# Novel Powerful Comprehensive Analytical Probabilistic Model of Random Variation in Subthreshold MOSFET's Performance

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*Abstract:* - In this research, the novel comprehensive probabilistic analytical model of the subthreshold MOSFET's performance affected by both random dopant fluctuation and process variation effects has been proposed. The up to dated Takeuchi's physical level random variation model has been adopted. The proposed model has been found to be analytic, powerful and comprehensive as it has been derived by using the subthreshold MOSFET's physical equation without any approximation. This model has been verified at the nanometer level i.e. 65 nm CMOS process, by using the BSIM4 based Monte-Carlo simulations. The verifications have been performed based on both NMOS and PMOS technologies. This model is very accurate since it can closely follow the Monte-Carlo based distributions with pleasant goodness of fit test results. Furthermore, the proposed model can also serve as the basis for the mismatch modeling and performance optimization. Hence, the proposed model has been found to be the potential mathematical tool for the statistical/variability aware analysis/design of various subthreshold region operated MOSFET based low voltage/low power.

*Key-Words:* - drain current, low power, low voltage, nanometer level, MOSFET, statistical design, subthreshold, variability aware design.

## 1 Introduction

Recently, the subthreshold (weak inversion) region operation has been adopted in many low voltage/low power circuits and systems with the inferior robustness to the super-threshold (strong inversion) operated counterparts as a penalty [1]. Due to such robustness reduction, these subthreshold operated circuits and systems are more susceptible to the imperfection in MOSFET's properties for example random dopant fluctuation, line edge roughness and gate length random fluctuation which cause the random variations in MOSFET's parameters such as drain current and transconductance etc. This susceptibility is a critical issue in the statistical/variability aware design of MOSFET based low voltage/ low power applications as the devices are extremely scaled down [2].

According to this motivation and the fact that the analytical model is a computationally efficient tool for circuit analysis and design [3, 4], there are many previous researches on the analytical modeling of such variation in the subthreshold region operated MOSFET for examples [1] and [3]-[7] etc. In these researches, the analytical modeling of variation in $I_d$ which is a key performance metric has been focused. The model of the relative standard deviation of $I_d$ has been proposed in [1]. In [3], the analytical models of means and variances of $I_d$ have been proposed. Later, the analytical modeling of the probability density function of $I_d$ has been performed in [4]. In [5]-[7], the drain current variation has been modeled in its per-unit basis given by $\Delta I_d/I_d$ as a normally distributed random variable.

Of course, the probability density function based model as those in [4]-[7] is more powerful as it can completely describe the distribution of the drain current variation, furthermore, any statistical parameters such as mean, variance and moments can be determined from the models by using the conventional mathematical statistics. On the other hand, those in [1] and [3] which have been derived based on the specific statistical parameters are less powerful since the related distribution is not revealed and the other statistical parameters cannot be determined. However, the model in [4] has been derived based on its own fitted empirical formula of $I_d$ not the physical one. So, it is not comprehensive as the related physical parameters are not completely shown. Furthermore, the normal distribution based models proposed in [5]-[7] are

practically inaccurate due to the loss of accuracy by the adopted Taylor's series based approximation of the subthreshold drain current.

However, it has been observed in [8] that the on current of any subthreshold digital circuit and system which is $I_d$ with $V_{gs}= V_{dd}$, under the effect of the process variation has a lognormal distribution. However, there is an approximation behind this observation that

$$1 >> \exp\left[-\frac{qV_{ds}}{kT}\right] \qquad (1)$$

For the up to dated low voltage/low power designing which $V_{ds}$ can be very small, this approximation is unfortunately ceased to be valid. Hence, the resulting observation becomes suspicious for such low voltage/low power scenario. Furthermore, the affected $I_d$ at any value of $V_{gs}$ apart from that at $V_{gs} = V_{dd}$ and one at $V_{gs} = 0$ which is defined as the leakage current, have not been mentioned in [8] at all. Motivated by [8], it is suspected that such affected $I_d$ is not normally distributed.

In [9], the comprehensive analytical models of random variations in the super-threshold (strong inversion region) operated nanoscale MOS transistor have been proposed in term of the powerful probability density functions incorporated by many related physical parameters. So, they have been found to be efficient for the statistical/variability aware design of various super-threshold operated MOSFET based analog/mixed signal circuits and systems in the nanoscale regime. Motivated by [8], [9] and many problems that have been arose in the previous attempts to perform the modeling for the subthreshold MOSFET's performance such as those in [1] and [3]-[7] etc., it has been found to be worthy to derive the comprehensive analytical model of such random dopant fluctuation/process variation affected subthreshold drain current in term of the powerful probability density function. This model must be in the similar manner to those in [9] but with the devotion to the subthreshold region operated MOSFET. In this model, many related physical level variables must be incorporated for the comprehensiveness. Any approximation cannot be allowed in the derivation in order to prevent the approximation related accuracy loss. Finally, this model must be applicable to the arbitrary affected $I_d$ not only the specific ones.

Hence, in this research, the novel comprehensive probabilistic analytical model of the subthreshold

MOSFET's $I_d$ under the effect of both random dopant fluctuation and process variation effects which are the major causes of the random variation in the MOSFET characteristic, have been proposed. Unlike [9] which oriented to the super-threshold MOSFET and adopts the classical Pelgrom's model [10], the up to dated Takeuchi's physical level random variation model [11] has been adopted for this subthrehold MOSFET devoted model. The proposed model has been found to be analytic and powerful as it is in term of the probability density function similarly to those in [4]-[7] but with more comprehensiveness than that in [4] without approximation related loss of accuracy as those in [5]-[7]. The reason for this is that the proposed model has been derived by using the subthreshold MOSFET's physical equation not the empirically fitted one like that in [4] and without any approximation similarly to those in [5]-[7]. So, both analyticity and comprehensiveness can be achieved where as the nonlinearity related accuracy can be preserved. By using this model, the distribution of randomly varied $I_d$ can be clearly explored and the related physical level parameters which affect such variations can be precisely revealed. Unlike [8] which concerns only the on and leakage currents, the proposed model is applicable to the arbitrary $I_d$ at any $V_{gs}$. Furthermore, this model is unsuspicious in the low voltage/low power scenario as the doubtful approximation given by (1) has not been used. This model has been verified at the nanometer level i.e. 65 nm CMOS process, by using the BSIM4 based Monte-Carlo simulations. The verifications have been performed based on both NMOS and PMOS technologies. This model is very accurate since it can closely follow the Monte-Carlo based distributions with enjoyable goodness of fit test results. Furthermore, the proposed model can also serve as the basis for the mismatch modeling and performance optimization as will be discussed later. So, the proposed models have been found to be the potential mathematical tool for the statistical/variability aware analysis/design of various subthreshold region operated MOSFET based low voltage/low power applications.

## 2 The Proposed Model

In this section, the proposed model will be discussed. Unlike the fitted formula based previous model in [4], this model has been derived based on the physical drain current of the subthreshold region operated MOSFET proposed in [12]. By taking the effect of the random dopant fluctuation and process variation into account, $I_d$ can be given by

$$I_d = \mu C_{dep} \frac{W}{L} \left(\frac{kT}{q}\right)^2 \exp\left[\frac{q(V_{gs}-V_{TH})}{nkT}\right]\left[1-\exp\left[-\frac{qV_{ds}}{kT}\right]\right] \quad (2)$$
$$\times \exp\left[\frac{q(V_t-V_{TH})}{nkT}\right]$$

where $C_{dep}$ and n denote the capacitance of the depletion region under the gate area and the subthreshold parameter respectively [12]. By using the state of the art Takeuchi's model of physical level variation [11] not the classical one in [10] which has been adopted in the super-threshold MOSFET's model [9], and the principle of random variable transformation [13] with (2) has been used as the mapping function, the probability density function of $I_d$ under the effect of both random dopant fluctuation and process variation can be given by

$$f_{I_d}(i_d) = \sqrt{\frac{3}{2\pi}} \frac{nkT}{|i_d|(V_{gs}-V_{TH})} \left\{\frac{\varepsilon_{ox}WL}{q^3 T_{INV}(V_{TH}-V_{FB}-\phi_s)}\right\}^{0.5}$$
$$\times \exp\left[-\frac{\sqrt{3}}{2} \frac{nkT}{q^{1.5}} \left\{\frac{\varepsilon_{ox}WL}{T_{INV}(V_{TH}-V_{FB}-\phi_s)}\right\}^{0.5}\right. \quad (3)$$
$$\times \ln\left[\left\{\mu C_{dep}\frac{W}{L}\left(\frac{kT}{q}\right)^2\left\{\exp\left[\frac{q(V_{gs}-V_{TH})}{nkT}\right]\right\}\right.\right.$$
$$\left.\left.\times\left\{1-\exp\left[\frac{qV_{ds}}{kT}\right]\right\}\right\}^{-1}|i_d|\right]$$

where $i_d$, $T_{INV}$, $V_{FB}$, $\varepsilon_{ox}$ and $\phi_s$ denote any sampled value of $I_d$ the electrical oxide thickness, flat band voltage, gate oxide permittivity and effective work function respectively [11]. At this point, it can be stated that the comprehensive analytical probabilistic model of the subthreshold region operated MOSFERT's $I_d$ under the effect of random dopant fluctuation process variation has been proposed. Unlike [4], this model is highly comprehensive because many physical level parameters such as $T_{INV}$, $V_{FB}$, $V_{TH}$ and $\phi_s$ etc., have been incorporated. As the powerful probability density functions similarly to those in [4]-[7] and [9], the probabilistic behavior of such affected $I_d$ can be clearly explained and many meaningful statistical parameters of $I_d$ such as its mean, variance and standard deviation etc., can be derived by applying the conventional mathematical statistics to the proposed model. These features enlighten the understanding of the performance variation of the subthreshold MOSFET which yields various benefits. More effective design/analysis involving these variations can be achieved as the related statistical parameters can be effectively computed

and the physical level parameters which affect the variations are precisely known. As the analytical model, it is a computational efficient designing tool [3, 4] since its resulting computational effort is potentially smaller than that of the brute force Monte-Carlo analysis based on the random variations of the physical parameters which is time expensive as stated in[14].

Unlike those in [5]-[7] which rely on the Taylor's series based approximation, this research directly uses the principle of random variable transformation without any approximation. So, the obtained model predicts that the distribution function of such affected $I_d$ is totally different from the Gaussian probability density function which is the prediction of those in [5]-[7] as the result of using the Taylor's series based approximation. Instead of the traditional normal distribution, the proposed model predicts that the subthreshold MOSFET's $I_d$ under the effect of random dopant fluctuation and process variation has a lognormal distribution with the parameters denoted by $\alpha_{Id}$ and $\beta_{Id}$ as given respectively below.

$$\alpha_{I_d} = \ln\left[\mu C_{dep}\frac{W}{L}\left(\frac{kT}{q}\right)^2\left\{\exp\left[\frac{q(V_{gs}-V_{TH})}{nkT}\right]\right\}\right.$$
$$\left.\times\left\{1-\exp\left[\frac{qV_{ds}}{kT}\right]\right\}\right] \quad (4)$$

and

$$\beta_{I_d} = \frac{1}{\sqrt{3}}\frac{q^{1.5}}{nkT}\left\{\frac{T_{INV}(V_{TH}-V_{FB}-\phi_s)}{\varepsilon_{ox}WL}\right\}^{0.5} \quad (5)$$

Actually, $\alpha_{Id}$ and $\beta_{Id}$ are the mean and the standard deviation of a dummy random variable defined as $\ln|I_d|$ which is expected to be normally distributed according to the properties of the lognormal random variable [15] as the proposed model predicts the lognormal distribution of $I_d$. The predicted lognormal distribution cannot be approximated by the normal one predicted by those in [5]-[7]. The reason for this can be seen from the consideration of mean, median and mode of such $I_d$ denoted by $\mu_{Id}$, $\tilde{I}_d$ and $M(I_d)$ respectively. With the proposed model, they can be analytically given by

$$\mu_{Id} = \mu C_{dep} \frac{W}{L}\left(\frac{kT}{q}\right)^2 \left\{\exp\left[\frac{q(V_{gs}-V_{TH})}{nkT}\right]\right\}$$
$$\times \left\{1 - \exp\left[\frac{qV_{ds}}{kT}\right]\right\} \qquad (6)$$
$$\times \exp\left\{\frac{1}{2}\left[\left(\frac{q^{1.5}}{nkT}\right)^2 \left\{\frac{T_{INV}(V_{TH}-V_{FB}-\phi_s)}{3\varepsilon_{ox}WL}\right\}\right]\right\}$$

$$\widetilde{I}_d = \mu C_{dep}\frac{W}{L}\left(\frac{kT}{q}\right)^2 \left\{\exp\left[\frac{q(V_{gs}-V_{TH})}{nkT}\right]\right\}\left\{1-\exp\left[\frac{qV_{ds}}{kT}\right]\right\} \quad (7)$$

and

$$M(I_d) = \mu C_{dep}\frac{W}{L}\left(\frac{kT}{q}\right)^2 \left\{\exp\left[\frac{q(V_{gs}-V_{TH})}{nkT}\right]\right\}$$
$$\times \left\{1 - \exp\left[\frac{qV_{ds}}{kT}\right]\right\} \qquad (8)$$
$$\times \exp\left\{-\frac{1}{6}\left(\frac{q^{1.5}}{nkT}\right)^2 \left\{\frac{T_{INV}(V_{TH}-V_{FB}-\phi_s)}{\varepsilon_{ox}WL}\right\}\right\}$$

For the extremely scaled MOSFET based low voltage/low power circuits and systems which the effect of the random dopant fluctuation and process variation are crucial [2] and the corresponding voltages are extremely low, it can be seen that these quantities are significantly different from each other as the random dopant fluctuation/process variation related terms are not negligible. Such terms of $\mu_{Id}$ and $M(I_d)$ can be respectively given by

$$\exp\left\{\frac{1}{6}\left(\frac{q^{1.5}}{nkT}\right)^2 \frac{T_{INV}(V_{TH}-V_{FB}-\phi_s)}{\varepsilon_{ox}WL}\right\} \qquad (9)$$

and

$$\exp\left\{-\frac{1}{6}\left(\frac{q^{1.5}}{nkT}\right)^2 \frac{T_{INV}(V_{TH}-V_{FB}-\phi_s)}{\varepsilon_{ox}WL}\right\} \qquad (10)$$

So, the subthreshold MOSFET's $I_d$ under the effect of random dopant fluctuation and process variation cannot be approximated by the normally distributed random variable which its mean, mode and variance are equal to each other, as does in [5]-[7]. According to this observation and the other mentioned above that (1) is ceased to be valid for low voltage/low power designing, the proposed model must be undeniably utilized in order to obtain the complete and accurate analytical description of the low voltage/low power subthreshold operated

MOSFET's $I_d$ affected by the random dopant fluctuation and process variation. By careful observation, it can be found that $M(I_{dr}) < \widetilde{I}_{dr} < \mu_{Idr}$ which resembles the conventional lognormal random variable.

Furthermore, unlike [8] which its results are applicable only to the on-current and leakage current, the proposed model is also applicable to arbitrary values of $I_d$ not only on-current and leakage one because the model derivation is based on arbitrary values of $V_{gs}$, not only $V_{gs} = V_{dd}$ and $V_{gs} = 0$. This is the generality of this model. If desired, the models for such on-current and leakage-current can be obtained from the proposed model as its special cases. By letting $V_{gs} = V_{dd}$, the model for the on-current ($I_{on}$) of the subthreshold MOSFET affected by the random dopant fluctuation and process variation can be obtained from the proposed model as in (11) where $i_{on}$ denotes any sampled value of $I_{on}$. On the other hand, the model for subthreshold MOSFET's leakage-current ($I_{leak}$) under the similar effect can also be obtained from the proposed model by letting $V_{gs} = 0$ as in (12) where $i_{leak}$ denotes any sampled value of $I_{leak}$.

$$f_{I_{on}}(i_{on}) = \sqrt{\frac{3}{2\pi}}\frac{nkT}{|i_{on}|(V_{dd}-V_{TH})}\left\{\frac{\varepsilon_{ox}WL}{q^3 T_{INV}(V_{TH}-V_{FB}-\phi_s)}\right\}^{0.5}$$
$$\times \exp\left[-\frac{\sqrt{3}}{2}\frac{nkT}{q^{1.5}}\left\{\frac{\varepsilon_{ox}WL}{T_{INV}(V_{TH}-V_{FB}-\phi_s)}\right\}^{0.5}\right] \qquad (11)$$
$$\times \ln\left[\left\{\mu C_{dep}\frac{W}{L}\left(\frac{kT}{q}\right)^2\left\{\exp\left[\frac{q(V_{dd}-V_{TH})}{nkT}\right]\right\}\right.\right.$$
$$\left.\left.\times\left\{1-\exp\left[\frac{qV_{ds}}{kT}\right]\right\}\right\}^{-1}|i_{on}|\right]$$

$$f_{I_{leak}}(i_{leak}) = -\sqrt{\frac{3}{2\pi}}\frac{nkT}{|i_{leak}|V_{TH}}\left\{\frac{\varepsilon_{ox}WL}{q^3 T_{INV}(V_{TH}-V_{FB}-\phi_s)}\right\}^{0.5}$$
$$\times \exp\left[-\frac{\sqrt{3}}{2}\frac{nkT}{q^{1.5}}\left\{\frac{\varepsilon_{ox}WL}{T_{INV}(V_{TH}-V_{FB}-\phi_s)}\right\}^{0.5}\right] \qquad (12)$$
$$\times \ln\left[\left\{\mu C_{dep}\frac{W}{L}\left(\frac{kT}{q}\right)^2\left\{\exp\left[-\frac{qV_{TH}}{nkT}\right]\right\}\right.\right.$$
$$\left.\left.\times\left\{1-\exp\left[\frac{qV_{ds}}{kT}\right]\right\}\right\}^{-1}|i_{leak}|\right]$$

Finally, the corresponding models for the other parameters such as $g_m$ and $g_{ds}$ etc., of the random dopant fluctuation/process variation affected subthreshold operated MOSFET can be found by using the proposed model. This is because these

parameters can be related to $I_d$ for examples $g_m = \partial I_d / \partial V_{gs}$ and $g_{ds} = \partial I_d / \partial V_{ds}$ etc., hence, the corresponding models of such parameters in term of the probability density function can be found by using the proposed model. As the illustration, it can be seen that $g_m$ under the effect of random dopant fluctuation and process variation can be respectively given based on (1) and the above definition as follows

$$g_m = \frac{dI_d}{dV_{gs}}$$
$$= \frac{\mu}{n} C_{dep} \frac{W}{L} \left( \frac{kT}{q} \right)^2 \exp\left[ \frac{q(V_{gs} - V_{TH})}{nkT} \right] \qquad (13)$$
$$\times \left[ 1 - \exp\left[ -\frac{qV_{ds}}{kT} \right] \right] \exp\left[ \frac{q(V_t - V_{TH})}{nkT} \right]$$

It can be seen that the corresponding model for $g_m$ denoted by $f_{gm}(\gamma_m)$ can be analytically derived by solving the following equation formulated via the usage of the proposed model i.e. $f_{Id}(i_d)$.

$$\int_{-\infty}^{\gamma_m} f_{g_m}(v) |dv| = \int_{-\infty}^{g_m^{-1}(\gamma_M)} f_{Id}(u) |du| \qquad (14)$$

where $\gamma_m$ denotes any sampled value of $g_m$. On the other hand, u and v are dummy variables. At this point, it can be seen that this $g_m$ modeling can be simply performed by the usage of the proposed model. In the subsequent section, the verification of the proposed model will be mentioned.

## 3  The Verification

The verification of the proposed model has been performed in both qualitative and quantitative aspects on both NMOS and PMOS technologies at the nanometer regime based on the 65 nm CMOS process technology. For the qualitative verification, the estimated distribution of the percentage that $\ln(I_d)$ deviated from its nominal value $\ln(I_{d,Nom})$, obtained from the model, has been graphically compared to that of the similar metric obtained from the Monte-Carlo SPICE simulation of the diode connected subthreshold region operated MOSFET modelled by BSIM4 (SPICE LEVEL 54) which have been chosen as the benchmark circuit. Let such verification metric be denoted by $\delta\ln(I_d)$, it can be mathematically defined as

$$\delta\ln(I_d) = \frac{\ln(I_d) - \ln(I_{d,Nom})}{\ln(I_{d,Nom})} \times 100\% \qquad (15)$$

where $I_{d,Nom}$ which denotes the nominal value of $I_d$ can be mathematically given by

$$I_{d,Nom} = \mu C_{dep} \frac{W}{L} \left( \frac{kT}{q} \right)^2 \exp\left[ \frac{q(V_{gs} - V_{TH})}{nkT} \right] \left[ 1 - \exp\left[ -\frac{qV_{ds}}{kT} \right] \right] \quad (16)$$

This metric has been chosen because it is convenient to perform the comparative distribution plots of $\delta\ln(I_d)$ as it is also expected to be normally distributed since it is a linear function of $\ln(I_d)$ which is expected to has a normal distribution as the proposed model predicts a lognormal distribution for $I_d$. It should be mentioned here that the benchmark circuit for the verification on the NMOS technology is depicted in Fig.1 while that with PMOS based diode replaces by the NMOS one is used for the verification on the PMOS technology as shown in Fig.2.

On the other hand, for the quantitative point of view, as the proposed model predicts that $I_d$ has a lognormal distribution with parameters given by (4) and (5), the chosen strategy is to validate the acceptance of the hypothesis that $\ln(I_d)$ is normally distributed with mean and standard deviation given by (4) and (5) respectively and the KS-test has been found to be applicable for this research according to suggestion in [16] that the KS-test is powerful for the normality test. Furthermore, its statistic is much simpler than the other powerful tests such as Cramer-von Mises test and Anderson-Darling test [17]. If the mentioned hypothesis is accepted at a certain confidence then it is verified that $I_d$ has a lognormal distribution with parameters given by (4) and (5) at the similar confidence. As this is the prediction obtained from the proposed model, the accuracy of the model is immediately verified with such confidence.

Since the KS-test relies on the cumulative distribution function, it is worthy to derive such function of $\ln(I_d)$ at this point. By using the proposed model, the probability density function of $\ln(I_d)$ and the cumulative distribution function of $\ln(I_d)$ can be respectively given by

$$f_{\ln(I_d)}[\ln(i_d)] = \sqrt{\frac{3}{2\pi}} nkT \left\{ \frac{\varepsilon_{ox} WL}{q^3 T_{INV}(V_{TH} - V_{FB} - \phi_s)} \right\}^{0.5}$$

$$\exp\left[ -\frac{1}{2} \left\{ \ln\left[ i_{dr} \left\{ \mu C_{dep} \frac{W}{L} \left( \frac{kT}{q} \right)^2 \right. \right. \right. \right. \quad (17)$$

$$\left\{ \exp\left[ \frac{q(V_{gs} - V_{TH})}{nkT} \right] \right\} \left\{ 1 - \exp\left[ \frac{qV_{ds}}{kT} \right] \right\}^{-1} \right\}^2$$

$$\left\{ \left( \frac{nkT}{q^3} \right)^2 \frac{WL}{3\varepsilon_{ox} T_{INV}(V_{TH} - V_{FB} - \phi_s)} \right\} \right]$$

and

$$F_{\ln(I_d)} \ln(i_d) = \int_{-\infty}^{\ln(i_d)} f_{\ln(I_d)}(x)dx$$

$$= \frac{1}{2} \left\{ 1 + erf\left[ \frac{nkT}{\sqrt{2}q^{1.5}} \left\{ \frac{3\varepsilon_{ox} WL}{T_{INV}(V_{TH} - V_{FB} - \phi_s)} \right\}^{0.5} \right. \right.$$

$$\ln\left[ i_{dr} \left\{ \mu C_{dep} \frac{W}{L} \left( \frac{kT}{q} \right)^2 \right. \right. \quad (18)$$

$$\left. \times \left\{ \exp\left[ \frac{q(V_{gs} - V_{TH})}{nkT} \right] \right\} \left\{ 1 - \exp\left[ \frac{qV_{ds}}{kT} \right] \right\}^{-1} \right] \right]$$

where $erf(x)$ denotes the error function of any arbitrary variable, $x$ which can be mathematically defined as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2)du. \quad (19)$$

According to [16], [18] and [19], the concept of the KS-test is to performed the comparison of the obtained test statistic (KS) and the critical value (c) where it can be stated that any model fits its target data set if and only if its KS is not exceed c. For this research, KS can be defined as

$$KS = \max_{\ln(i_d)} \left\{ \left| F_{\ln(I_d)}[\ln(i_d)] \right|_{circuit} - \left| F_{\ln(I_d)}[\ln(i_d)] \right|_{model} \right\} \quad (20)$$

where $F_{\ln(Id)}[\ln(i_d)]|_{circuit}$ and $F_{\ln(Id)}[\ln(i_d)]|_{model}$ represent the estimated cumulative distribution function of $\ln(I_d)$ obtained from the proposed model and that of the similar quantity obtained from the benchmark circuit respectively. Alternatively, if the Cramer-von Mises test has been chosen, the corresponding test statistic will be given according to [17] by

$$W^2 = N \int_{<all\ln(i_d)>} [F_{\ln(I_d)}[\ln(i_d)]|_{circuit} - F_{\ln(I_d)}[\ln(i_d)]|_{model}]^2$$

$$d F_{\ln(I_d)}[\ln(i_d)]|_{model} \quad (21)$$

where N denotes the numbers of observation. On the other hand, if the Anderson-Darling test has been chosen, the test statistic for this research can be given according to [17] as follows

$$A^2 = N \int_{<all\ln(i_d)>} [F_{\ln(I_d)}[\ln(i_d)]|_{circuit} - F_{\ln(I_d)}[\ln(i_d)]|_{model}]^2$$

$$\Psi[\ln(i_d)] d F_{\ln(I_d)}[\ln(i_d)]|_{model} \quad (22)$$

where $\Psi[\ln(i_d)]$ which denotes the weighting function can be given for this research based on [17] by

$$\Psi[\ln(i_d)] = \frac{1}{F_{\ln(I_d)}[\ln(i_d)]|_{model} \{1 - F_{\ln(I_d)}[\ln(i_d)]|_{model}\}} \quad (23)$$

It can be suddenly seen that the statistic of the Kolmogorov-Smirnov test given by (20) is the simplest one so, this test has been chosen due to such simplicity. Furthermore, as the confidence level of the test is 99% or α = 0.01 in the other words, c can be given by [19]

$$c = \frac{1.63}{\sqrt{n_M}} \quad (24)$$

where $n_M$ denotes the number of Monte-Carlo analysis runs. Before proceed further, it should be mentioned here that $n_M = 1000$ which yields c = 0.0163. In the upcoming sections, the verification of the model on both NMOS and PMOS technologies will be discussed.
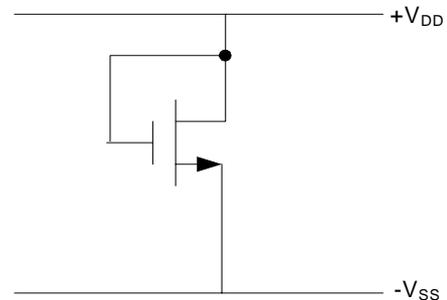


Fig.1 The NMOS diode connected transistor
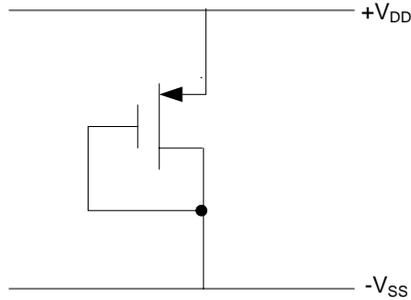
Fig.2 The PMOS diode connected transistor

### 3.1 NMOS based model verification

As the qualitative verification, the graphical comparison for the distribution of $\delta\ln(I_d)$, is depicted in Fig.3 which a strong agreement between the model based distribution and the interpolated benchmark circuit based one can be observed. Hence, the NMOS based qualitative verification of the proposed model at the nanometer level gives a result with satisfaction.

For the quantitative verification, it can be seen by using (20) that the resulting test statistic can be found as $KS = 0.01377$ which is smaller than $c = 0.0163$ This can be interpret in the straight forward manner that the hypothesis that $\ln(I_d)$ is normally distributed with mean and standard deviation given by (4) and (5) respectively is accepted with 99% confidence. According to the interpretation of the verification result mentioned above, it can be seen that the accuracy of the proposed model has been verified with 99% confidence. At this point, the nanometer regime NMOS based verification of the proposed model has been accomplished in both aspects.

### 3.2 PMOS based model verification

As the qualitative verification, the similar graphical comparison for the distribution of $\delta\ln(I_d)$, is depicted in Fig.3 which a strong agreement between the model based distribution and the interpolated benchmark circuit based one can also be seen. Hence, the nanometer level PMOS based qualitative verification of the model also gives a result with satisfaction.

For the quantitative verification, it can be seen by also using (20) that the resulting test statistic can be given by $KS = 0.01193$ which is also smaller than $c = 0.0163$. By the interpretation in the similar manner to the previous subsection, the proposed model is accurate with 99% confidence. At this

point, the PMOS based verification of the proposed model based on the nanometer level CMOS technology has been accomplished in both aspects.

Finally, it can be concluded that the proposed model is better fit to the PMOS technology than the NMOS one due to the closer agreement seen in the PMOS based comparative plot and the corresponding smaller KS statistic.
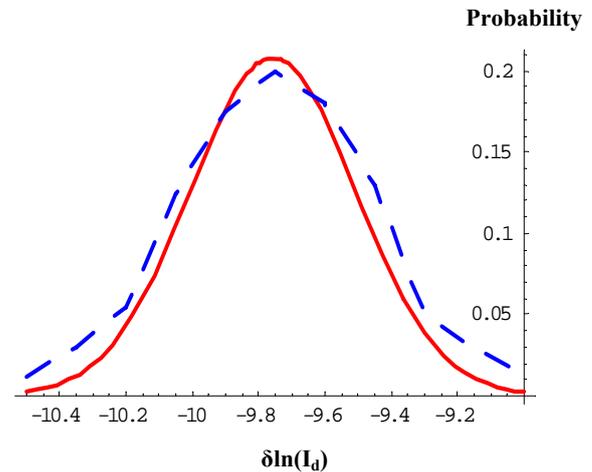


Fig.2. NMOS based comparative distribution plot: The model based (line) v.s. The transistor based (dash)
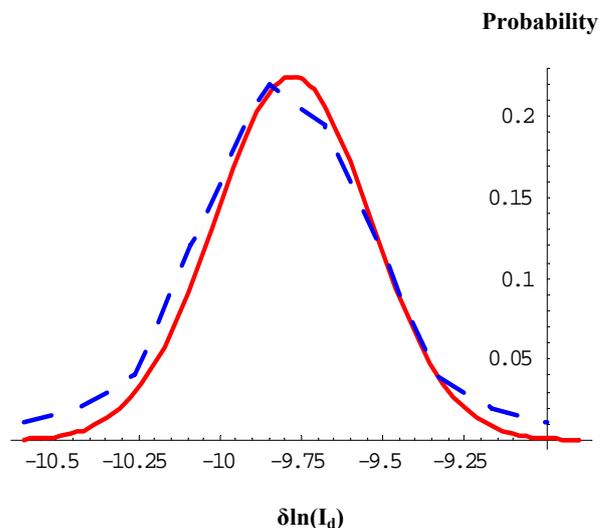


Fig.3. PMOS based comparative distribution plot: The model based (line) v.s. The transistor based (dash)

## 4 Application for Mismatch Modeling

By using the proposed model, various statistical information of the subthreshold MOSFET's $I_d$ such as mean, median, mode, variance and moments, can be obtained analytically. However, these quantities are for a single transistor. For the mismatch in $I_d$ of two or more MOSFETs, the modeling can be performed by using the proposed model as the

mathematical foundation. This is an interesting application of the proposed model apart from the simple ones such as being the basis for the calculation of statistical parameters etc. In order to do so, let such mismatch in $I_d$ be defined as $\Delta I_d$, its variance which is used for this mismatch modeling can be given by

$$\sigma_{\Delta Id}^2 = 2\sigma_{Id}^2(1 - \rho_{Id,circ}) \qquad (25)$$

where $\sigma_{Id}^2$ and $\rho_{Id,circ}$ denote variance of $I_d$ for a single transistor and the drain current correlation coefficient of the MOSFETs within any interested circuit respectively. With the proposed model, $\sigma_{Id}^2$ can be found as follows

$$\sigma_{Id}^2 = \left\{ \exp\left[ \left( \frac{q^{1.5}}{nkT} \right)^2 \left\{ \frac{T_{INV}(V_{TH} - V_{FB} - \phi_s)}{3\varepsilon_{ox}WL} \right\} \right] - 1 \right\}$$
$$\times \exp\left[ 2\ln\left[ \mu C_{dep} \frac{W}{L} \left( \frac{kT}{q} \right)^2 \left\{ \exp\left[ \frac{q(V_{gs} - V_{TH})}{nkT} \right] \right\} \right. \right. \qquad (26)$$
$$\left. \left. \times \left\{ 1 - \exp\left[ \frac{qV_{ds}}{kT} \right] \right\} \right] + \left\{ \left( \frac{q^{1.5}}{nkT} \right)^2 \left\{ \frac{T_{INV}(V_{TH} - V_{FB} - \phi_s)}{3\varepsilon_{ox}WL} \right\} \right\} \right]$$

On the other hand, $\rho_{Id,circ}$ can also be determined by using the proposed model. For the closely spaced transistors which are strongly correlated, $\rho_{Id,circ}$ can be given by (27). At this point, $\sigma_{\Delta Id}^2$ which model the subthreshold drain current mismatch can be determined by using (25)-(27). It should be mentioned here that this model is suitable for the closely spaced transistors.

$$\rho_{Id,circ}$$
$$= \frac{\mu}{\sigma_{Id}^2} C_{dep} \frac{W}{L} \left( \frac{kT}{q} \right)^2 \left\{ \exp\left[ \frac{q(V_{gs} - V_{th})}{nkT} \right] \right\}$$
$$\left\{ 1 - \exp\left[ \frac{qV_{ds}}{kT} \right] \right\} \exp\left[ 2\left( \frac{q^{1.5}}{nkT} \right)^2 \left\{ \frac{T_{INV}(V_{TH} - V_{FB} - \phi_s)}{3\varepsilon_{ox}WL} \right\} \right] \qquad (27)$$
$$- \left\{ \mu C_{dep} \frac{W}{L} \left( \frac{kT}{q} \right)^2 \left\{ \exp\left[ \frac{q(V_{gs} - V_{th})}{nkT} \right] \right\} \right.$$
$$\left. \left\{ 1 - \exp\left[ \frac{qV_{ds}}{kT} \right] \right\} \exp\left[ \frac{1}{2}\left( \frac{q^{1.5}}{nkT} \right)^2 \left\{ \frac{T_{INV}(V_{TH} - V_{FB} - \phi_s)}{3\varepsilon_{ox}WL} \right\} \right] \right\}^2$$

For the distanced transistors which are lowly correlated, $\rho_{Id,circ}$ can be given by

$$\rho_{Id,circ}$$
$$= \frac{\mu}{\sigma_{Id}^2} C_{dep} \frac{W}{L} \left( \frac{kT}{q} \right)^2 \left\{ \exp\left[ \frac{q(V_{gs} - V_{TH})}{nkT} \right] \right\}$$
$$\left\{ 1 - \exp\left[ \frac{qV_{ds}}{kT} \right] \right\} \exp\left[ \left( \frac{q^{1.5}}{nkT} \right)^2 \left\{ \frac{T_{INV}(V_{TH} - V_{FB} - \phi_s)}{3\varepsilon_{ox}WL} \right\} \right] \qquad (28)$$
$$- \left\{ \mu C_{dep} \frac{W}{L} \left( \frac{kT}{q} \right)^2 \left\{ \exp\left[ \frac{q(V_{gs} - V_{TH})}{nkT} \right] \right\} \right.$$
$$\left. \left\{ 1 - \exp\left[ \frac{qV_{ds}}{kT} \right] \right\} \exp\left[ \frac{1}{2}\left( \frac{q^{1.5}}{nkT} \right)^2 \left\{ \frac{T_{INV}(V_{TH} - V_{FB} - \phi_s)}{3\varepsilon_{ox}WL} \right\} \right] \right\}^2$$

So, $\sigma_{\Delta Id}^2$ for such distanced transistors can be determined by using (25), (26) and (28).

It can be observed that $\rho_{Id,\ circ}$ is maximized where as $\sigma_{\Delta Id}^2$ is minimized for closely spaced transistors. For the distanced devices, the opposite results are obtained. This is because the following term is included in $\rho_{Id,\ circ}$ for such closely spaced devices i.e. (27).

$$\exp\left[ 2\left( \frac{q^{1.5}}{nkT} \right)^2 \left\{ \frac{T_{INV}(V_{TH} - V_{FB} - \phi_s)}{3\varepsilon_{ox}WL} \right\} \right] \qquad (29)$$

On the other hand, the corresponding term in $\rho_{Id,\ circ}$ for the distanced transistors i.e. (28), is

$$\exp\left[ \left( \frac{q^{1.5}}{nkT} \right)^2 \left\{ \frac{T_{INV}(V_{TH} - V_{FB} - \phi_s)}{3\varepsilon_{ox}WL} \right\} \right] \qquad (30)$$

From this observation, it can be seen that the best matching can be achieved for the closely spaced devices and getting worse as the distance increased

# 5 Application for Optimization

In this section, some discussion regarding to the other interesting application of the proposed model apart from those mentioned above such as being the basis for the calculation of statistical parameters and the mathematical modeling of the mismatch etc., will be given. Such interesting application is being the mathematical basis for the optimization schemes with the minimization of variation in $I_d$ of the subthreshold MOSFET as the goal. A candidate simple objective function can be given based on the proposed model as follows

$$\min[\left\{\exp\left[\left(\frac{q^{1.5}}{nkT}\right)^2\left\{\frac{T_{INV}(V_{TH}-V_{FB}-\phi_s)}{3\varepsilon_{ox}WL}\right\}\right]-1\right\}$$

$$\times\exp\left[2\ln\left[\mu C_{dep}\frac{W}{L}\left(\frac{kT}{q}\right)^2\left\{\exp\left[\frac{q(V_{gs}-V_{TH})}{nkT}\right]\right]\right.\right. \quad (31)$$

$$\times\left\{1-\exp\left[\frac{qV_{ds}}{kT}\right]\right\}\right]$$

$$+\left\{\left(\frac{q^{1.5}}{nkT}\right)^2\left\{\frac{T_{INV}(V_{TH}-V_{FB}-\phi_s)}{3\varepsilon_{ox}WL}\right\}\right\}\right]]$$

If the target is to minimize the mismatch in $I_d$ among various transistors then $\sigma^2_{\Delta Id}$ can be used which yields the following objective function.

$$\min[\sigma^2_{\Delta Id}] \qquad (32)$$

where $\sigma^2_{\Delta Id}$ can be derived from the proposed model as mentioned above.

For the optimization of a single MOSFET, a more rigorous objective function in term of probability can be given as follows

$$\max[\Pr\{I_d=I_{d,Nom}\}] \qquad (33)$$

where $\Pr\{I_d=I_{d,Nom}\}$ can be interpreted as the probability of obtaining no variation in $I_d$. According to [13], it can be mathematically defined by using the proposed model as follows

$$\Pr\{I_d=I_{d,Nom}\}=\int_{-\infty}^{I_{d,Nom}}f_{Id}(i_d)di_d-\lim_{\chi\to0}[\int_{-\infty}^{-\chi}f_{Id}(i_d)di_d] \quad (34)$$

where $\chi\neq0$ even though it is very closed to zero. In order to save the computational effort, $\Pr\{I_d=I_{d,Nom}\}$ can be given by

$$\Pr\{I_d=I_{d,Nom}\}=\lim_{\chi\to0}[\int_{-\chi}^{I_{d,Nom}}f_{Id}(i_d)di_d] \qquad (35)$$

In a more relax manner, the goal of the optimization can be simply keeping $I_d$ within its acceptable predetermined boundaries denoted by $I_{d,min}$ and $I_{d,max}$ for the lower and upper bounds respectively. Based on the proposed model, these boundaries can be given by

$$I_{d,min}=\mu C_{dep}\frac{W}{L}\left(\frac{kT}{q}\right)^2\exp\left[\frac{q(V_{gs}-V_{TH})}{nkT}\right]$$

$$\times\left[1-\exp\left[-\frac{qV_{ds}}{kT}\right]\right] \qquad (36)$$

$$\times\exp[-\frac{\sqrt{3}q^{1.5}}{nkT}\left\{\frac{T_{INV}(V_{TH}-V_{FB}-\phi_s)}{\varepsilon_{ox}WL}\right\}^{0.5}]$$

and

$$I_{d,max}=\mu C_{dep}\frac{W}{L}\left(\frac{kT}{q}\right)^2\exp\left[\frac{q(V_{gs}-V_{TH})}{nkT}\right]$$

$$\times\left[1-\exp\left[-\frac{qV_{ds}}{kT}\right]\right] \qquad (37)$$

$$\times\exp[\frac{\sqrt{3}q^{1.5}}{nkT}\left\{\frac{T_{INV}(V_{TH}-V_{FB}-\phi_s)}{\varepsilon_{ox}WL}\right\}^{0.5}]$$

For this less rigorous optimization, the following objective function has been found to be promising

$$\max[\Pr\{I_{d,min}\le I_d\le I_{d,max}\}] \qquad (38)$$

where $\Pr\{I_{d,min}\le I_d\le I_{d,max}\}$ denotes the probability of obtaining $I_d$ within the acceptable boundary. $\Pr\{I_{d,min}\le I_d\le I_{d,max}\}$ can be simply derived by using the proposed models as follows

$$\Pr\{I_{d,min}\le I_d\le I_{d,max}\}=\int_{I_{d,min}}^{I_{d,max}}f_{Id}(i_d)di_d \qquad (39)$$

At this point, it can be seen that the proposed model can serve as the efficient mathematical basis for the optimization in the designing of subthreshold MOSFET based circuits and systems affected by the random dopant fluctuation and process variation

# 6 Conclusion

The novel comprehensive probabilistic analytical model of the subthreshold MOSFET's $I_d$ under the effect of both random dopant fluctuation and process variation effects which are the major causes of the random variation in the MOSFET characteristic, have been proposed. The up to dated Takeuchi's physical level random variation model [9] has been adopted as the modeling basis. The proposed model has been found to be analytic and powerful as it is in the form of the probability density function but with more comprehensiveness

than that in [4] and without loss of accuracy as those in [5]-[7]. The proposed model is applicable to the arbitrary $I_d$ at any $V_{gs}$ unlike [8]. This model is also unsuspicious in the low voltage/low power scenario as the doubtful approximation has not been used.

This model has been verified with the 65 nm CMOS process by using the BSIM4 based Monte-Carlo simulations. The verifications have been performed based on both NMOS and PMOS technologies. These models are very accurate since they can closely follow the Monte-Carlo based distributions with satisfaction guaranteed goodness of fit test results. Furthermore, the proposed model can also serve as the basis for the mismatch modeling and performance optimization as mentioned above. Hence, the proposed models have been found to be the potential mathematical tool for the statistical/variability aware analysis/design of various subthreshold region operated MOSFET based low voltage/low power circuits and systems.

Finally, as this research is focused on the conventional MOSFET, it is worthy to perform the similar modeling for those unconventional such as multiple input floating-gate MOSFET [20] etc., which have been adopted in various subthreshold region operation based applications for example [21]-[23] etc.

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