Adaptive Sliding-Mode Load-Torque Observer: Its Stability Aspects

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Abstract: - This article reveals the detailed analyses of the uniform stability and the transient stability of an adaptive sliding-mode load-torque observer. Using the Lyapunov's direct method, the stability can be concluded only in some time intervals during transient state. Using the LaSalle's invariance principle, it can be concluded that by the end of the transient state the observer definitely enters the stable steady-state.

Key-Words: - Asymptotically autonomous system; Invariant set; LaSalle's principle; Load-torque observer; Lyapunov's direct method; Quadratic inequality; Transient stability; Uniform stability.

Notation Lists:

a , b , c	mechanical parameters of the motor-load system,
$\hat{a}$, $\hat{b}$	online estimated mechanical parameters,
$\hat{a}_0$, $\hat{b}_0$, $\hat{T}_{L0}$	initial values of the estimated mechanical parameters,
B_t	viscous friction coefficient (N·m·s/rad),
$e_{\psi\alpha}$, $e_{\psi\beta}$	errors of the estimated rotor flux,
e_ω	error of the estimated rotor speed obtained from the double estimation,
h_ω	integral of the e_ω,
i_{eq} , $\hat{i}_{eq}$	equivalent current proportional to the electromagnetic torque, and the corresponding estimated current, respectively,
$i_{s\alpha}$, $i_{s\beta}$	two-phase stator currents (A) in horizontal and vertical axes, respectively,
J_t	moment of inertia (kg·m^2),
k_{ia} , k_{ib} , k_{il}	integral gains,
k_{pa} , k_{pb} , k_{pl}	proportional gains,
K_T	electromagnetic torque constant (N·m /A),
k_3	surface gain,
p	number of motor poles,
s_ω	surface signal,
t	time (s),
T_e	electromagnetic torque of the motor (N·m),
T_L , $\hat{T}_L$	actual load-torque (N·m) and online estimated load-torque, respectively,
V	Lyapunov function,
α , β	subscripts indicating the horizontal and the vertical axes, respectively, of the stator reference frame,
Δa , Δb	errors of the mechanical parameters obtained from online estimation,
Δi_{eq}	error of the equivalent current proportional to the electromagnetic torque,
ΔT_L	error of the estimated load-torque,
$\Delta\omega_r$	error of the estimated rotor speed,
δ_o	correction signal,
τ	integration time-constant,
ϕ_3, ϕ_4, Λ_3	correction gains,
$\hat{\psi}_{r\alpha}$, $\hat{\psi}_{r\beta}$	estimated rotor flux (Wb),
ω_r , $\hat{\omega}_r$	rotor speed (rad/s), and estimated rotor speed, respectively,
$\hat{\omega}_r''$	double estimated rotor speed (rad/s), and
$2\rho_l \Delta t$	equivalent quantity of the rate of change of load-torque with respect to time (N·m).

1 Introduction

Control community has been familiar with the sliding-mode approach for many years. New developments in control can usually be found, e.g. control of buck converters [1], position servo control [2, 3], minimum energy control of PMSM drives [4], etc. Not many applications in observer development can be found. Recently, researchers [5] have published their work on the development of a sliding-mode observer, and [6] has presented an adaptive flux observer considered linear. For motor operation in general, load-torque dynamic is not usually known. Acquiring the load-torque needs an expensive transducer as a common practice. An alternative is to use a good load-torque observer for an economical reason. Even though the recent work [5] confirms steady-state stability, it is questionable whether the observer still performs stably well during the transient state. So far, there has not been a previous work answering this question. This article provides an extension of our previous work to explain the uniform stability and transient stability analyses. The analysis has applied the Lyapunov's direct method, the Lyapunov's theorem under relaxation, and the LaSalle's invariance principle. Simulation results are also presented.

2 Adaptive Sliding-Mode Load-Torque Observer – A Brief Review

The load-torque observer of our previous work delineated in Fig. 1 operates in a cascade connection with a speed observer that provides the estimation of the rotor flux and speed of an induction motor. The load-torque observer has the estimated rotor speed and the equivalent current ($\hat{i}_{eq} = i_{s\beta}\hat{\psi}_{r\alpha} - i_{s\alpha}\hat{\psi}_{r\beta}$) proportional to the electromagnetic torque of the motor ($T_e = K_T i_{eq}$) as its inputs. The observer performs an online estimation according to (1)

$$\dot{\hat{\omega}}''_r = \hat{a}\hat{\omega}''_r + \hat{b}\hat{i}_{eq} + c\hat{T}_L + \delta_o \tag{1}$$

in which $\hat{\omega}''_r$ is the double estimated rotor speed, $\hat{a}$, $\hat{b}$ are the estimated mechanical parameters, $\hat{T}_L$ is the estimated load-torque, and δ_o is the correction signal. This δ_o is to compensate for the estimation errors. The equation (2) expresses the integral of the error resulted from the double estimated rotor speed. The equation (3) expresses the surface signal in which $k_3 > 0$ is the surface gain

$$\dot{h}_\omega = -e_\omega = \hat{\omega}''_r - \hat{\omega}_r \qquad (2),$$

$$s_\omega = e_\omega - k_3 h_\omega \qquad (3).$$

The composition of the correction signal is as shown in (4) in which ϕ_3, ϕ_4 and Λ_3 are the correction gains

$$\delta_o = \phi_3 e_\omega + \phi_4 k_3 h_\omega + \Lambda_3 \qquad (4).$$

The surface signal, s_ω, must converge to zero, and hence six logical rules designed are as follows

$$\begin{array}{llll}
\text{if} & s_\omega e_\omega > 0 & \text{then} & \phi_3 > |a + k_3|, \\
\text{if} & s_\omega e_\omega < 0 & \text{then} & \phi_3 < -|a + k_3|, \\
\text{if} & s_\omega h_\omega > 0 & \text{then} & \phi_4 > 0, \\
\text{if} & s_\omega h_\omega < 0 & \text{then} & \phi_4 < 0, \\
\text{if} & s_\omega > 0 & \text{then} & \Lambda_3 > |f_3|, \text{ and} \\
\text{if} & s_\omega < 0 & \text{then} & \Lambda_3 < -|f_3|
\end{array} \qquad (5).$$

According to these rules, the terms
$a = -(B_t/J_t) < 0$, $b = (p/2J_t)K_T > 0$,
$f_3 = a\Delta\omega_r + b\Delta i_{eq} - [d(\Delta\omega_r)/dt]$,
and $\Delta i_{eq} = i_{s\beta} e_{\psi\alpha} - i_{s\alpha} e_{\psi\beta}$. The last two terms are unknown since $e_{\psi\alpha}$, $e_{\psi\beta}$, and $\Delta\omega_r = \omega_r - \hat{\omega}_r$ are unreachable. Three PI adaptive laws for $\hat{a}$, $\hat{b}$, and $\hat{T}_L$ are shown in (6)-(8), respectively

$$\hat{a} = \hat{a}_0 + k_{pa}\Theta_a + k_{ia}\int_0^t \Theta_a(\tau)\,d\tau \qquad (6),$$

$$\hat{b} = \hat{b}_0 + k_{pb}\Theta_b + k_{ib}\int_0^t \Theta_b(\tau)\,d\tau \qquad (7),$$

$$\hat{T}_L = \hat{T}_{L0} - k_{pl}s_\omega - k_{il}\int_0^t s_\omega(\tau)\,d\tau \qquad (8).$$

According to the equations (6)-(8), the terms
$\Theta_a = \Theta_a(s_\omega, \hat{\omega}''_r) = \hat{\omega}''_r s_\omega$,
$\Theta_b = \Theta_b(s_\omega, \hat{i}_{eq}) = \hat{i}_{eq} s_\omega$,
k_{pa}, k_{pb}, k_{pl} are the proportional gains (positive), k_{ia}, k_{ib}, k_{il} are the integral gains (positive), $\hat{a}_0$, $\hat{b}_0$ and $\hat{T}_{L0}$ are the initial estimated values of the terms $\hat{a}$, $\hat{b}$ and $\hat{T}_L$, respectively. To conduct the stability analysis, it is assumed that the mechanical parameters are almost constant ($\dot{a} \approx 0$, $\dot{b} \approx 0$ and $\dot{T}_L \approx 0$), and the Lyapunov function can be written as in (9)

$$V = \frac{1}{2k_{ia}}\left(\Delta a + k_{pa}\Theta_a\right)^2 + \frac{1}{2k_{ib}}\left(\Delta b + k_{pb}\Theta_b\right)^2 - \frac{c}{2k_{il}}\left(\Delta T_L - k_{pl}s_\omega\right)^2 + \frac{1}{2}s_\omega^2 \geq 0 \qquad (9),$$

where $\Delta a = a - \hat{a}$, $\Delta b = b - \hat{b}$, and $\Delta T_L = T_L - \hat{T}_L$ are the errors of the estimated mechanical parameters, and the online estimated

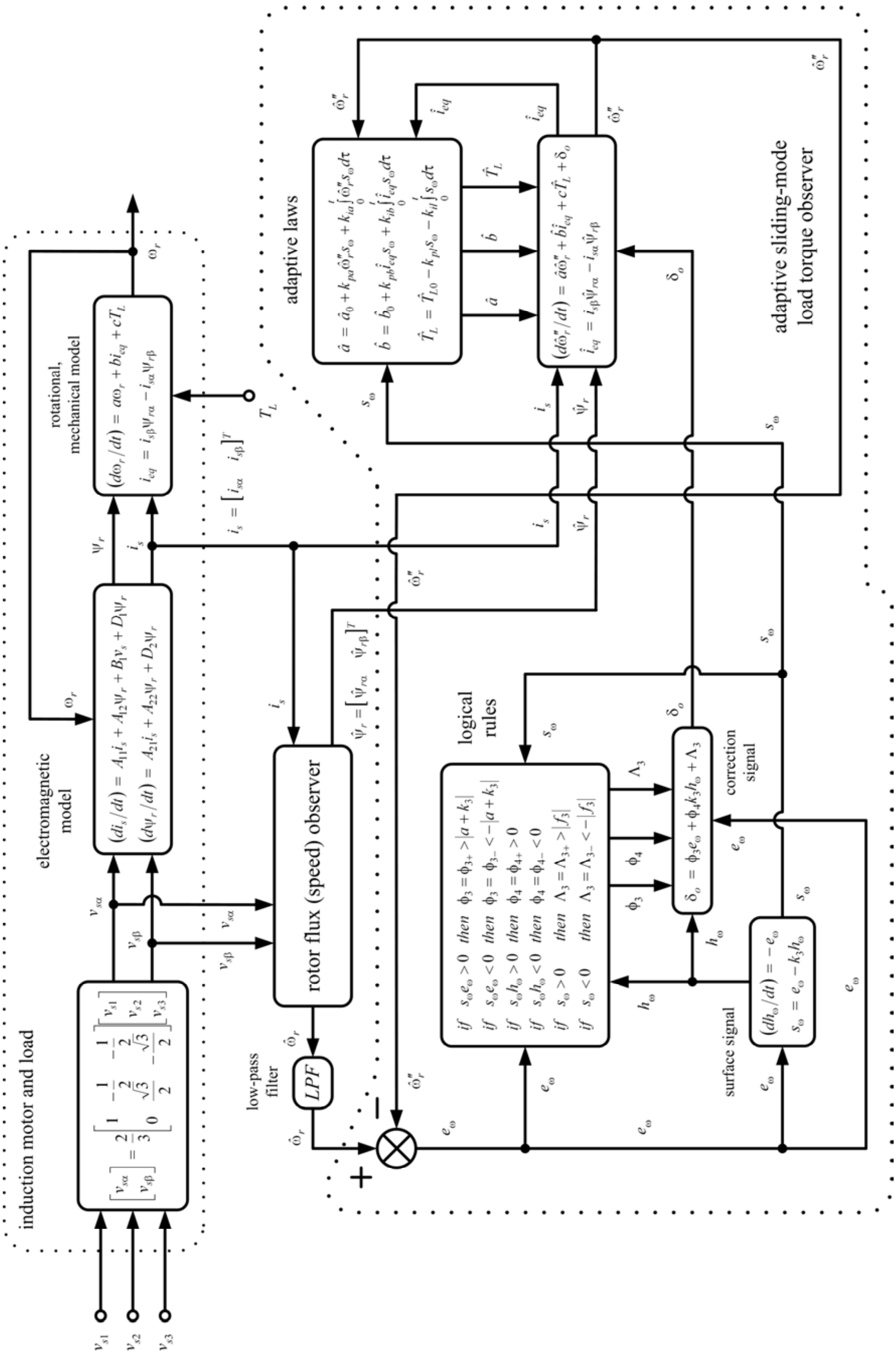

Fig. 1 Block diagram of the adaptive sliding-mode speed-torque observer.

load-torque, respectively. Furthermore, $c = -\left(p/2J_t\right) < 0$, and henceforth the time derivative of the Lyapunov function is expressed by (10)

$$\dot{V} = \left(a + k_3 - \phi_3\right) s_\omega e_\omega - \phi_4 k_3 s_\omega h_\omega + \left(f_3 - \Lambda_3\right) s_\omega - k_{pa}\Theta_a^2 - k_{pb}\Theta_b^2 + ck_{pl} s_\omega^2 \leq 0 \quad (10),$$

where $(a + k_3 - \phi_3)s_\omega e_\omega \leq 0$, $-\phi_4 k_3 s_\omega h_\omega \leq 0$ and $(f_3 - \Lambda_3)s_\omega \leq 0$. The equations (9) and (10) suggest that if the actual load-toque is continuously constant, the load-torque observer always remains stable because $\dot{V}$ is negative semi-definite. The analysis follows in the next section.

3 Uniform Stability of the Proposed Observer

The whole system comprising a plant (an induction motor, its mechanical load, and the speed observer) and the load-torque observer shown in Fig. 1 is classified as non-autonomous. Stability analysis of the system can be conducted according to the Lyapunov's theorem under relaxation [7]. Regarding this, three error expressions of online parameter estimations can be written as

$$\Delta a = \Delta a + k_{pa}\Theta_a - k_{pa}\Theta_a \quad (11),$$

$$\Delta b = \Delta b + k_{pb}\Theta_b - k_{pb}\Theta_b \quad (12),$$

$$\Delta T_L = \Delta T_L - k_{pl}\, s_\omega + k_{pl}\, s_\omega \quad (13).$$

Every composition in the right-hand side of the correction signal in the equation (4) can be rewritten as

$$\delta_o\left(s_\omega\right) = \varphi_\delta s_\omega = \left|\phi_3\right| \frac{s_\omega e_\omega}{\left|s_\omega e_\omega\right|} e_\omega + \left|\phi_4\right| \frac{s_\omega h_\omega}{\left|s_\omega h_\omega\right|} k_3 h_\omega + \left|\Lambda_3\right| \frac{s_\omega}{\left|s_\omega\right|} \quad (14),$$

where $\phi_3 = \left|\phi_3\right| \operatorname{sgn}\left(s_\omega e_\omega\right) = \left|\phi_3\right| \frac{s_\omega e_\omega}{\left|s_\omega e_\omega\right|}$,

$\phi_4 = \left|\phi_4\right| \operatorname{sgn}\left(s_\omega h_\omega\right) = \left|\phi_4\right| \frac{s_\omega h_\omega}{\left|s_\omega h_\omega\right|}$,

$\Lambda_3 = \left|\Lambda_3\right| \operatorname{sgn}\left(s_\omega\right) = \left|\Lambda_3\right| \frac{s_\omega}{\left|s_\omega\right|}$, and

$$\varphi_\delta = \frac{\left|\phi_3\right|}{\left|s_\omega e_\omega\right|} e_\omega^2 + \frac{k_3\left|\phi_4\right|}{\left|s_\omega h_\omega\right|} h_\omega^2 + \frac{\left|\Lambda_3\right|}{\left|s_\omega\right|} \geq 0 .$$

The equation (15) denotes the time derivative of the surface signal,

$$\dot{s}_\omega = \left(a + k_3\right) e_\omega + \Delta a \hat{\omega}_r'' + \Delta b \hat{i}_{eq} + c\Delta T_L + f_3 - \delta_o \quad (15).$$

The surface signal in the equation (3) is rewritten as

$$e_\omega = s_\omega + k_3 h_\omega \quad (16).$$

By substituting the equation (16) into the equation (15), the time derivative of the surface signal becomes

$$\dot{s}_\omega = \left(a + k_3\right) s_\omega + \left(a + k_3\right) k_3 h_\omega + \Delta a \hat{\omega}_r'' + \Delta b \hat{i}_{eq} + c\Delta T_L + f_3 - \delta_o \quad (17).$$

Afterwards, the terms Δa, Δb, ΔT_L, and δ_o in the equation (17) are substituted by the equations (11) to (14) and then the resultant equation is rearranged into the form of

$$\dot{s}_\omega = \left\{a + k_3 - \varphi_\delta - k_{pa}\left(\hat{\omega}_r''\right)^2 - k_{pb}\hat{i}_{eq}^2 + ck_{pl}\right\} s_\omega + \left(a + k_3\right) k_3 h_\omega + \left(\Delta a + k_{pa}\Theta_a\right)\hat{\omega}_r'' + \left(\Delta b + k_{pb}\Theta_b\right)\hat{i}_{eq} + c\left(\Delta T_L - k_{pl} s_\omega\right) + f_3 \quad (18).$$

Differentiating three PI adaptive laws in the equations (6) to (8) and rewriting them gives

$$\Delta\dot{a} + k_{pa}\dot{\Theta}_a = -\dot{\hat{a}} + k_{pa}\dot{\Theta}_a = -k_{ia}\Theta_a \quad (19),$$

$$\Delta\dot{b} + k_{pb}\dot{\Theta}_b = -\dot{\hat{b}} + k_{pb}\dot{\Theta}_b = -k_{ib}\Theta_b \quad (20),$$

$$\Delta\dot{T}_L - k_{pl}\dot{s}_\omega = -\dot{\hat{T}}_L - k_{pl}\dot{s}_\omega = k_{il} s_\omega \quad (21).$$

Thereafter, the last four equations (18)-(21) are collected into a single form of

$$\dot{\chi}_l = F_l\left(t, \chi_l\right) \quad (22),$$

where

$$\chi_l = \left[\, s_\omega \quad \Delta a + k_{pa}\Theta_a \quad \Delta b + k_{pb}\Theta_b \quad \Delta T_L - k_{pl} s_\omega \,\right]^T$$

is regarded as a state vector.

Anyone can comfortably choose $\gamma_2^{\min} = \min\left(\frac{1}{2}, \frac{1}{2k_{ia}}, \frac{1}{2k_{ib}}, -\frac{c}{2k_{il}}\right)$ and $\gamma_2^{\max} = \max\left(\frac{1}{2}, \frac{1}{2k_{ia}}, \frac{1}{2k_{ib}}, -\frac{c}{2k_{il}}\right)$.

Such choice generally exhibits a certain boundary of the Lyapunov function along

$$\gamma_2^{\min}\|\chi_l\|^2 \le V \le \gamma_2^{\max}\|\chi_l\|^2 \qquad (23),$$

where $\|\chi_l\| = \sqrt{\chi_l^T\chi_l} \ge 0$, and $\|\chi_l\|^2$ is a strictly non-decreasing quadratic scalar function. Thus, at a point in time, real value of the Lyapunov function is always bounded within this sector. Moreover, the time derivative of the Lyapunov function can be expressed as the next equation (24) and the inequality (25), respectively,

$$\begin{aligned}\dot{V} &= -\left\{|\phi_3||s_\omega e_\omega| - (a+k_3)s_\omega e_\omega\right\} - k_3|\phi_4||s_\omega h_\omega| \\ &\quad - \left(|\Lambda_3||s_\omega| - f_3 s_\omega\right) - k_{pa}\Theta_a^2 - k_{pb}\Theta_b^2 \\ &\quad + ck_{pl}s_\omega^2 \le 0\end{aligned} \qquad (24),$$

$$\dot{V} \le -\left(|\Lambda_3||s_\omega| - f_3 s_\omega\right) \le 0 \qquad (25).$$

From the inequality (25), one can conclude that the proposed load-torque observer is uniformly stable.

4 Analysis

Considering the expression (10), if the load-torque is not steady ($\dot{T}_L \ne 0$), the term $\dot{T}_L$ will appear in the relation. The expression (10) becomes (26) indicating that the time derivative of the Lyapunov function can be either positive or negative

$$\begin{aligned}\dot{V} &= \left(a+k_3-\phi_3\right)s_\omega e_\omega - \phi_4 k_3 s_\omega h_\omega + \left(f_3 - \Lambda_3\right)s_\omega \\ &\quad - k_{pa}\Theta_a^2 - k_{pb}\Theta_b^2 + ck_{pl}s_\omega^2 \\ &\quad - \frac{c}{k_{il}}\left(\Delta T_L - k_{pl}s_\omega\right)\dot{T}_L\end{aligned} \qquad (26).$$

Considering an infinitesimal time interval t to $t + \Delta t$, the time rate of change of load-torque (ρ_l) during the transient state is assumed constant, i.e. $\dot{T}_L \approx \rho_l$ or $T_L(t+\Delta t) = T_L(t) + \rho_l\Delta t$ for a very small positive value of $\Delta t \to 0$. The derivative of the Lyapunov function can be rewritten as

$$\begin{aligned}\dot{V}(t+\Delta t) &= \dot{\bar{V}}(t+\Delta t) \\ &\quad - \frac{c}{k_{il}}\left[T_L(t+\Delta t) - \hat{T}_L(t+\Delta t) - k_{pl}s_\omega(t+\Delta t)\right] \\ &\quad \times \dot{T}_L(t+\Delta t)\end{aligned} \qquad (27),$$

where

$$\begin{aligned}\dot{\bar{V}}(t+\Delta t) = \dot{\bar{V}} &= \left(a+k_3-\phi_3\right)s_\omega e_\omega - \phi_4 k_3 s_\omega h_\omega \\ &\quad + \left(f_3-\Lambda_3\right)s_\omega - k_{pa}\Theta_a^2 - k_{pb}\Theta_b^2 + ck_{pl}s_\omega^2 \le 0\end{aligned}$$

is the time derivative of the Lyapunov function when the load-torque is constant. By substituting the terms $\dot{T}_L(t+\Delta t) \approx \rho_l$ and $T_L(t+\Delta t)$ into (27), and re-arranging the terms, one could obtain (28)

$$\dot{V}(t+\Delta t) = -\frac{c\Delta t}{k_{il}}\rho_l^2 - \frac{c\Omega_L}{k_{il}}\rho_l + \dot{\bar{V}}(t+\Delta t) \qquad (28),$$

where

$$\begin{aligned}\Omega_L &= \Omega_L\left(T_L(t), \hat{T}_L, s_\omega\right) \\ &= T_L(t) - \hat{T}_L(t+\Delta t) - k_{pl}s_\omega(t+\Delta t).\end{aligned}$$

One could obtain the two inequalities (29) and (30), respectively from rearranging the (28). The inequality (30) is useful for the determination of stability of the observer in such a way that the rate of change of load-torque ($2\dot{T}_L\Delta t \approx 2\rho_l\Delta t$) must be bounded within the region indicated by the relation (31)

$$-c\Delta t\rho_l^2 - c\Omega_L\rho_l + k_{il}\dot{\bar{V}}(t+\Delta t) \le 0 \qquad (29),$$

$$\rho_l^2 + \frac{\Omega_L}{\Delta t}\rho_l - \frac{k_{il}}{c\Delta t}\dot{\bar{V}}(t+\Delta t) \le 0 \qquad (30),$$

$$\begin{aligned}-\Omega_L - \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t\dot{\bar{V}}}{c}} &\le 2\rho_l\Delta t \\ &\le -\Omega_L + \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t\dot{\bar{V}}}{c}}\end{aligned} \qquad (31).$$

The relation (31) expresses the transient stability condition, in which $\Omega_L^2 + \frac{4k_{il}\Delta t\dot{\bar{V}}}{c}$ is the discriminant, $-\Omega_L - \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t\dot{\bar{V}}}{c}}$ is the lower bound, and $-\Omega_L + \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t\dot{\bar{V}}}{c}}$ is the upper bound. Within a period of time and under the positive discriminant, if the inequality (31) is true, $\dot{V}$ in (26) is negative. Thus, the load-torque observer is stable. However, if the inequality (31) becomes false, i.e.

$2\rho_l \Delta t \geq -\Omega_L + \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t\dot{\bar{V}}}{c}}$ or

$2\rho_l \Delta t \leq -\Omega_L - \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t\dot{\bar{V}}}{c}}$, $\dot{V}$ in (26) is positive. Then, the stability conclusion cannot be drawn. A negative discriminant may lead the inequality (32) to become false

$$\left(\rho_l + \frac{\Omega_L}{2\Delta t}\right)^2 - \frac{1}{4(\Delta t)^2}\left(\Omega_L^2 + \frac{4k_{il}\Delta t\dot{\bar{V}}}{c}\right) \leq 0 \qquad (32).$$

An unconcluded situation again occurs because $\dot{V}$ in (26) is positive. Practically, only some certain time intervals are subjected to the bounds. The existence of the bounds depends on the load characteristics, and the observer gains. Once the discriminant is successively negative or the actual load-torque is consistently constant, the bounds become meaningless.

The equation (33) represents the error due to the observer, and assuming that it is satisfied with the Lipschitz's condition [8]

$$\dot{e}_\omega = ae_\omega + \Delta a\hat{\omega}_r'' + \Delta b\hat{i}_{eq} + c\Delta T_L + f_3 - \delta_o \qquad (33).$$

Whenever all the observer gains are properly adjusted such that $\int_0^\infty \left| f_3(t) - \delta_o(t)\right| dt < \infty$, and the quantities of Δa , Δb , and ΔT_L converge to a very small constant ε, $\varepsilon \to 0$ as $t \to \infty$, the dynamic system (33) becomes asymptotically autonomous [9]. Hence, a nonempty set Ξ_C governing the derivative of the Lyapunov function to be negative semi-definite in transient and steady states is expressed as

$$\Xi_C = \Big\{ e_\omega \in \Re \;\Big|\; \exists\Delta a, \exists\Delta b, \exists\Delta T_L, \exists\Delta t,\ \rho_l \neq 0,\ \Omega_L^2 + \frac{4k_{il}\Delta t\dot{\bar{V}}}{c} \geq 0,\ \left|2\rho_l\Delta t + \Omega_L\right| \leq \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t\dot{\bar{V}}}{c}} \Big\} \cup \Big\{ e_\omega \in \Re \;\Big|\; \exists\Delta a, \exists\Delta b, \exists\Delta T_L,\ \dot{T}_L = 0,\ \ddot{T}_L = 0,\ \dot{V} \leq 0 \Big\} \qquad (34),$$

where $\Re$ is a set of real numbers. Whenever $|e_\omega|$ is bounded, there is a nonempty, compact, and invariant set confined inside Ξ_C. While the load and motor set rotates at a constant speed, and the load-torque is also constant, the load-torque observer is operating in a steady state mode as well as V is a decreasing function of t so long as $\dot{V} < 0$. Thus, V declines until $\dot{V} \to 0^-$, and thereby V becomes constant. The equations (9) and (10) represent this case, i.e. $t \to \infty$ and hence $s_\omega \to 0$, $e_\omega \to 0$, $\Theta_a \to 0$, $\Theta_b \to 0$ while Δa, Δb and ΔT_L converge to a fixed tiny constant resulting in a compact set Ξ_I as follows

$$\Xi_I = \Big\{ e_\omega \in \Re \;\Big|\; \exists\Delta a, \exists\Delta b, \exists\Delta T_L,\ \dot{T}_L = 0,\ \ddot{T}_L = 0,\ e_\omega \to 0,\ \dot{e}_\omega \to 0,\ \dot{V} \to 0^- \Big\} \qquad (35).$$

Whenever the trajectory e_ω lies in Ξ_I, it is equal to 0 exactly and also $\dot{V} = 0$. So far, Ξ_I is an invariant set because the trajectory is yet trapped inside Ξ_I until the rotor speed varies again due to either T_e or T_L changing. Through the inequality (30), at the instance of $\rho_l \neq 0$, when $\rho_l^2 + \frac{\Omega_L}{\Delta t}\rho_l - \frac{k_{il}}{c\Delta t}\dot{\bar{V}} = 0$ temporarily, $\dot{V}$ also equals zero momentarily. Therefore, a set Ξ_O governing the derivative of the Lyapunov function to be zero in both transient and steady states is written as

$$\Xi_O = \Big\{ e_\omega \in \Re \;\Big|\; \exists\Delta a, \exists\Delta b, \exists\Delta T_L, \exists\Delta t,\ \rho_l \neq 0,\ \Omega_L^2 + \frac{4k_{il}\Delta t\dot{\bar{V}}}{c} \geq 0,\ \left|2\rho_l\Delta t + \Omega_L\right| = \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t\dot{\bar{V}}}{c}} \Big\} \cup \Xi_I \qquad (36).$$

According to the LaSalle's theorem of invariant set [8], if $\Xi_I \subset \Xi_O \subset \Xi_C$ then $e_\omega \in \Xi_C$ moves to $e_\omega \in \Xi_I$ as $t \to \infty$. Therefore, the invariant set Ξ_I possesses the property of asymptotic stability. Hence, the load-torque observer converges to a stable region by the final stage of the transient period, and becomes surely stable throughout the steady state period.

5 Simulation Results

The load-torque observer of our previous work performs an online estimation by acquiring from the speed observer the estimated rotor speed, and the equivalent current proportional to the electromagnetic torque of an induction motor coupled with an inertia load. The motor and two observers form an open-loop system. According to direct-on-line starting, at the initial instant of time (t = 0) the motor previously de-energized at standstill is connected directly to a 220 V, 50 Hz three-phase ac supply. All initial conditions of state variables of both the motor-load system and the observers are zeroed. A first-order low-pass filter having 7.95-Hz cut-off frequency is used to denoise the estimated speed signal. The surface gains k_1 and k_2 of the speed observer, and k_3 of the torque observer are equal to 5. The correction gains for the speed observer are $|\phi_{1\alpha}| = |\phi_{1\beta}| = 290$, $|\phi_{2\alpha}| = |\phi_{2\beta}| = 1$, and $|\Lambda_\alpha| = |\Lambda_\beta| = 10$, whereas those of the torque observer are $|\phi_3| = 6$, $|\phi_4| = 1$, and $|\Lambda_3| = 0.2$. The PI gains of the adaptive laws of the speed observer are set to $k_{sp} = k_{si} = k_{rp} = k_{ri} = k_{mp} = k_{mi} = 0.00001$, $k_{\omega p} = 10$, and $k_{\omega i} = 800$ while those of the adaptive laws of the torque observer are set to $k_{pa} = k_{ia} = 0.000001$, $k_{pb} = k_{ib} = 0.001$, $k_{pl} = 2$, and $k_{il} = 40$, respectively.

The results in Figs 2 and 3 show that very large errors of the estimated rotor speed and flux only occur at the beginning of the estimation process. After about 0.5 seconds, the estimated load-torque converges to the actual value as shown by Fig. 4 with its magnified picture shown in Fig. 5. When the speed observer more accurately estimates the rotor speed and flux, the load-torque estimation is likewise valid. Although the actual load-torque changes in a step-ramp manner at the starting and during the constant speed operation of the motor, the observer still satisfactorily tracks this load as shown in Fig. 5. During the ramp period, the load-torque estimation is more erroneous than the one during the constant period as shown in Fig. 6.

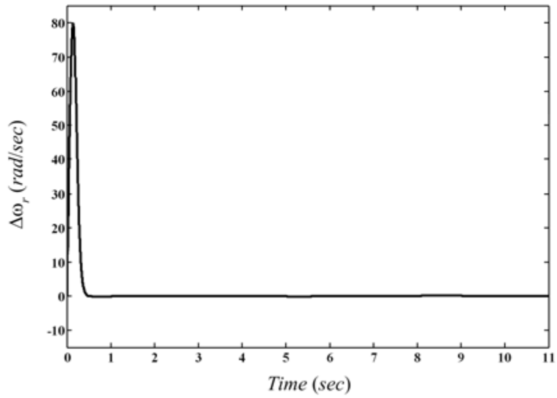

Fig. 2 Errors of the estimated rotor speed.

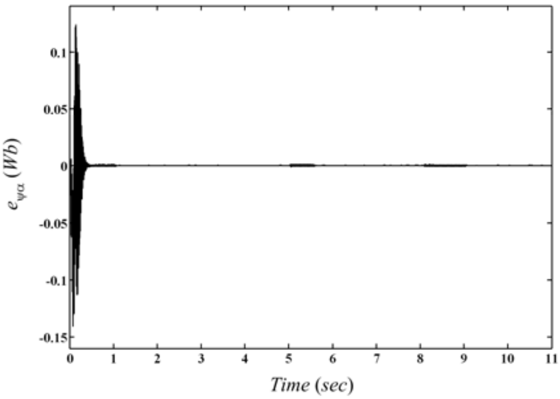

Fig. 3 Errors of the estimated rotor flux.

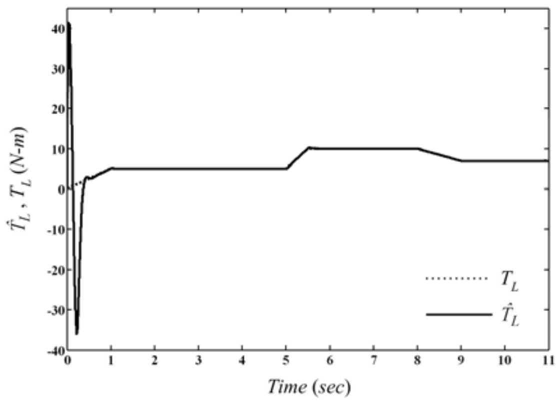

Fig. 4 Actual and estimated load-torques.

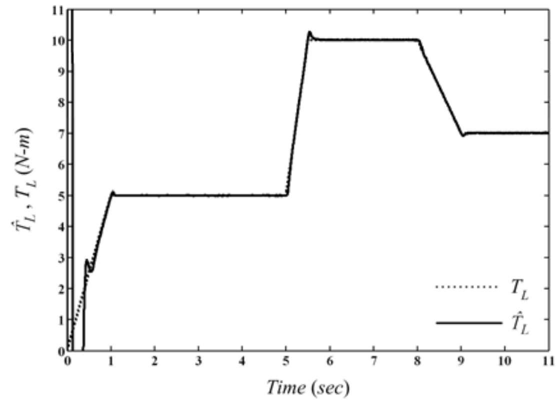

Fig. 5 The vertical axis of Fig. 4 magnified within 0–11 N·m.

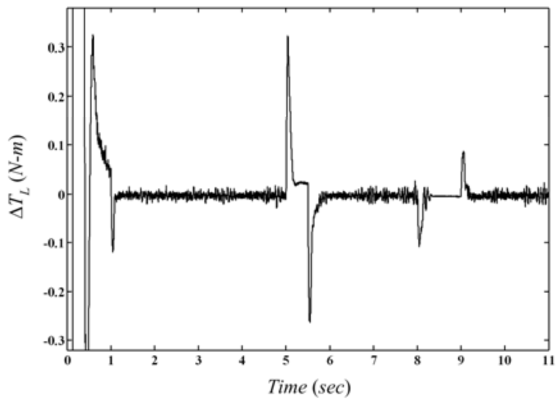

Fig. 6 Errors of the estimated load-torque.

The set Ξ_C contains an invariant set because $|e_\omega|$ is bounded. Fig. 7 illustrates the waveform of $|e_\omega|$. Figs. 8 and 9 indicate that the nonzero equivalent quantity of the load-torque derivative ($2\rho_l\Delta t$) is outside the bounds during the time 5.0–5.12 seconds while the discriminant is progressively positive ($e_\omega \notin \Xi_C$). As a result, the derivative of the Lyapunov function becomes positive and V grows positively during this time interval as illustrated in Figs. 10 and 11. This means that the observer's stability cannot be concluded momentarily. Thereafter, the quantity $2\rho_l\Delta t$ is within the bounds temporarily and the observer becomes stable for a certain period of time. Once the actual load-torque is constant ($\dot{T}_L = 0$ and $\ddot{T}_L = 0$), the quantity $2\rho_l\Delta t$ locates within the bounds again, and the observer resumes its stability ($e_\omega \in \Xi_C$). The derivative of V becomes negative definite ($\dot{V} < 0$) and further remains negative semi-definite ($\dot{V} \leq 0$) throughout the steady-state operation of the observer. Consequently, V becomes a non-increasing function and $e_\omega \in \Xi_C \rightarrow e_\omega \in \Xi_I$. In the meantime, the estimated load-torque is nearly equal to the actual value as shown in Figs. 5 and 6. In addition, the results shown in Figs. 12 to 16 illustrate some similar situations during the time 7.9–9.4 seconds. In steady state, the trajectory e_ω sinks within Ξ_I as indicated by V becoming constant as shown in Figs. 17 and 18.

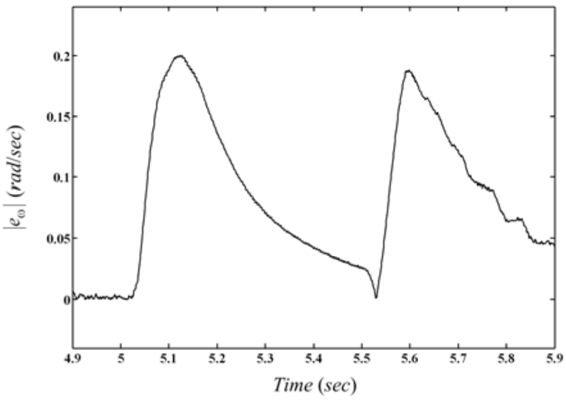

Fig. 7 $|e_\omega|$ during transient state and approaching steady state (4.9–5.9 seconds).

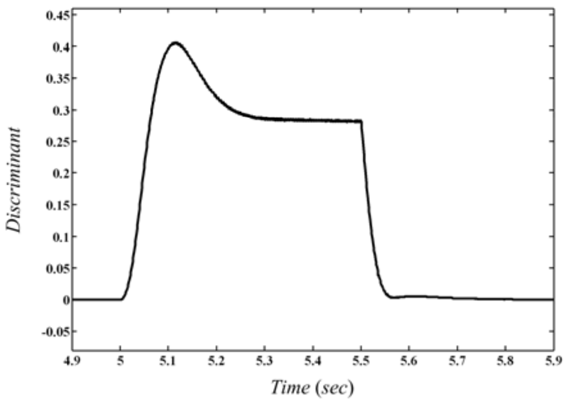

Fig. 8 Discriminant of the quantity $2\rho_l\Delta t$ (4.9–5.9 seconds).

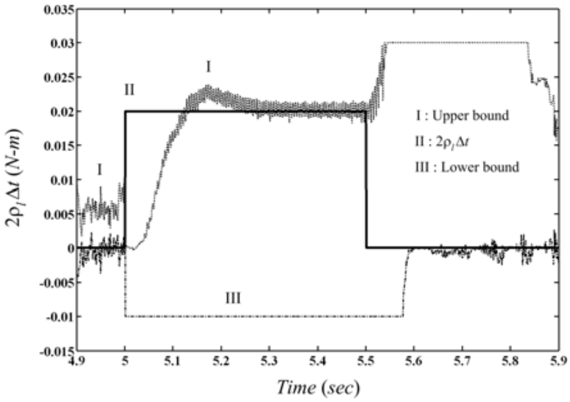

Fig. 9 Rate of change of load-torque, upper and lower bounds (4.9–5.9 seconds).

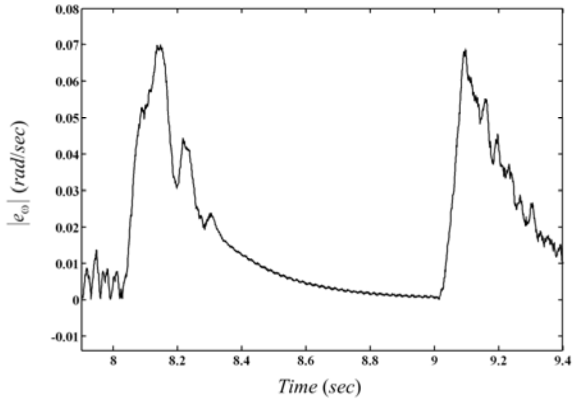

Fig. 12 $|e_{\omega}|$ during transient state and approaching steady state (7.9–9.4 seconds).

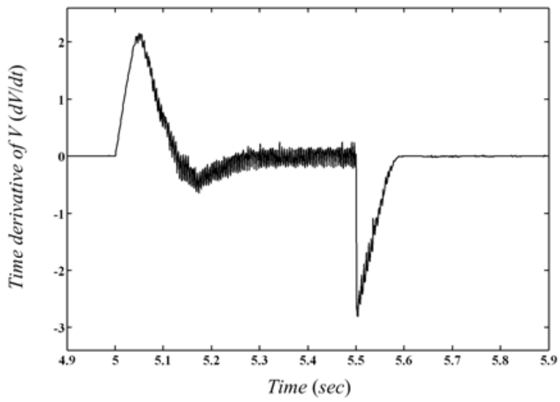

Fig. 10 Time derivative of the Lyapunov function in (26) (4.9–5.9 seconds).

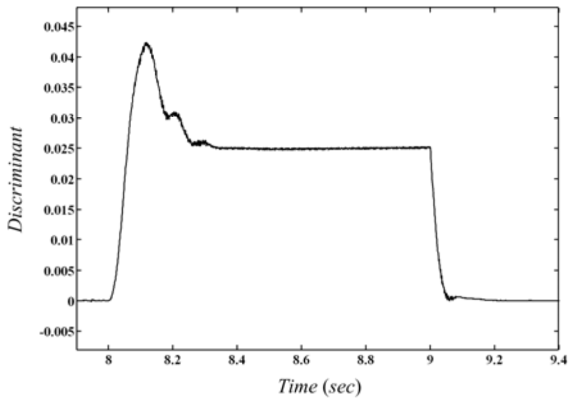

Fig. 13 Discriminant of the quantity $2\rho_l\Delta t$ (7.9–9.4 seconds).

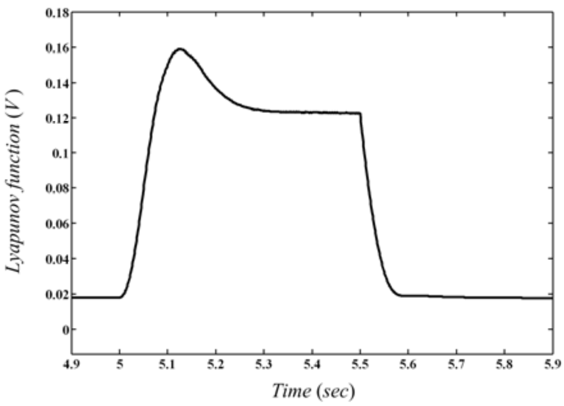

Fig. 11 The Lyapunov function in (9) (4.9–5.9 seconds).

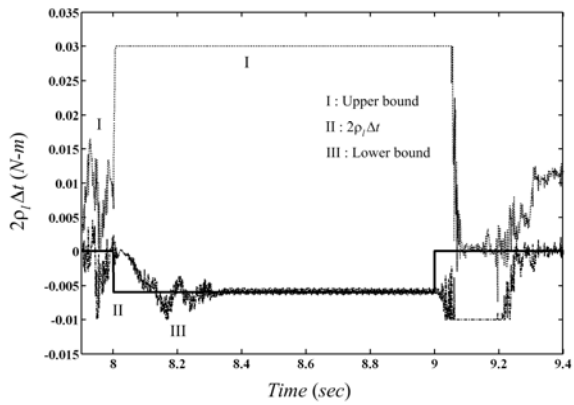

Fig. 14 Rate of change of load-torque, upper and lower bounds (7.9–9.4 seconds).

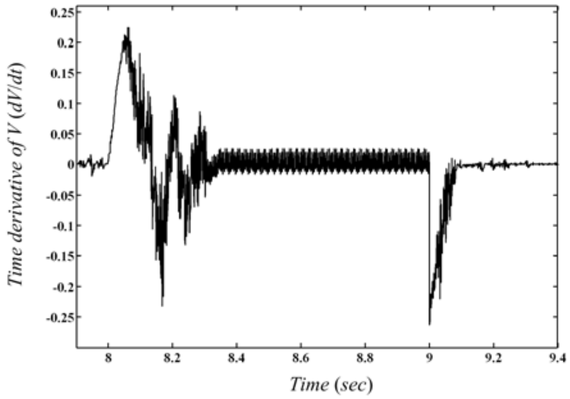

Fig. 15 Time derivative of the Lyapunov function in (26) (7.9–9.4 seconds).

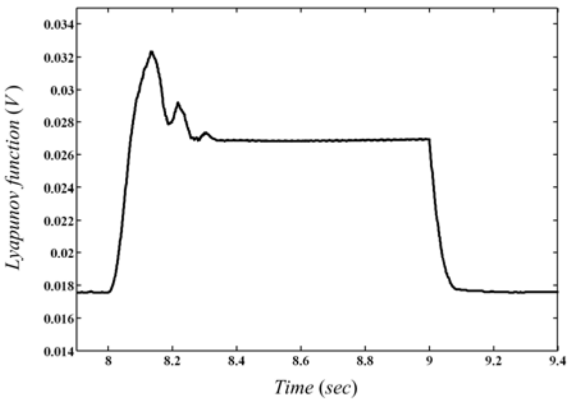

Fig. 16 The Lyapunov function in (9) (7.9–9.4 seconds).

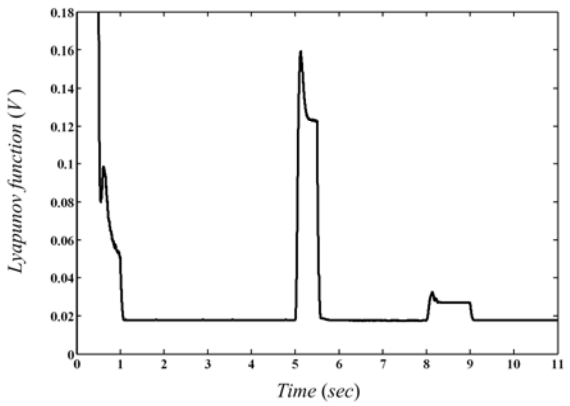

Fig. 17 The Lyapunov function in (9) throughout the simulation time of 11 seconds.

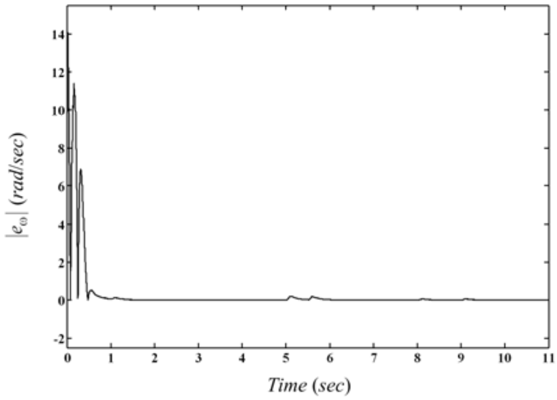

Fig. 18 $|e_\omega|$ throughout the simulation time of 11 seconds.

6 Conclusions

During the transient state, the adaptive sliding-mode load-torque observer is stable in the Lyapunov's sense for some time intervals, while during some intervals the stability cannot be concluded. The observer's uniform stability is guaranteed for steady-state period. Based on the Lyapunov's direct method, the above conclusion has been drawn using the quadratic inequality describing the stability without the knowledge of the load-torque dynamic as a priori. The LaSalle's theorem of invariant sets has been applied to confirm that the observer, by the end of the transient state, certainly enters the stable steady-state. To make the load-torque observer become widely accepted for industrial practice, its transient stability and accuracy problems must be solved carefully.

Acknowledgements

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Appendix:

Derivation towards the equation (10).

Multiplying both sides of the equation (15) by s_ω and rearranging the resultant equation lead to

$$s_\omega \dot{s}_\omega = (a+k_3)s_\omega e_\omega + \Delta a\hat{\omega}_r'' s_\omega + \Delta b\hat{i}_{eq}s_\omega + c\Delta T_L s_\omega + f_3 s_\omega - s_\omega \delta_o \quad ,$$

$$s_\omega \dot{s}_\omega = s_\omega\{(a+k_3)e_\omega + f_3 - \delta_o\} + \Delta a\hat{\omega}_r'' s_\omega + \Delta b\hat{i}_{eq}s_\omega + c\Delta T_L s_\omega \quad ,$$

$$s_\omega \dot{s}_\omega = s_\omega\{(a+k_3)e_\omega + f_3 - \delta_o\} + \Delta a\Theta_a + \Delta b\Theta_b + c\Delta T_L s_\omega \quad (37).$$

Then, differentiating the equation (9) with respect to time yields

$$\begin{aligned}\dot{V} &= \frac{1}{2}(2s_\omega\dot{s}_\omega) \\ &+ \frac{1}{2k_{ia}}\left[2\left(\Delta a + k_{pa}\Theta_a\right)\left(\Delta\dot{a} + k_{pa}\dot{\Theta}_a\right)\right] \\ &+ \frac{1}{2k_{ib}}\left[2\left(\Delta b + k_{pb}\Theta_b\right)\left(\Delta\dot{b} + k_{pb}\dot{\Theta}_b\right)\right] \\ &- \frac{c}{2k_{il}}\left[2\left(\Delta T_L - k_{pl}s_\omega\right)\left(\Delta\dot{T}_L - k_{pl}\dot{s}_\omega\right)\right]\end{aligned} \quad ,$$

$$\begin{aligned}\dot{V} &= s_\omega\dot{s}_\omega + \frac{1}{k_{ia}}\left(\Delta a + k_{pa}\Theta_a\right)\left(\Delta\dot{a} + k_{pa}\dot{\Theta}_a\right) \\ &+ \frac{1}{k_{ib}}\left(\Delta b + k_{pb}\Theta_b\right)\left(\Delta\dot{b} + k_{pb}\dot{\Theta}_b\right) \\ &- \frac{c}{k_{il}}\left(\Delta T_L - k_{pl}s_\omega\right)\left(\Delta\dot{T}_L - k_{pl}\dot{s}_\omega\right)\end{aligned} \quad (38).$$

Three errors due to the online mechanical parameter and load-torque estimations as well as their derivatives are written together as follows: $\Delta a = a - \hat{a}$, $\Delta b = b - \hat{b}$, and $\Delta T_L = T_L - \hat{T}_L$, as well as $\Delta\dot{a} \approx -\dot{\hat{a}}$, $\Delta\dot{b} \approx -\dot{\hat{b}}$, and $\Delta\dot{T}_L \approx -\dot{\hat{T}}_L$ owing to $\dot{a} \approx 0$, $\dot{b} \approx 0$, and $\dot{T}_L \approx 0$. Then, three PI adaptive laws through the equations (6)-(8) are rearranged into

$$-\Delta\dot{a} \approx \dot{\hat{a}} = k_{pa}\dot{\Theta}_a + k_{ia}\Theta_a \quad ,$$

$$-\Delta\dot{b} \approx \dot{\hat{b}} = k_{pb}\dot{\Theta}_b + k_{ib}\Theta_b \quad ,$$

$$-\Delta\dot{T}_L = \dot{\hat{T}}_L = -k_{pl}\dot{s}_\omega - k_{il}s_\omega \quad .$$

Three derivatives in the relevant equations above become

$$\Delta\dot{a} = -k_{pa}\dot{\Theta}_a - k_{ia}\Theta_a \quad ,$$

$$\Delta\dot{b} = -k_{pb}\dot{\Theta}_b - k_{ib}\Theta_b \quad ,$$

$$\Delta\dot{T}_L = k_{pl}\dot{s}_\omega + k_{il}s_\omega \quad .$$

So far, the three above derivatives are rewritten into

$$\Delta\dot{a} + k_{pa}\dot{\Theta}_a = -k_{ia}\Theta_a \qquad (39),$$

$$\Delta\dot{b} + k_{pb}\dot{\Theta}_b = -k_{ib}\Theta_b \qquad (40),$$

$$\Delta\dot{T}_L - k_{pl}\dot{s}_\omega = k_{il}s_\omega \qquad (41).$$

Thereafter, substituting the last three equations (39)-(41) and the term $s_\omega\dot{s}_\omega$ from the equation (37) into the equation (38) brings about

$$\begin{aligned}\dot{V} &= s_\omega\dot{s}_\omega + \frac{1}{k_{ia}}\left(\Delta a + k_{pa}\Theta_a\right)\left(-k_{ia}\Theta_a\right)\\ &+ \frac{1}{k_{ib}}\left(\Delta b + k_{pb}\Theta_b\right)\left(-k_{ib}\Theta_b\right)\\ &- \frac{c}{k_{il}}\left(\Delta T_L - k_{pl}s_\omega\right)\left(k_{il}s_\omega\right)\end{aligned},$$

$$\begin{aligned}\dot{V} &= s_\omega\dot{s}_\omega - \Theta_a\left(\Delta a + k_{pa}\Theta_a\right)\\ &- \Theta_b\left(\Delta b + k_{pb}\Theta_b\right) - cs_\omega\left(\Delta T_L - k_{pl}s_\omega\right)\end{aligned},$$

$$\begin{aligned}\dot{V} &= s_\omega\dot{s}_\omega - \Delta a\Theta_a - k_{pa}\Theta_a^2 - \Delta b\Theta_b\\ &- k_{pb}\Theta_b^2 - c\Delta T_L s_\omega + ck_{pl}s_\omega^2\end{aligned},$$

$$\begin{aligned}\dot{V} &= s_\omega\dot{s}_\omega - \Delta a\Theta_a - \Delta b\Theta_b - c\Delta T_L s_\omega\\ &- k_{pa}\Theta_a^2 - k_{pb}\Theta_b^2 + ck_{pl}s_\omega^2\end{aligned},$$

$$\begin{aligned}\dot{V} &= s_\omega\left\{\left(a + k_3\right)e_\omega + f_3 - \delta_o\right\} + \Delta a\Theta_a + \Delta b\Theta_b\\ &+ c\Delta T_L s_\omega - \Delta a\Theta_a - \Delta b\Theta_b - c\Delta T_L s_\omega\\ &- k_{pa}\Theta_a^2 - k_{pb}\Theta_b^2 + ck_{pl}s_\omega^2\end{aligned},$$

$$\begin{aligned}\dot{V} &= s_\omega\left\{\left(a + k_3\right)e_\omega + f_3 - \delta_o\right\} - k_{pa}\Theta_a^2\\ &- k_{pb}\Theta_b^2 + ck_{pl}s_\omega^2 \le 0\end{aligned},$$

$$\begin{aligned}\dot{V} &= s_\omega\left\{\left(a + k_3\right)e_\omega + f_3 - \left(\phi_3 e_\omega + \phi_4 k_3 h_\omega + \Lambda_3\right)\right\}\\ &- k_{pa}\Theta_a^2 - k_{pb}\Theta_b^2 + ck_{pl}s_\omega^2 \le 0\end{aligned},$$

$$\begin{aligned}\dot{V} &= \left(a + k_3 - \phi_3\right)s_\omega e_\omega - \phi_4 k_3 s_\omega h_\omega + \left(f_3 - \Lambda_3\right)s_\omega\\ &- k_{pa}\Theta_a^2 - k_{pb}\Theta_b^2 + ck_{pl}s_\omega^2 \le 0\end{aligned} \qquad (42).$$

Finally, the equation (42) is identical with the one (10).

Derivation towards the equation (26).

Whenever the load-torque is not constant ($\dot{T}_L \ne 0$), the derivative of the error due to the load-torque estimation becomes $\Delta\dot{T}_L = \dot{T}_L - \dot{\hat{T}}_L$ and is rewritten as $\Delta\dot{T}_L = \dot{T}_L + k_{pl}\dot{s}_\omega + k_{il}s_\omega$ and then

$$\Delta\dot{T}_L - k_{pl}\dot{s}_\omega = \dot{T}_L + k_{il}s_\omega \qquad (43),$$

respectively. Afterwards, the last expression in the right-hand side of the equation (38) is substituted from the equation (43). This expression would be rewritten as

$$\begin{aligned}&-\frac{c}{k_{il}}\left(\Delta T_L - k_{pl}s_\omega\right)\left(\Delta\dot{T}_L - k_{pl}\dot{s}_\omega\right)\\ &\quad = -\frac{c}{k_{il}}\left(\Delta T_L - k_{pl}s_\omega\right)\left(\dot{T}_L + k_{il}s_\omega\right)\end{aligned},$$

$$\begin{aligned}&-\frac{c}{k_{il}}\left(\Delta T_L - k_{pl}s_\omega\right)\left(\Delta\dot{T}_L - k_{pl}\dot{s}_\omega\right)\\ &\quad = -\frac{c}{k_{il}}\left(\Delta T_L - k_{pl}s_\omega\right)\dot{T}_L - cs_\omega\left(\Delta T_L - k_{pl}s_\omega\right)\end{aligned},$$

$$\begin{aligned}&-\frac{c}{k_{il}}\left(\Delta T_L - k_{pl}s_\omega\right)\left(\Delta\dot{T}_L - k_{pl}\dot{s}_\omega\right)\\ &\quad = -c\Delta T_L s_\omega + ck_{pl}s_\omega^2 - \frac{c}{k_{il}}\left(\Delta T_L - k_{pl}s_\omega\right)\dot{T}_L\end{aligned} \qquad (44).$$

Thereafter, by substituting two expressions from the equations (39)-(40), the term $s_\omega\dot{s}_\omega$ from the equation (37), and the expression from the equation (44) into the equation (38), this procedure brings about

$$\begin{aligned}\dot{V} &= \left(a + k_3 - \phi_3\right)s_\omega e_\omega - \phi_4 k_3 s_\omega h_\omega + \left(f_3 - \Lambda_3\right)s_\omega\\ &- k_{pa}\Theta_a^2 - k_{pb}\Theta_b^2 + ck_{pl}s_\omega^2\\ &- \frac{c}{k_{il}}\left(\Delta T_L - k_{pl}s_\omega\right)\dot{T}_L\end{aligned} \qquad (45).$$

Eventually, the equation (45) is identical with the one (26).

Derivation towards the equation (27).
When the time derivative of the Lyapunov function is evaluated at a point of time $t + \Delta t$, the equation (26) would be rewritten in a form of

$$\dot{V} = \dot{\bar{V}} - \frac{c}{k_{il}}\left(T_L - \hat{T}_L - k_{pl}s_\omega\right)\dot{T}_L ,$$

$$\dot{V}(t+\Delta t) = \dot{\bar{V}}(t+\Delta t) - \frac{c}{k_{il}}\left[T_L(t+\Delta t) - \hat{T}_L(t+\Delta t) - k_{pl}s_\omega(t+\Delta t)\right] \times \dot{T}_L(t+\Delta t) \quad (46).$$

Finally, the equation (46) is identical with the one (27).

Derivation towards the equation (28).
By substituting the two terms $T_L(t + \Delta t) = T_L(t) + \rho_l\Delta t$ and $\dot{T}_L(t+\Delta t) \approx \rho_l$ into the equation (27) and then rearranging the resultant equation, these procedures yield

$$\dot{V}(t+\Delta t) = \dot{\bar{V}}(t+\Delta t) - \frac{c}{k_{il}}\left[T_L(t) + \rho_l\Delta t - \hat{T}_L(t+\Delta t) - k_{pl}s_\omega(t+\Delta t)\right]\rho_l ,$$

$$\dot{V}(t+\Delta t) = \dot{\bar{V}}(t+\Delta t) - \frac{c}{k_{il}}\rho_l\left\{\left[T_L(t) - \hat{T}_L(t+\Delta t) - k_{pl}s_\omega(t+\Delta t)\right] + \rho_l\Delta t\right\} ,$$

$$\dot{V}(t+\Delta t) = \dot{\bar{V}}(t+\Delta t) - \frac{c\Delta t}{k_{il}}\rho_l^2 - \frac{c}{k_{il}}\left[T_L(t) - \hat{T}_L(t+\Delta t) - k_{pl}s_\omega(t+\Delta t)\right]\rho_l ,$$

$$\dot{V}(t+\Delta t) = \dot{\bar{V}}(t+\Delta t) - \frac{c\Omega_L(t+\Delta t)}{k_{il}}\rho_l - \frac{c\Delta t}{k_{il}}\rho_l^2 ,$$

$$\dot{V}(t+\Delta t) = -\frac{c\Delta t}{k_{il}}\rho_l^2 - \frac{c\Omega_L}{k_{il}}\rho_l + \dot{\bar{V}}(t+\Delta t) \quad (47).$$

Eventually, the equation (47) is identical with the one (28).

Derivation towards the solution of the inequality (30).
When the quadratic inequality (30) is arranged into type of completing the square, it could be written as

$$\rho_l^2 + \frac{\Omega_L}{\Delta t}\rho_l + \left(\frac{\Omega_L}{2\Delta t}\right)^2 - \left(\frac{\Omega_L}{2\Delta t}\right)^2 - \frac{k_{il}}{c\Delta t}\dot{\bar{V}}(t+\Delta t) \le 0 ,$$

$$\left(\rho_l + \frac{\Omega_L}{2\Delta t}\right)^2 - \frac{\Omega_L^2}{4(\Delta t)^2} - \frac{k_{il}}{c\Delta t}\dot{\bar{V}}(t+\Delta t) \le 0 ,$$

$$\left(\rho_l + \frac{\Omega_L}{2\Delta t}\right)^2 - \frac{\Omega_L^2}{4(\Delta t)^2} - \frac{4k_{il}\Delta t}{4c(\Delta t)^2}\dot{\bar{V}}(t+\Delta t) \le 0 ,$$

$$\left(\rho_l + \frac{\Omega_L}{2\Delta t}\right)^2 - \frac{1}{4(\Delta t)^2}\left[\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\bar{V}}(t+\Delta t)\right] \le 0 ,$$

$$\left(\frac{2\rho_l\Delta t + \Omega_L}{2\Delta t}\right)^2 - \frac{1}{4(\Delta t)^2}\left[\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\bar{V}}(t+\Delta t)\right] \le 0 ,$$

$$\frac{(2\rho_l\Delta t + \Omega_L)^2}{4(\Delta t)^2} - \frac{1}{4(\Delta t)^2}\left[\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\bar{V}}(t+\Delta t)\right] \le 0 ,$$

$$(2\rho_l\Delta t + \Omega_L)^2 - \left(\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\bar{V}}\right) \le 0 ,$$

$$(2\rho_l\Delta t + \Omega_L)^2 - \left(\sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\bar{V}}}\right)^2 \le 0 ,$$

$$\left(2\rho_l\Delta t + \Omega_L - \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\bar{V}}}\right) \times \left(2\rho_l\Delta t + \Omega_L + \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\bar{V}}}\right) \le 0 \quad (48).$$

The either of two solutions taken from the above inequality is

$2\rho_l\Delta t + \Omega_L - \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\bar{V}}} \le 0$ together with

$2\rho_l\Delta t + \Omega_L + \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\bar{V}}} \ge 0$ or

$2\rho_l \Delta t + \Omega_L - \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\overline{V}}} \ \geq \ 0$ together with

$2\rho_l \Delta t + \Omega_L + \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\overline{V}}} \ \leq \ 0$.

These solutions could be rewritten as

$2\rho_l \Delta t \ \leq \ -\Omega_L + \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\overline{V}}}$ together with

$2\rho_l \Delta t \ \geq \ -\Omega_L - \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\overline{V}}}$ or

$2\rho_l \Delta t \ \geq \ -\Omega_L + \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\overline{V}}}$ together with

$2\rho_l \Delta t \ \leq \ -\Omega_L - \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\overline{V}}}$,

where $\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\overline{V}} \ \geq \ 0$. However, owing to

$$-\Omega_L - \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\overline{V}}} \ \leq \ -\Omega_L \ \leq \ -\Omega_L + \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\overline{V}}},$$

thus the relation

$$-\Omega_L - \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\overline{V}}} \ \leq \ 2\rho_l \Delta t \ \leq \ -\Omega_L + \sqrt{\Omega_L^2 + \frac{4k_{il}\Delta t}{c}\dot{\overline{V}}}$$

is the correct solution of the inequality (30). It is corresponding to the relation (31).