Multitime Models of Optimal Growth*

CONSTANTIN UDRISTE
University Politehnica of Bucharest
Department of Mathematics
Splaiul Indpendentei 313
060042, Bucharest
ROMANIA
udriste@mathem.pub.ro

MASSIMILIANO FERRARA
University Mediterranea of Reggio Calabria
Faculty of Law
Via dei Bianchi 2-Palazzo Zani
89100 Reggio Calabria
ITALY
massimiliano.ferrara@unirc.it

Abstract: Section 1 underline the limitations of standard multi-variable variational calculus and the sense of multitime. Section 1 formulates the controllability problem for a multiple integral functional or for a path independent curvilinear integral subject to a multitime evolution of flow type. Section 2 describes a two-time optimal economic growth modelled by Euler-Lagrange PDEs associated to a double integral functional or to a path independent curvilinear integral in two dimensions. Section 3 motivates the optimal economic growth by two-time maximum principles. Section 4 studies the two-time optimal economic growth with bang-bang policy based on a curvilinear integral action.

Key–Words: multitime maximum principle, multitime optimal economic growth, bang-bang policy.

1 Multitime optimal control theory

The interval $[0, T] = \Omega_{0,T}$, in R^m with product order, is called *planning horizon*. Geometrically, it is a hyperparallelepiped fixed by the diagonal opposite points 0 and T. Consider a dynamic system evolving over multi-time $t = (t^1, \dots, t^m) \in \Omega_{0,T}$ and an agent (planner) who has the task to control the evolution of m-sheets. We assume $T = (T^1, \dots, T^m)$ has finite norm, but sometimes we can relax this assumption. The dynamic behaviour of the system is described by the state variables $x = (x^1, \dots, x^n) : \Omega_{0,T} \to R^n$, $x(t) \in SV$ (*state variables*). The planner knows the *initial state* of the system $x(0) = x_0$ and the *final state* of the system $x(T) = x_T$ (*boundary conditions*).

We accept that the state variables are affected through a set of control variables $c = (c_1, \dots, c_q) : \Omega_{0,T} \to R^q$, $c(t) \in CV$ (*control variables*). The planner knows the relationship between the actions taken and evolution of the states, which are summarized by a "law of evolution" of the states, a (non-autonomous) PDEs system of the type

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(x(t), c(t), t) \qquad (PDE)$$

$$i = 1, \dots, n;\ \alpha = 1, \dots, m,$$

defined by the vector fields

$$X_\alpha : SV \times CV \times \Omega_{0,T} \to R^n$$

satisfying the complete integrability conditions

$$[X_\alpha, X_\beta] = \frac{\partial X_\alpha}{\partial c_a}\frac{\partial c_a}{\partial t^\beta} - \frac{\partial X_\beta}{\partial c_a}\frac{\partial c_a}{\partial t^\alpha} + \frac{\partial X_\alpha}{\partial t^\beta} - \frac{\partial X_\beta}{\partial t^\alpha},$$

$$a = 1, ..., q.$$

Fixing the control variables at a given multi-instant t, the evolution of the state variables at point t are obtained as solutions of the previous (PDE). Also given the value of the state at point t, the future values are determined.

Controllability problem: We are allowed to act on the m-sheets of the (PDE) system by means of a suitable control (included in the right hand side, in the boundary conditions, etc). Then, given a multitime $t \in \Omega_{0,T}$, and initial and final states, we have to find a control such that the solution matches both the initial state at multi-time $t = 0$ and the final one at multi-time $t = T$.

A way to choose properly the controls is to introduce:

1) either a *multiple integral functional*

$$I(c(\cdot)) = \int_{\Omega_{0,T}} L(x(t), c(t), t) dt^1 ... dt^m,$$

2) or a *path independent objective functional*

$$J(c(\cdot)) = \int_{\Gamma_{0,T}} L_\beta(x(t), c(t), t) dt^\beta,$$

where $\Gamma_{0,T}$ is a C^1 path joining the diagonal opposite points 0 and T.

*

Each functional summarizes the values of any given sheet of states and control on extremal points $0, T$. The function L (or the 1-form L_β) is called *instantaneous return or utility function (1-form)*.

The general control problem faced by the planner is

$$\max I(c(\cdot)) \text{ or } \max J(c(\cdot))$$

subject to

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(x(t), c(t), t) \qquad (PDE)$$

$$x(0) = x_0,\ x(T) = x_T, x(t) \in SV,\ c(t) \in CV.$$

This kind of research started with [11], as application of the theory from [6]-[9] to practical problems suggested by [3], [4]. On the other hand, the theory in [6]-[9], [11]-[12] follows the point of view in [1]. This theory can be extended for the PDEs in [4], [5], [10].

2 Two-time optimal economic growth

The theory of optimal economic growth starts with the following question: how much should be consumed and how much should be invested for future consumption? To formulate an answer, we accept that the evolution is 2-dimensional. That is why we introduce the following variables and functions:

$t = (t^1, t^2) = 2$ - moment of the economical effect;

$K(t) =$ capital;

$L(t) =$ labour force; with partial growing at a constant exogenous rate n_α, i.e., $\frac{\partial}{\partial t^\alpha} \ln L = n_\alpha$, $\alpha = 1, 2$ or equivalently $L = c_1 e^{n_\alpha t^\alpha}$;

$Y_\alpha = F_\alpha(K, L) =$ homogeneous commodities (production functions).

Each commodity $Y_\alpha(t) = F_\alpha(K(t), L(t))$ decomposes as sum of consumed part $c_\alpha(t)$, partial velocity of capital $\frac{\partial K}{\partial t^\alpha}(t)$ (further capital) and depreciation capital $\mu_\alpha K(t)$, where μ_α is a constant rate:

$$Y_\alpha(t) = c_\alpha(t) + \frac{\partial K}{\partial t^\alpha}(t) + \mu_\alpha K(t), \quad \alpha = 1, 2.$$

The production functions $Y_\alpha = F_\alpha(K, L)$, assumed homogeneous of degree one, could be written $Y_\alpha = LF_\alpha\left(\frac{K}{L}, 1\right) = Lf_\alpha(k)$, $k = \frac{K}{L}$. Putting $y_\alpha = \frac{Y_\alpha}{L}$, it follows $y_\alpha = f_\alpha(k)$, where each function $f_\alpha(k)$ is a strictly concave monotonically increasing function of k, with slope $f'_\alpha(k)$ decreasing from $\lim_{k\to 0} f'_\alpha(k) = \infty$ to $\lim_{k\to\infty} f'_\alpha(k) = 0$. In this way we obtain a two-time evolution

$$\frac{\partial k}{\partial t^\alpha}(t) = f_\alpha(k(t)) - (\mu_\alpha + n_\alpha)k(t) - c_\alpha(t), \quad \alpha = 1, 2.$$

Also we accept that this PDEs system satisfies the complete integrability conditions.

Let us apply the multi-time Euler-Lagrange theory: let $D = (D_1, D_2)$ be a constant positive rate vector of future discount; let $\lambda_\alpha = \mu_\alpha + n_\alpha$ and $g_\alpha(k) = f_\alpha(k) - \lambda_\alpha k$.

2.1 Case of double integral functional

Let $u(c)$ be the utility function which obeys the law of diminishing marginal utility $d^2u(c) < 0$ (concave function), $\frac{\partial u}{\partial c_\gamma} > 0$. Maximize the functional

$$I(c(\cdot)) = \int_{\Omega_{0,T}} e^{-D_\lambda t^\lambda} u(c(t)) dt^1 dt^2, \quad c = (c_1, c_2),$$

subject to

$$c_\alpha(t) = g_\alpha(k(t)) - \frac{\partial k}{\partial t^\alpha}(t),$$

$$k(0) = k_0,\ k(T) = k_T,\ 0 = (0, 0),\ T = (T^1, T^2).$$

Eliminating $c_\alpha(t)$, we find the Lagrangian

$$L(k(t), k_\gamma(t), t) = e^{-D_\lambda t^\lambda} u(c(t)) =$$

$$= e^{-D_\lambda t^\lambda} u\left(g_1(k(t)) - \frac{\partial k}{\partial t^1}(t), g_2(k(t)) - \frac{\partial k}{\partial t^2}(t)\right).$$

The extremals are solutions of the *multi-time Euler-Lagrange equation*

$$\frac{\partial L}{\partial k} - \frac{\partial}{\partial t^\gamma}\frac{\partial L}{\partial k_\gamma} = 0.$$

It follows the PDEs system

$$\frac{\partial^2 u}{\partial c_\alpha \partial c_\gamma}\frac{\partial c_\alpha}{\partial t^\gamma} + \frac{\partial u}{\partial c_\gamma}\left(\frac{dg_\gamma}{dk} - D_\gamma\right) = 0$$

$$\frac{\partial k}{\partial t^\alpha}(t) = g_\alpha(k(t)) - c_\alpha(t).$$

First we obtain an *equilibrium point* (k^*, c^*) at which $\frac{\partial k}{\partial t^\alpha} = 0$, $\frac{\partial c_\lambda}{\partial t^\sigma} = 0$. It follows

$$\frac{\partial u}{\partial c_\gamma}\left(\frac{dg_\gamma}{dk} - D_\gamma\right) = 0, \quad g_\alpha(k(t)) - c_\alpha(t) = 0,$$

which must produce k^* and $c^*_\alpha = g_\alpha(k^*)$.

Second, an *analytical solution* is possible when $f_\alpha(k)$ and $u(c)$ are explicitly given. For example, $f_\alpha(k) = a_\alpha k$, i.e., $g_\alpha(k) = (a_\alpha - \lambda_\alpha)k$, and $u(c) = c_1^2 + c_2^2$. Then the previous PDEs system is reduced to

$$\frac{\partial c_1}{\partial t^1} + \frac{\partial c_2}{\partial t^2} + c_1(a_1 - \lambda_1 - D_1) + c_2(a_2 - \lambda_2 - D_2) = 0$$

$$\frac{\partial k}{\partial t^\alpha}(t) = (a_\alpha - \lambda_\alpha)k - c_\alpha(t).$$

A particular solution of the first PDE is

$$c_1(t) = c_2(t) = e^{-(a_\alpha - \lambda_\alpha - D_\alpha)t^\alpha}.$$

In the complete integrability conditions of the second PDEs,

$$2a_1 - 2\lambda_1 - 2a_2 + 2\lambda_2 + D_2 - D_1 = 0,$$

we obtain the corresponding solution $k(t)$.

2.2 Case of path independent integral functional

Let $u_\beta(c)$ be the utility 1-form whose elements obey the law of diminishing marginal utility $d^2u_\beta(c) < 0$ (concave functions), $\frac{\partial u_\beta}{\partial c_\gamma} > 0$. Maximize the functional

$$J(c(\cdot)) = \int_{\Gamma_{0,T}} e^{-D_\lambda t^\lambda} u_\beta(c(t))dt^\beta, \quad c = (c_1, c_2),$$

subject to

$$c_\alpha(t) = g_\alpha(k(t)) - \frac{\partial k}{\partial t^\alpha}(t),$$
$$k(0) = k_0,\ k(T) = k_T,\ 0 = (0,0),\ T = (T^1, T^2).$$

Eliminating $c_\alpha(t)$, we find the Lagrangian 1-form

$$L_\beta(k(t), k_\gamma(t), t) = e^{-D_\lambda t^\lambda} u_\beta(c(t)) =$$
$$= e^{-D_\lambda t^\lambda} u_\beta\left(g_1(k(t)) - \frac{\partial k}{\partial t^1}(t), g_2(k(t)) - \frac{\partial k}{\partial t^2}(t)\right)$$

that must satisfy the complete integrability conditions. The extremals are solutions of the *multi-time Euler-Lagrange equations*

$$\frac{\partial L_\beta}{\partial k} - \frac{\partial}{\partial t^\gamma}\frac{\partial L_\beta}{\partial k_\gamma} = a_\beta.$$

It follows the PDEs system

$$\frac{\partial^2 u_\beta}{\partial c_\gamma \partial c_\alpha}\frac{\partial c_\alpha}{\partial t^\gamma} + \frac{\partial u_\beta}{\partial c_\gamma}\left(\frac{\partial g_\gamma}{\partial k} - D_\gamma\right) = a_\beta$$

$$\frac{\partial k}{\partial t^\alpha}(t) = g_\alpha(k(t)) - c_\alpha(t).$$

First we obtain an *equilibrium point* (k^*, c^*) at which $\frac{\partial k}{\partial t^\alpha} = 0$, $\frac{\partial c_\lambda}{\partial t^\sigma} = 0$. It follows

$$\frac{\partial u_\beta}{\partial c_\gamma}\left(\frac{\partial g_\gamma}{\partial k} - D_\gamma\right) = a_\beta,\ g_\alpha(k(t)) - c_\alpha(t) = 0,$$

which must produce k^* and $c_\alpha^* = g_\alpha(k^*)$.

Second, an *analytical solution* is possible when $f_\alpha(k)$ and $u_\beta(c)$ are explicitly given. For example, $f_\alpha(k) = a_\alpha k$, i.e., $g_\alpha(k) = (a_\alpha - \lambda_\alpha)k$, and

$$u_\beta(c) = \begin{cases} \dfrac{c_\beta^{1-\nu}}{1-\nu} & \text{if} \quad \nu > 0,\ \nu \neq 1 \\ \ln c_\beta & \text{if} \quad \nu = 1. \end{cases}$$

3 Reformulation as an optimal control

Let us formulate the optimal growth as a multi-time optimal control model (see [2], [6]-[9]) starting with $\lambda_\alpha = n_\alpha + \mu_\alpha$ (constant population growth rates + constant depreciation rates).

3.1 Case of double integral functional

For that we choose a rate of per capita consumption $c(t) = (c_1(t), c_2(t))$ which satisfies the multi-time growth law

$$\frac{\partial k}{\partial t^\alpha}(t) = f_\alpha(k(t)) - \lambda_\alpha k(t) - c_\alpha(t), \quad \alpha = 1, 2$$

and which minimizes the functional

$$I(c(\cdot)) = \int_{\Omega_{0,T}} e^{-D_\lambda t^\lambda} u(c(t))dt^1dt^2.$$

The nonautonomous control Hamiltonian is

$$H = e^{-D_\lambda t^\lambda}\left(u(c) + q^\alpha(f_\alpha(k) - \lambda_\alpha k - c_\alpha)\right),$$

where the *co-states variables* $p^\alpha(t) = q^\alpha(t)e^{-D_\lambda t^\lambda}$ mean the discounted values of additional investment. For an interior maximum with respect to the control c we must have $\frac{\partial H}{\partial c_\gamma} = 0$, i.e., $\frac{\partial u}{\partial c_\gamma} = p^\gamma$. The adjoint equation

$$\frac{\partial p^\alpha}{\partial t^\alpha} = -\frac{\partial H}{\partial k} = -(f'_\alpha - \lambda_\alpha)p^\alpha$$

and transversality condition

$$p^1(t)n^1(t) + p^2(t)n^2(t)|_{\partial\Omega_{0,T}} = 0$$

are equivalent to

$$\frac{\partial q^\alpha}{\partial t^\alpha} = -(f'_\alpha - \lambda_\alpha - D_\alpha)q^\alpha,$$

$$q^1(t)n^1(t) + q^2(t)n^2(t)|_{\partial\Omega_{0,T}} = 0.$$

These PDEs produce the same information as those in the previous paragraph.

3.2 Case of path independent integral functional

For that we choose a rate of per capita consumption $c(t) = (c_1(t), c_2(t))$ which satisfies the multi-time growth law

$$\frac{\partial k}{\partial t^\alpha}(t) = f_\alpha(k(t)) - \lambda_\alpha k(t) - c_\alpha(t), \quad \alpha = 1, 2$$

and which minimizes the functional

$$J(c(\cdot)) = \int_{\Gamma_{0,T}} e^{-D_\lambda t^\lambda} u_\beta(c(t))dt^\beta.$$

The nonautonomous control 1-form is

$$S_\alpha = e^{-D_\lambda t^\lambda}\left(u_\alpha(c) + q(f_\alpha(k) - \lambda_\alpha k - c_\alpha)\right),$$

where the *co-states variable* $p(t) = q(t)e^{-D_\lambda t^\lambda}$ means the discounted value of additional investment. For an interior maximum with respect to the control c we must have $\frac{\partial S_\alpha}{\partial c_\gamma} = 0$, i.e., $\frac{\partial u_\alpha}{\partial c_\gamma} = p\delta^\gamma_\alpha$. The adjoint equation

$$\frac{\partial p}{\partial t^\alpha} = -\frac{\partial S_\alpha}{\partial k} = -(f'_\alpha - \lambda_\alpha)p, \; p(T) = 0$$

is equivalent to

$$\frac{\partial q}{\partial t^\alpha} = -(f'_\alpha - \lambda_\alpha - D_\alpha)q, \; q(T) = 0.$$

Of course, here we need the complete integrability conditions. These PDEs produce the same information as those in the previous paragraph.

4 Optimal economic growth with bang-bang policy

In this section we adapt the multi-time controllability, observability and bang-bang principle [8] to the context of this paper. For that, let us accept that $u_\beta(c) = c_\beta$, $a^\alpha = \text{const}$, $c_\alpha(t) =$ per capita consumptions, $||T|| = \infty$ and that we use the path independent curvilinear integral. Then

$$\text{maximize } J(c(\cdot)) = \int_{\Gamma_{0,\infty}} e^{-D_\lambda t^\lambda} c_\beta(t)dt^\beta$$

subject to

$$\frac{\partial k}{\partial t^\alpha}(t) = f_\alpha(k(t)) - \lambda_\alpha k(t) - c_\alpha(t),$$

where $k =$ capital, and $k(0) = k_0$, D_α, λ_α are positive constants.

The nonautonomous control 1-form is

$$S_\alpha = e^{-D_\lambda t^\lambda} c_\alpha + p(f_\alpha(k) - \lambda_\alpha k - c_\alpha)$$

or, with the definition $p(t) = q(t)e^{-D_\lambda t^\lambda}$, we can write

$$S_\alpha = e^{-D_\lambda t^\lambda}(1-q)c_\alpha + e^{-D_\lambda t^\lambda} q(f_\alpha(k) - \lambda_\alpha k).$$

We remark that the control tensor is linear in the control variables $c_\alpha(t)$. Also we accept $\bar{c}_\alpha \leq c^*_\alpha \leq f_\alpha(k)$, i.e., $\bar{c}_\alpha$ is the minimum level and $f_\alpha(k)$ is the maximum level. The switching functions $\sigma_\alpha = e^{-D_\lambda t^\lambda}(1-q)c_\alpha$ shows that the optimal policy is to choose

$$c^*_\alpha = \left\{ \begin{array}{c} \bar{c}_\alpha(=0) \\ 0 < c_\alpha < f_\alpha(k) \\ f_\alpha(k) \end{array} \right\} \quad \text{if} \quad q = \left\{ \begin{array}{c} > 1 \\ = 1 \\ < 1 \end{array} \right\}.$$

The dynamic state and adjoint systems are

$$\frac{\partial k}{\partial t^\alpha} = f_\alpha(k) - \lambda_\alpha k - c^*_\alpha(q),$$

$$\frac{\partial q}{\partial t^\alpha} = -q^\beta(f'_\alpha - \lambda_\alpha - D_\alpha),$$

which are solved after substituting the optimal control $c^* = (c^*_1, c^*_2)$.

Cases:

1) The first bang-bang policy should be used when $q < 1$, $c^*_\alpha = f_\alpha(k) = c_{\alpha\,\max}$. The above dynamic system becomes

$$\frac{\partial k}{\partial t^\alpha} = -\lambda_\alpha k, \quad \frac{\partial q}{\partial t^\alpha} = (\lambda_\alpha + D_\alpha - f'_\alpha(k))q.$$

In this way the capital stock decreases at 2-rate (λ_1, λ_2), i.e., $k(t) = ce^{-\lambda_\alpha t^\alpha}$.

2) The second bang-bang policy should be used when $q > 1$, $c^*_\alpha = \bar{c}_\alpha = c_{\alpha\,\min} = 0$. The multi-time dynamic system is

$$\frac{\partial k}{\partial t^\alpha} = f_\alpha(k) - \lambda_\alpha k, \quad \frac{\partial q}{\partial t^\alpha} = (\lambda_\alpha + D_\alpha - f'_\alpha(k))q.$$

3) A singular control would be the appropriate policy if $q \equiv 1$, $\sigma_\alpha = 0$.

Acknowledgements. Partially supported by Grant CNCSIS 86/ 2007 and by 15-th Italian-Romanian Executive Programme of S&T Co-operation for 2006-2008.

References:

[1] L. C. Evans, An Introduction to Mathematical Optimal Control Theory, *Lecture Notes, University of California, Department of Mathematics*, Berkeley, 2005.

[2] S. Pickenhain, M. Wagner, Piecewise continuous controls in Dieudonne-Rashevsky type problems, *JOTA*, 127 (2005), pp.145-163.

[3] P. N. V. Tu, Introductory Optimization Dynamics, *Springer-Verlag*, 1991.

[4] C. Udrişte, M. Ferrara, D. Opris, Economic Geometric Dynamics, *Monographs and Textbooks 6, Geometry Balkan Press*, 2004.

[5] C. Udrişte, A. M. Teleman, Hamiltonian approaches of field theory, *IJMMS*, 57 (2004), 3045-3056.

[6] C. Udrişte, Multi-time maximum principle, short communication at *International Congress of Mathematicians*, Madrid, August 22-30, 2006; Plenary Lecture at *6-th WSEAS International Conference on Circuits, Systems, Electronics, Control&Signal Processing (CSECS'07) and 12-th WSEAS International Conference on Applied Mathematics*, Cairo, Egypt, December 29-31, 2007.

[7] C. Udrişte, I. Ţevy, Multi-Time Euler-Lagrange-Hamilton Theory, *WSEAS Transactions on Mathematics*, 6, 6 (2007), 701-709.

[8] C. Udrişte, Multi-Time Controllability, Observability and Bang-Bang Principle, *6th Congress of Romanian Mathematicians*, June 28 - July 4, 2007, Bucharest, Romania.

[9] C. Udrişte, I. Ţevy, Multi-Time Euler-Lagrange Dynamics, *Proceedings of the 7th WSEAS International Conference on Systems Theory and Scientific Computation (ISTASC'07)*, Vouliagmeni Beach, Athens, Greece, August 24-26 (2007), 66-71.

[10] C. Udrişte, Maxwell geometric dynamics, *European Computing Conference*, Vouliagmeni Beach, Athens, Greece, September 24-26, 2007.

[11] C. Udrişte, M. Ferrara, Multi-time optimal economic growth, *Journal of the Calcutta Mathematical Society*, 3, 1 (2007), 1-6.

[12] C. Udrişte, Multi-time Stochastic Control Theory, *Selected Topics on Circuits, Systems, Electronics, Control&Signal Processing, Proceedings of the 6-th WSEAS International Conference on Circuits, Systems, Electronics, Control&Signal Processing (CSECS'07)*, pp. 171-176; Cairo, Egypt, December 29-31, 2007.