On a theorem of Anosov and its generalization

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Abstract: - D. Anosov showed that for any selfmap $f : X \to X$ of a nilmanifold X, $N(f) = \pm L(f)$ where $N(f)$ and $L(f)$ denote the Nielsen and the Lefschetz number of f, respectively. We introduce generalized Lefschetz and Nielsen type numbers for maps of pairs, denoted by $\mathfrak{L}(f; X, A)$ and $\mathfrak{N}(f; X, A)$ so that if $f : (X, A) \to (X, A)$ is a map of a pair of nilmanifolds, then $\mathfrak{N}(f; X, A) = \pm\mathfrak{L}(f; X, A) =$ provided $L(f) \cdot L(f_A) \geq 0$.

Key-Words: - Relative Nielsen fixed point theory, nilmanifolds, solvmanifolds, Nielsen numbers, generalized Lefschetz numbers.

1 Introduction

For a selfmap $f : X \to X$ of a compact connected polyhedron X, the Lefschetz number $L(f)$ gives an algebraic count of the number of fixed points of f. The Nielsen number $N(f)$, on the other hand, often yields the minimal number of fixed points in the homotopy class of f. When X is a nilmanifold, D. Anosov [1] showed that these two homotopy invariants coincide, up to a sign. Thus, on nilmanifolds, the computation of the Nielsen number, which is difficult in general, reduces to the homological calculation of the Lefschetz trace. A strengthened version of Anosov's theorem was obtained by B. Norton-Odenthal [8] by proving a product formula for the Reidemeister trace or equivalently, the generalized Lefschetz number $\mathfrak{L}(f)$ [6]. For selfmaps on solvmanifolds, only the inequality $|L(f)| \leq N(f)$ holds [7] in general.

H. Schirmer introduced a Nielsen type number $N(f; X, A)$ in [10] for a map $f : (X, A) \to (X, A)$ of a polyhedral pair. Under mild conditions, $N(f; X, A)$ can be realized as the minimal number of fixed points in the relative homotopy class of f. A generalized Lefschetz number for a map of pairs was introduced in [9] so that $N(f; X, A)$ is the number of terms with nonzero coefficients, analogous to the same relationship between $N(f)$ and $\mathfrak{L}(f)$ in the absolute case. More recently, F. Cardona [4] extended a relative Reidemeister theory initiated by A. Schusteff [12], introducing relative Reidemeister numbers which are upper bounds for the relative Nielsen number and the relative Nielsen number on the complement.

The relative Reidemeister number $R(f; X, A)$ was defined [4] in the same fashion as the relative Nielsen number of Schirmer, based on the inclusion-exclusion principle. The relative generalized Lefschetz number of [9] was defined in a different way and it is an element in a free $\mathbb{Z} \times \mathbb{Z}$ module whose generating set does not easily yield $R(f; X, A)$ as $\mathfrak{L}(f)$ does in the absolute case.

The purpose of this note is to introduce an appropriate relative generalized Lefschetz number $\mathfrak{L}(f; X, A)$ and a relative generalized Nielsen number $\mathfrak{N}(f; X, A)$, following Schirmer [10] and Cardona [4], so that when (X, A) is a pair of nilmanifolds, we generalize Anosov's theorem using these new homotopy invariants.

2 The combinatorial relative generalized Lefschetz number

The classical Reidemeister trace or the generalized Lefschetz number is the twisted conjugacy class of the trace in the integral group ring generated by the fundamental group. It also has a different representation as follows. Let $f : X \to X$ be a selfmap of a compact connected polyhedron and $\varphi = f_{\#} : \pi_1(X) \to \pi_1(X)$ be the induced homomorphism by choosing appropriate base points. The set of orbits of the action of $\pi_1(X)$ on $\pi_1(X)$

via $\sigma \bullet \alpha \mapsto \sigma\alpha\varphi(\sigma)^{-1}$ is the set of Reidemeister classes, denoted by $\mathfrak{R}(\varphi, \pi)$ where $\pi = \pi_1(X)$. To each Reidemeister class ρ, we associate the usual fixed point index $i_X(\rho)$ (see e.g. [2]). If ρ corresponds to an empty fixed point class of f, then we set $i_X(\rho) = 0$. Then the generalized Lefschetz number $\mathfrak{L}(f)$ can be represented by

$$\mathfrak{L}(f) = \sum_{\rho \in \mathfrak{R}(\varphi,\pi)} i_X(\rho)\rho \in \mathbb{Z}\mathfrak{R}(\varphi, \pi)$$

as an element of the free abelian group generated by the set $\mathfrak{R}(\varphi, \pi)$ (see [6]).

Suppose that $f : (X, A) \to (X, A)$ is a map of a compact polyhedral pair and X is connected. By choosing base points as in [5] or [9], we have the Reidemeister actions of

1. π_A on π_A:

$$\overline{\sigma} \bullet \overline{\alpha} \mapsto \overline{\sigma\alpha}\varphi_A(\overline{\sigma})^{-1}$$

and

2. π on π:

$$\sigma \bullet \alpha \mapsto \sigma\alpha\varphi(\sigma)^{-1}$$

where φ, φ_A are the induced homomorphisms of f and f_A respectively. Thus, we can define

$$\mathfrak{L}(f_A) = \sum_{\overline{\rho} \in \mathfrak{R}(\varphi_A,\pi_A)} i_A(\overline{\rho})\overline{\rho} \in \mathbb{Z}\mathfrak{R}(\varphi_A, \pi_A).$$

As in [4] and [5], the set of *common* Reidemeister classes is defined as

$$\mathfrak{R}(\varphi, \varphi_A) = \{\rho \in \mathfrak{R}(\varphi, \pi) | \exists \overline{\rho} \in \mathfrak{R}(\varphi_A, \pi_A) \text{ such that } \widehat{j_A}(\overline{\rho}) = \rho\}$$

where $\widehat{j_A} : \mathfrak{R}(\varphi_A, \pi_A) \to \mathfrak{R}(\varphi, \pi)$ is the function induced by the inclusion $j_A : A \hookrightarrow X$.

Define the common generalized Lefschetz number to be the element

$$\mathfrak{L}(f, f_A) = \sum_{\rho \in \mathfrak{R}(\varphi,\varphi_A)} I_{X,A}(\rho)\rho \in \mathbb{Z}\mathfrak{R}(\varphi, \varphi_A)$$

where

$$I_{X,A}(\rho) = \begin{cases} i_X(\rho), & \text{if } \exists \overline{\rho} \in \widehat{j_A}^{-1}(\rho) \text{ such} \\ & \text{that } i_A(\overline{\rho}) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Define the *combinatorial relative generalized Lefschetz number* to be

$$\mathfrak{L}(f; X, A) := \mathfrak{L}(f_A) + \mathfrak{L}(f) - \mathfrak{L}(f, f_A)$$

as an element of $\mathbb{Z}\mathfrak{R}(\varphi, \pi) \oplus \mathbb{Z}\mathfrak{R}(\varphi_A, \pi_A)$.

Remark 1 *We use the term combinatorial since $\mathfrak{L}(f; X, A)$ is defined using the inclusion-exclusion principle of combinatorics.*

Theorem 1 *(1) If $A = \emptyset$ or $A = X$, then $\mathfrak{L}(f; X, A) = \mathfrak{L}(f)$; (2) $\mathfrak{L}(f; X, A)$ is invariant under homotopy and homotopy type; and it possesses the usual commutativity property; (3) the number of non-zero terms in $\mathfrak{L}(f; X, A)$ is equal to the relative Nielsen number $N(f; X, A)$.*

3 A relative Anosov's theorem

We present a relative version of Anosov's theorem in this section and its generalization to solvmanifolds. First, we introduce some terminology and notation.

First, we generalize the relative Nielsen number by defining

$$\begin{aligned}
\mathfrak{N}(f) &= \sum_{i_X(\rho)\neq 0} \rho; \\
\mathfrak{N}(f_A) &= \sum_{i_A(\overline{\rho})\neq 0} \overline{\rho}; \\
\mathfrak{N}(f, f_A) &= \sum_{I_{X,A}(\rho)\neq 0} \rho; \text{ and} \\
\mathfrak{N}(f; X, A) &= \mathfrak{N}(f) + \mathfrak{N}(f_A) - \mathfrak{N}(f, f_A).
\end{aligned}$$

The following is our main theorem.

Theorem 2 *Let (X, A) be a pair of nilmanifolds. For any $f : (X, A) \to (X, A)$, if $L(f) \cdot L(f_A) \geq 0$ then we have*

$$\mathfrak{N}(f; X, A) = \pm\mathfrak{L}(f; X, A)$$

as elements in $\mathbb{Z}\mathfrak{R}(\varphi, \pi) \oplus \mathbb{Z}\mathfrak{R}(\varphi_A, \pi_A)$.

Remark 2 *Note that in the extreme cases where $A = \emptyset$ or $A = X$, the condition $L(f) \cdot L(f_A) \geq 0$ always holds and thus Theorem 2 reduces to Anosov's theorem [1].*

There is another way to extend Anosov's theorem. Analogous to $|L(f)|$, we define the *transversal generalized Lefschetz number* to be

$$|\mathfrak{L}(f)| = \sum_{\rho \in \mathfrak{R}(\varphi, \pi)} |i_X(\rho)| \rho.$$

Similarly, we define

$$|\mathfrak{L}(f_A)| = \sum_{\overline{\rho} \in \mathfrak{R}(\varphi_A, \pi_A)} |i_A(\overline{\rho})| \overline{\rho};$$

$$|\mathfrak{L}(f, f_A)| = \sum_{\rho \in \mathfrak{R}(\varphi, \varphi_A)} |I_{X,A}(\rho)| \rho;$$

and the *relative transversal generalized Lefschetz number* is given by

$$|\mathfrak{L}(f; X, A)| := |\mathfrak{L}(f_A)| + |\mathfrak{L}(f)| - |\mathfrak{L}(f, f_A)|.$$

Next, we extend Theorem 2 to maps of pairs of solvmanifolds, without any hypotheses on the Lefschetz numbers of f and f_A.

Theorem 3 *If (X, A) is a pair of solvmanifolds, then for any $f : (X, A) \to (X, A)$, we have*

$$|\mathfrak{L}(f; X, A)| = \mathfrak{N}(f; X, A).$$

Remark 3 *The number $\sum |i_X(\rho)|$ in the definition of $|\mathfrak{L}(f)|$ coincides with the transversal Nielsen number $N_{\pitchfork}(f)$ as introduced in [11]. Therefore $|\mathfrak{L}(f)|$ is a generalization of $N_{\pitchfork}(f)$ and thus the terminology "transversal" is justified. Moreover, consideration of the sum of the absolute value of the indices had already been used by H. Hopf in the 1930's in the context of the absolute degree or Absolutgrad (see [3]).*

In conclusion, we observe that Anosov's theorem can be generalized in many ways. In particular, Theorem 3 takes a very simple form when $|L(f)|$ is replaced by the relative transversal generalized Lefschetz number $|\mathfrak{L}(f; X, A)|$. For further generalization to coincidences of maps on solvmanifolds and related results, see [14].

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