$, $T(A)$ is $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}0\\1\\1\end{bmatrix}$, or the zero matrix, and is zero only if $A = 0$ or $E_{3,1}$.

As the following lemma illustrates, characterizing linear operators which preserve the set of matrices of zero-term rank 1 can be assisted by first looking at linear operators on $M_{m,n}(\mathbf{B})$.

Lemma 2.2 If F is a field with at least $mn+1$ elements and $T\colon M_{m,n}(F) \to M_{m,n}(F)$ preserves zero-term rank 1, then $\overline{T}$ is bijective on the set of cells in $M_{m,n}(\mathbf{B})$.

Proof. From the above proof of Lemma 2.1, there exists $X \in \mathrm{M}_{m,n}(F)$ such that $T(X)$ has all nonzero entries. That is $\#T(X) = mn$. Since F has at least $mn+1$ elements, there exists a pair (k,l) such that for some $x_{k,l} \in F$,

$$T(E_{i,j} + x_{k,l}E_{k,l}) > T(E_{i,j})$$

unless $\#T\ (E_{i,j}) = mn$. Let $Y_1 = E_{i,j} + x_{k,l}E_{k,l}$. If $\#T(Y_1) \neq mn$ there is some cell $E_{r,s}$ such that for some $x_{r,s} \in F$, $T(Y_1 + x_{r,s}E_{r,s}) > T(Y_1)$. Repeating this process, one finds a matrix Y such that $\#Y < mn$ while $\#T(Y) = mn$. Since $\#Y < mn$, we may assume without loss of generality that $y_{1,1} = 0$. Since F has at least $mn+1$ elements, there is $x \in F$ such that

$$\#T(Y + x(J - E_{1,1})) = mn$$

and

$$\#(Y + x(J - E_{1,1})) = mn - 1.$$

That is,

$$z(T(Y + x(J - E_{1,1}))) = 0$$

while $z(Y + x(J - E_{1,1})) = 1$, a contradiction. Thus $\#T(E_{i,j}) \leq 1$, for all cells $E_{i,j}$. Further, if $T(E_{i,j}) = 0$, then since $\#T(X) = mn$, $T(E_{r,s})$ must have at least 2 entries for some (r,s), a contradiction. That is $\overline{T}$ is bijective on the set of cells □$M_{m,n}(\mathbf{B})$.

Theorem 2.1 If F is a field with at least $mn+1$ elements and $T\colon M_{m,n}(F) \to M_{m,n}(F)$ preserves zero-term rank 1, then T is a (P,Q,B)-operator.

Proof. From lemma 2.2, $\overline{T}$ is bijective on the set of cells in $M_{m,n}(\mathbf{B})$. Since T preserves zero-term rank 1 we have that $\overline{T}$ does also. By Theorem 1.1, $\overline{T}$ is a (P,Q,B)-operator, where $B = J$. Thus, $\overline{P^tTQ^t}$ is the identity linear operator on $M_{m,n}(\mathbf{B})$. That is, $P^tT(E_{i,j})Q^t = b_{i,j}E_{j,i}$ for each pair (i,j) (or perhaps $P^tT(E_{i,j})Q^t = b_{i,j}E_{j,i}$ in the case $m = n$). Then, $T(X) = P(X \circ B)Q$ for all $X \in \mathrm{M}_{m,n}(F)$ or $m = n$ and $T(X) = P(X \circ B)^tQ$ for all $X \in \mathrm{M}_{m,n}(F)$. □

By application of theorem 2.1 to theorems 1.1 and 1.2 we obtain the characterizations of the linear operators that preserve zero-term rank of matrices over fields .

Theorem 2.2 If F is a field with at least $mn+1$ elenents and $T\colon M_{m,n}(F) \to M_{m,n}(F)$ then the following are equivalent:

) T is a (P,Q,B)-operator;

) T preserves zero-term rank;

) T preserves zero-term rank 1.

) T preverves term rank;

) T preserves term rank 1 and term rank 2.

Proof. Obviously) implies) and) implies). Theorem 2.1 shows that) implies). Since),) and) are equivalent by theorem 1.2, we have done.

3 Zero-term rank preservers over rings

In this section, we obtain the characterizations of the linear operators that preserve zero-term tank of matrices over rings.

Lemma 3.1. Let $\Re$ be any ring whose characteristic is not 2. If $T: M_{m,n}(\Re) \to M_{m,n}(\Re)$ preserves zero-term ranks 0 and 1 then T maps each cell to a nonzero multiple of some cell which induces a bijection on the set of indices $\{1,\cdots,m\}\times\{1,\cdots,n\}$.

Proof. Since T preserves zero-term rank 0, We have $T(J)=K$ for some $K \in M_{m,n}(\Re)$ with $k_{i,j} \neq 0$ for all (i,j). If $T(E_{i,j})=0$ then $T(J \setminus E_{i,j}) = T(J)$. But $z(T(J)) = z(K) = 0$ while $z(T(J \setminus E_{i,j})) = 1$ since T preserves zero-term rank 1. This contradiction implies that $T(E_{i,j}) \neq 0$ for all (i,j). Since $z(T(J \setminus E_{i,j})) = 1$, there is some pair (r,s) such that $T(J \setminus E_{i,j})$ has (r,s) entry zero. Let
$T(E_{i,j}) = X = (x_{c,d})$. Then
$K = T(J) = T(J \setminus E_{i,j}) + T(E_{i,j})$
and hence $k_{r,s} = x_{r,s}$. If $x_{c,d} \neq 0$ and $x_{c,d} \neq k_{c,d}$, then $z(x_{c,d}J - k_{c,d}E_{i,j}) = 0$ while the (c,d) entry of $T(x_{c,d}J - k_{c,d}E_{i,j}) = x_{c,d}T(J) - k_{c,d}T(E_{i,j})$ is $x_{c,d}k_{c,d} - k_{c,d}x_{c,d} = 0$, a contradiction. Thus if $x_{c,d} \neq 0$, then $x_{c,d} = k_{c,d}$ for all (c,d). Further for $T(J \setminus E_{i,j}) = C$, we must have $C + X = K$. Hence if $x_{r,s} \neq 0$, we have $c_{r,s} + x_{r,s} = k_{r,s}$ or $c_{r,s} + k_{r,s} = k_{r,s}$. Necessarily, $c_{r,s} = 0$. Since $z(T(J \setminus E_{i,j})) = 1$, we must have all the zero entries of C lie in a single row or column. For our purpose we may assume all zero entries of C and hence all nonzero entries of X lie in row r.

Suppose that $T(E_{i,j}) = X = (x_{c,d})$ and $T(E_{h,l}) = Y = (y_{e,f})$. If the (r,s) entries of both X and Y are not zero, then $k_{r,s} = x_{r,s} = y_{r,s}$ and hence $T(E_{i,j} + E_{h,l})$ has (r,s) entry $2k_{r,s}$. Since the characteristic of $\Re$ is not 2, $2k_{r,s} \neq 0$. Hence $T(2J - E_{i,j} - E_{h,l})$ has zero in the (r,s) entry so $z(T(2J - E_{i,j} - E_{h,l})) \geq 1$ while $z(2J - E_{i,j} - E_{h,l}) = 0$, a contradiction. By the pigeon hole principle, since $T(E_{i,j}) \neq 0$ for all (i,j), $T(E_{i,j})$ must be a single weighted cell. Since $T(J) = K$ has zero-term rank 0, the mapping T must induce a bijection on the set of indices $\{1,\cdots,m\}\times\{1,\cdots,n\}$.

Now, we have the following theorems 3.1 and 3.2 by the similar methods to those of theorems 2.1 and 2.2

Theorem 3.1. Let $\Re$ be any ring whose characteristic is not 2. If $T: M_{m,n}(\Re) \to M_{m,n}(\Re)$ preserves zero-term ranks 0 and 1, then T is a (P,Q,B)-operator. □

Theorem 3.2. Let $\Re$ be any ring whose characteristic is not 2, and $T: M_{m,n}(\Re) \to M_{m,n}(\Re)$ be a linear operator. Then the following are equivalent :

) T preserves zero-term ranks 0 and 1;

) T is a (P,Q,B)-operator;

) T preserves zero-term rank;

) T preverves term rank;

) T preserves term rank 1 and term rank 2. □

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