# Euler-Lagrange Characterization of Area-Minimizing Graphs in Randers Spaces with Non-constant Potential

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Abstract: Within the framework of Randers spaces with non-constant potential, the mean curvature form of hypersurfaces is determined, and the equations which characterize the area-minimizing graph surfaces are explicitly derived as associated Euler-Lagrange PDEs. By applying the extended framework to the particular case of graphs in Randers spaces with constant potential, the known result of Souza-Spruck-Tenenblat ([27]) is confirmed. It is shown that the only linear affine potentials for which the Randers metric admits all the affine planes as minimal graphs, are necessarily constant. As well, the ODE which characterizes the generating curve of surfaces of revolution is derived, this extending the result obtained by Souza-Tenenblat in [26] in the constant potential case.

*Key–Words:* Euler-Lagrange equations, mean curvature 1-form, Finsler structure, Randers metric, hypersurfaces, minimal graphs, minimal surfaces of revolution.

## 1 Introduction

The recent numerous and proficient applications of minimal and CMC (constant mean curvature) surfaces in biology, engineering, bionics, relativity and nanotechnology have drawn a significant interest towards the open problem of classifying CMC and minimal hypersurfaces both in isotropic and anisotropic media. Regarding the latter concern, the foundations of the related Finslerian BH (Busemann-Hausdorff) mean curvature theory go back to H.Rund ([19, 20, 21]) and M.Matsumoto ([13, 12]). Recently Z.Shen ([24]) obtained for the first time a feasible expression for the BH mean curvature form, related to the area-minimizing variational problem. As well, an alternative approach based on the Holmes-Thompson volume (e.g., in [17, 18]) has been developed. However, in both cases, the technical obstructions related to the volume computation of the induced hypersurface indicatrix have recommended a limited number of classes of Finsler spaces for deriving explicit results regarding the minimal BH submanifolds, especially surfaces. Significant progress has been obtained for $(\alpha, \beta)$ locally Minkowski Finsler metrics, in the Randers case by M. de Souza, J. Spruck and K. Tenenblat ([26, 25, 27]) and in the Kropina case by the author ([1, 2]).

In the present note, we extend the results obtained in [26] to the more general (*non-locally Minkowski* Randers) case, recovering the originar ones as particular cases when the potential form $\beta = b(x)dx^{n+1}$ has constant coefficient.

Let hereafter $(\tilde{M}, \tilde{F})$ be a Finsler structure, i.e., there exists a function (the Finsler norm) $\tilde{F} : T\tilde{M} \to \mathbb{R}_+$ which obeys the following properties:
i) $\tilde{F}$ is continuous on $T\tilde{M}$ and $C^\infty$ outside the null section of $T\tilde{M}$;
ii) $\tilde{F}$ is positively homogeneous of first order on the fibers of the tangent bundle $(T\tilde{M}, \pi, \tilde{M})$, i.e.,

$$\tilde{F}(x, \lambda y) = \lambda \tilde{F}(x, y), \ \forall \lambda \in \mathbb{R}_+;$$

iii) the halved $y$-Hessian of $\tilde{F}^2$, $g_{ij}(x, y) = \dfrac{1}{2}\dfrac{\partial^2 \tilde{F}^2}{\partial y^i \partial y^j}$ is positively definite for all $(x, y) \in T\tilde{M}$.

Let further $\varphi : (M, F) \to (\tilde{M}, \tilde{F})$ be an isometric immersion, with $F$ induced by $\tilde{F}$. Then the following result proved by Z. Shen ([24, (57), p.563]) holds true:

**Theorem 1.** *The mean curvature of the isometrically imbedded submanifold $M \subset \tilde{M}$ is given by*

$$\tilde{H}_\varphi(X) = \frac{G_{x^i} - G_{z^i_\varepsilon z^j_\eta} \cdot \varphi^j_{u^\varepsilon u^\eta} - G_{x^j z^i_\varepsilon} \cdot \varphi^j_{u^\eta}}{G} X^i, \quad (1)$$

*where lower indices stand for corresponding partial derivatives and:*

- $(u^a, v^b)_{a,b\in\overline{1,n}}$ *are local coordinates in* $TM$ *(*$\dim M = n$*);*
- $(x^i, y^j)_{i,j\in\overline{1,m}}$ *are local coordinates in* $T\tilde{M}$ *(*$\dim \tilde{M} = m$*);*
- $z^i_a$ *are the entries of the Jacobian matrix* $[J(\varphi)] = (\partial\varphi^i/\partial u^a)_{a=\overline{1,n},i=\overline{1,m}}$*;*
- $\varphi_t : M \to \tilde{M}, t \in (-\varepsilon, \varepsilon), \varphi_0 = \varphi$, *is a variation of the surface;*
- $X$ *is the vector field* $X_x = \left.\dfrac{\partial\varphi_t}{\partial t}\right|_{t=0}(x)$ *induced along* $\varphi$ *attached to the variation;*
- *$G$ is the Finsler induced volume form*

$$G_{\tilde{e}}(z) = \frac{vol[B^n]}{vol\{(v^a) \in \mathbb{R}^n \mid \tilde{F}(v^a z^i_a \tilde{e}_i) \le 1\}}, \qquad (2)$$

*where* $z = (z^i_a)_{a=\overline{1,n},i=\overline{1,m}} \in GL_{m\times n}(\mathbb{R})$, $\tilde{e} = \{\tilde{e}_i\}_{i=\overline{1,m}}$ *is an arbitrary basis in* $\mathbb{R}^m$ *and* $B^n \subset \mathbb{R}^n$ *is the standard Euclidean ball.*

As well, in the same cornerstone work [24], it was proved that the variation of the volume of $M$ reaches a minimum while the associated Euler-Lagrange PDEs $H_\varphi(X) = 0$ hold true for all $X \in \mathcal{X}(\tilde{M})$. Recent advances in constructing BH minimal surfaces based on the relation 2 were provided in ([25, 26]), by characterizing the minimal surfaces of revolution in Finsler $(\alpha, \beta)-$spaces $(\tilde{M} = \mathbb{R}^3, \tilde{F})$ with the Randers fundamental function

$$\tilde{F}(x, y) = \alpha(x, y) + \beta(x, y),$$

where $\alpha(x, y) = \sqrt{a_{ij}(x)^i y^j}$, $\beta(x, y) = b_i(x)y^i$, for the particular case when $a_{ij} = \delta_{ij}$ (the Euclidean metric) and constant potential $\beta = b\cdot \mathrm{d}x^3$, with $b \in [0, 1)$. As well, in [27], were derived the minimality equations for 2-dimensional graphs, in the case of constant potential form $\beta$.

## 2 Free potential Randers hypersurfaces

Recently, within the framework of Geometric Dynamics ([28, 29]), promising extensions of the Plateau problem ([10]) have been derived in [16]. As well, within the extension provided by the background of Finslerian Geometry, we shall derive in the following the mean curvature form and the explicit Euler-Lagrange equations for the Randers case with non-constant potential, then rewrite them in a form convenient for further technical processing, and specify them for graphs and surfaces of revolution in the case $n = 2$.

First, we consider the case when $\tilde{M} = \mathbb{R}^{n+1}$ is a Randers space endowed with the fundamental function

$$\tilde{F}(x, y) = \sqrt{\delta_{ij}y^i y^j} + b_{n+1}(x)y^{n+1}, \qquad (3)$$

with $b_{n+1}(x) \in [0, 1), \forall x \in \tilde{M}$, which is reducible for $b_{n+1} = const. = b \in [0, 1)$ to the case studied by M. Souza and K. Tenenblat ([25, 26]).

Let $M = Im\ \varphi$, $\varphi : D \subset \mathbb{R}^n \to \tilde{M} = \mathbb{R}^{n+1}$ be a isometrically immersed simple hypersurface. We denote $z^i_\alpha = \dfrac{\partial\varphi^i}{\partial u^\alpha}$, $u = (u^1, \ldots, u^n) \in D$, and $h_{\alpha\beta} = \sum\limits_{i=1}^{n+1} z^i_\alpha z^j_\beta$. From ([25, p.627],[26]), it is known that the defining function in (1) is in this case given by

$$G = C \cdot (1 - B)^{(n+1)/2},$$

where $B = b^2(x) z^{n+1}_a z^{n+1}_b h^{ab}$, $b = b_{n+1}$, $C = \sqrt{\det(h_{\alpha\beta})}$, and the volume form of the hypersurface $M$ is $dV_F = G \cdot du^1 \wedge \ldots \wedge du^n$.

Then we have the following:

**Theorem 2. ([3]).** *The components of the mean curvature 1-form of the hypersurface $M$, isometrically immersed in the Randers space $\tilde{M} = \mathbb{R}^{n+1}$ endowed with the fundamental function (3), are given by*

$$\begin{aligned}\tilde{H}_i = &-\frac{(n+1)B\omega_i}{1-B} - \frac{1}{C(1-B)^2}\cdot\\ &\cdot\left[\frac{(n^2-1)}{4}B_{z^i_\varepsilon}B_{z^j_\eta}C + (1-B)^2 C_{z^i_\varepsilon z^j_\eta} - \frac{n+1}{2}(1-B)\cdot\right.\\ &\left.\cdot\left(B_{z^i_\varepsilon z^j_\eta}C + B_{z^j_\eta}C_{z^i_\varepsilon} + B_{z^i_\varepsilon}C_{z^j_\eta}\right)\right]\frac{\partial^2\varphi^j}{\partial u^\varepsilon \partial u^\eta} +\\ &+\frac{(n+1)\omega_j}{C(1-B)}\left[BC_{z^i_\varepsilon} + \left(1 - \frac{B(n-1)}{2(1-B)}\right)CB_{z^i_\varepsilon}\right]\cdot\\ &\cdot\frac{\partial\varphi^j}{\partial u^\varepsilon},\quad i = \overline{1, n+1},\quad \textit{with } \omega_i = \frac{\partial \ln b}{\partial x^i}.\end{aligned} \qquad (4)$$

We note that in the case $b = const.$ (i.e., for $\omega_i = 0, \forall i \in \overline{1, n+1}$), $\tilde{H}_i v^i = 0$ becomes the minimality equation obtained in [26, Theorem 2, p. 629] by Souza-Tenenblat. As well, using laborious computations, one infers a useful alternate form of the mean curvature, as follows:

**Corollary 1.** *The mean curvature form components rewrite in terms of $C$ and $E = B^2C$ as follows:*

$$\begin{aligned}\tilde{H}_i = &-\tfrac{1}{C(C^2-E)}\Big[(C^2)_{z^i_\varepsilon z^j_\eta}\cdot k_{c_2} - E_{z^i_\varepsilon z^j_\eta}\cdot k_{e_2} + \\ &+E_{z^i_\varepsilon}E_{z^j_\eta}\cdot k_{e_{12}} + (C_{z^i_\varepsilon}E_{z^j_\eta} + C_{z^j_\eta}E_{z^i_\varepsilon})\cdot k_{ce} + \\ &+C_{z^i_\varepsilon}C_{z^j_\eta}\cdot k_{cc}\Big]\varphi^j_{\varepsilon\eta} + \tfrac{(n+1)}{C(C^2-E)}\big[EC_{z^i_\varepsilon} + \\ &+k_m\cdot(CE_{z^i_\varepsilon} - 2EC_{z^i_\varepsilon})\big]\omega_j\varphi^j_\varepsilon - \frac{(n+1)E}{C^2-E}\omega_i.\end{aligned} \quad (5)$$

*where*

$$k_{c_2} = \tfrac{C^2+nE}{2C}, \quad k_{e_2} = \tfrac{(n+1)C}{2},$$
$$k_{e_{12}} = \tfrac{(n^2-1)C}{4(C^2-E)}, \quad k_{ce} = \tfrac{(n+1)(C^2-nE)}{2(C^2-E)},$$
$$k_{cc} = \tfrac{n(n+2)E^2-2nEC^2-C^4}{C(C^2-E)}, \quad k_m = 1 - \tfrac{(n-1)E}{2(C^2-E)}$$

In particular, for surfaces ($n = 2$), we infer the following

**Corollary 2.** *For $M$ Randers surface isometrically immersed in $\tilde{M} = \mathbb{R}^3$, the mean curvature form $\tilde{H} = \tilde{H}_i dx^i$ has the components:*

$$\begin{aligned}\tilde{H}_i = &-\tfrac{3E}{C^2-E}\omega_i + \tfrac{3}{2C(C^2-E)^2}\big[2(2E-C^2)EC_{z^i_\varepsilon} + \\ &+(2C^2-3E)CE_{z^i_\varepsilon}\big]\omega_j\varphi^j_\varepsilon - \tfrac{1}{C(C^2-E)}\Big\{-\tfrac{3C}{2}E_{z^i_\varepsilon z^j_\eta} + \\ &+\tfrac{3C}{4(C^2-E)}E_{z^i_\varepsilon}E_{z^j_\eta} + \tfrac{3(C^2-2E)}{2(C^2-E)}(C_{z^i_\varepsilon}E_{z^j_\eta} + C_{z^j_\eta}E_{z^i_\varepsilon}) + \\ &+\tfrac{8E^2-4EC^2-C^4}{C(C^2-E)}C_{z^i_\varepsilon}C_{z^j_\eta} + \tfrac{C^2+2E}{2C}(C^2)_{z^i_\varepsilon z^j_\eta}\Big\}\varphi^j_{\varepsilon\eta},\end{aligned} \quad (6)$$

As consequence, one can characterize the CMC surfaces in a Randers space with nonconstant potential, as satisfying the PDE $\tilde{H}_iX^i = k$, with $\tilde{H}$ given by (6), and

$$X = ||N||^{-1}_{\tilde{F}}\cdot N, \quad N^i = \varepsilon^{ijk}(G_*Z^1)_j(G_*Z^2)_k,$$

where $Z^\alpha = z^i_\alpha\frac{\partial}{\partial x^i}, \alpha \in \overline{1,2}$, with $\varepsilon^{ijk}$ the skew-symmetrization symbol and $G_*v$ defined by the equality

$$(G_*v)(v') = \langle v, v'\rangle_{\tilde{F}} = \frac{1}{2}\frac{\partial\tilde{F}^2}{\partial y^i\partial y^j}v^iv'^j.$$

We note that $N \equiv *((G_*Z^1)\wedge(G_*Z^2))$ and $X \in Ker(G_*Z^1)\cap Ker(G_*Z^2)\cap\{y \in T_{\varphi(u)}\tilde{M} \mid \tilde{F}(y) = 1\}$, where "*" is the Euclidean Hodge operator.

Then, in the particular case $k = 0$ one obtains the minimal surfaces equation, and further, for $b = const$ and $X = Z^1\times Z^2$ is recaptured the minimality equation obtained by Souza and Tenenblat ([26, Corollary 3, p. 630]).

# 3 Free potential Randers minimal graphs

We shall further consider the case when the immersion $\varphi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ which provides the surface $M = \varphi(D)$ is a graph (a Monge patch) defined by $\varphi : D \subset \mathbb{R}^2 \to \mathbb{R}$,

$$\begin{aligned}\varphi(u^1,u^2) &= (u^1,u^2,f(u^1,u^2)) \\ &= \sum_{i=1}^{3}(u^1\delta_{i1} + u^2\delta_{i2} + f\delta_{i3}).\end{aligned}$$

Then, by choosing $X = \varphi_{u^1}\times\varphi_{u^2} = -\delta_{i1}f_1 - \delta_{i2}f_2 + \delta_{i3}$, the minimal graph equation $\tilde{H}_iX^i = 0$ can be explicitly derived, as follows:

**Theorem 3.** *The minimal graph equation for the Randers space with free potential $\beta = b(x)\,dx^3, b(x)\in[0,1), \forall x \in \mathbb{R}^3$ has the form:*

$$\begin{aligned}&\frac{3b^2}{\tau^2(1-b^2)}\cdot(b^2+2\tau-3b^2\tau)[f_1\omega_1 + f_2\omega_2 + \\ &+\omega_3(f_1^2+f_2^2)] - \frac{1-b^2}{\tau^2(\tau-b^2)}\ \{\tau(\tau-3b^2)\cdot \\ &\cdot[(1+f_1^2)f_{22} - 2f_1f_2f_{12} + (1+f_2^2)f_{11}] + \\ &+3b^2(\tau+b^2)[f_1^2f_{11} + 2f_1f_2f_{12} + f_2^2f_{22}]\} + \\ &+\frac{3b^2(1-\tau)}{\tau(1-b^2)}(-f_1\omega_1 - f_2\omega_2 + \omega_3) = 0,\end{aligned} \quad (7)$$

*where $\tau = b^2 + C^2(1-b^2)$, $C = \sqrt{det(h_{\alpha\beta})} = \sqrt{1+f_1^2+f_2^2}$, $f_i = \partial f/\partial x^i$ and $f_{ij} = \partial^2 f/\partial x^i\partial x^j$, $i,j = \overline{1,2}$.*

**Remark.** The equation (7) can be rewritten in the condensed form:

$$\begin{aligned}&\frac{3b^2}{\tau^2(1-b^2)}\cdot(b^2+2\tau-3b^2\tau)[f_1\omega_1 + f_2\omega_2 + \\ &+\omega_3(f_1^2+f_2^2)] - \frac{1-b^2}{\tau^2(\tau-b^2)}\Bigg[\sum_{i,j\in\overline{1,2}}\tau(\tau-3b^2)\cdot \\ &\cdot(\delta_{ij} - f_if_jC^{-2}) + 3b^2(\tau+b^2)f_if_jC^{-2}\Bigg] + \\ &+\frac{3b^2(1-\tau)}{\tau(1-b^2)}(-f_1\omega_1 - f_2\omega_2 + \omega_3) = 0.\end{aligned} \quad (8)$$

As consequence of Theorem 3, for $b = const.$, one easily infers the known result:

**Corollary 3. ([27])** *Any minimal graph immersed in a Randers space with constant potential* $\beta = bdx^3, b \in [0,1)$ *satisfies the equation:*

$$\tau(\tau - 3b^2)[(1+f_1^2)f_{22} - 2f_1f_2f_{12} + (1+f_2^2)f_{11}]+ \\ +3b^2(\tau(\tau+b^2)[f_1^2f_{11} + 2f_1f_2f_{12} + f_2^2f_{22}] = 0, \tag{9}$$

*or, equivalently,*

$$\sum_{i,j\in\overline{1,2}} \tau(\tau-3b^2)\left(\delta_{ij} - \frac{f_if_j}{C^2}\right)+3b^2(\tau+b^2)\frac{f_if_j}{C^2} = 0. \tag{10}$$

As well, more particular, for $b = 0$, one obtains the classic result:

**Corollary 4.** *Any minimal graph immersed in the Euclidean space satisfies the minimal surfaces PDE ([10]):*

$$(1+f_1^2)f_{22} - 2f_1f_2f_{12} + (1+f_2^2)f_{11} = 0. \tag{11}$$

Regarding the simplest Euclidean minimal graphs - the affine planes, one can prove the following result in the framework of Randers spaces with non-constant potential:

**Corollary 5.** *The only Randers spaces (3) endowed with* affine linear *potential function* $b(x)$*, for which all the affine planes are minimal, are the ones with constant potential.*

*Proof.* After replacing in (8) $u(x) = Ax^1+Bx^2+Cx^3+D$ with $x = (x^1,x^2,x^3) = (u^1,u^2,f(u^1,u^2))$ and $f(u) = mu^1+nu^2+p$ (where $A$, $B$, $C$, $D$, $m$, $n$, $p \in \mathbb{R}$), the left-hand side of the equation properly scaled provides a polynomial $P \in \mathbb{R}[A,B,C]$. The arbitrariness of $m,n,p,u^1,u^2$ for which $P = 0$ holds, leads subsequently to:

$$K_{4,0,3,2,0}P(A,B,C) = -8A^5 \quad \Rightarrow A = 0;$$
$$K_{3,1,8,0,0}P(0,B,C) = 13C^5 \quad \Rightarrow C = 0;$$
$$K_{0,4,4,1,0}P(0,B,0) = -4B^5 \quad \Rightarrow B = 0,$$

where $K_{i_1,i_2,i_3,i_4,i_5}P(a,b,c)$ is the coefficient of $(u^1)^{i_1}(u^2)^{i_2}m^{i_3}n^{i_4}p^{i_5}$ in $P(a,b,c)$. Hence $A = B = C = 0$ and $b(x) = D = const.$, which reduces the linearly affine function $b(x)$ to the particular case considered in [26]. □

# 4 Free potential Randers surfaces of revolution

For the case of surfaces of revolution, in the case of nonconstant Randers potential, Theorem 3 leads to an extension of the result obtained in [26], as follows:

**Corollary 6.** *Let* $M = \Sigma = Im\ \varphi \subset \mathbb{R}^3$ *be a surface of revolution of Randers type (3) described by*

$$\varphi(t,\theta) = (f(t)\cos\theta, f(t)\sin\theta, t),$$

*for* $(t,\theta) \in D = \mathbb{R}\times[0,2\pi)$. *Then* $M$ *is minimal iff the function* $f$ *satisfies the ODE*

$$\frac{-3b^2}{1+g^2-b^2}(-\omega_1\cos\theta - \omega_2\sin\theta + \omega_3 g)+ \\ +\frac{6b^2g[2(1+g^2)-3b^2]}{(1+g^2-b^2)^2}(\omega_1 g\cos\theta + \omega_2 g\sin\theta+ \\ +\omega_3) - \frac{1}{f(1+g^2)(1+g^2-b^2)^2}\{-fh\cdot \\ \cdot[(1-b^2+g^2)(1+2b^2+(1-3b^2)g^2)+ \\ +3b^4g^2] + [(1+g^2)(1-b^2+g^2)(1-b^2+ \\ +(1-3b^2)g^2)]\} = 0, \tag{12}$$

*where* $g = f'$ *and* $h = f''$.

Specifying this result to the particular subcase of constant potential, we infer:

**Corollary 7.** *Let* $M$ *be a minimal surfaces of revolution* $M$ *in the Randers space* $\mathbb{R}^3$ *endowed with the metric (3). Then:*

*a) for* $b = const$*, the minimality equation for surfaces of revolution (12) becomes ([26]):*

$$-ff''[3b^4f'^2 + (1-b^2+f'^2)(1+2b^2+ \\ +(1-3b^2)f'^2)] + (1+f'^2)(1-b^2+f'^2)\cdot \\ \cdot[1-b^2+(1-3b^2)f'^2] = 0. \tag{13}$$

*b) for* $b = 0$*, the ODE (13) leads to the classical minimality equation of surfaces of revolution isometrically immersed in the Euclidean space ([10]):* $1+f'^2 - ff'' = 0$.

It should be finally emphasized that, though the complexity degree in the PDEs (7)-(9)-(11) which characterize minimal graphs in Randers spaces ($b(x)$, $b = constant$ and Euclidean $b = 0$ cases, respectively), decreases, the problem of classification of minimal surfaces is still open, even in the simplest classical case (11). Among notable recent approaches in this respect, it is the DPW method (e.g., [8, 11, 4].

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