The Modified Möbius Function.

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Abstract: The Möbius function $\mu(n)$ arises naturally in Number Theory when one inverts the classical Riemann Zeta function.
In my paper *Modifying Möbius* [1], I modified the classical Möbius function and produced a number of interesting results such as

$$\left|\sum_{n=1}^{\infty} \frac{(-i)^{\Omega(n)}}{n^2}\right|^2 = \frac{\pi^4}{105},$$

where $\Omega(n)$ counts, with multiplicity, the number of prime factors of n, and

$$\left|\sum_{n=1}^{\infty} \frac{(1+i)^{\omega(n)}}{n^2}\right|^2 = \frac{35}{12},$$

where $\omega(n)$ counts the number of distinct prime factors of n.
In this paper, I present some further arithmetic and analytic results based on these ideas.

Key–Words: Möbius function, arithmetic functions, Riemann-Zeta function.

1 Introduction

Write, once and for all, $n = \prod_{j}^{r} p_j^{\alpha_j}$ as the canonical prime factorisation of a positive integer $n > 1$. For a given n, this defines the numbers r, α_j and primes p_j.

The classical Möbius function, $\mu(n)$, is defined by

$$\mu(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n \text{ has a square factor (except 1)} \\ (-1)^r & \text{if} \quad n = p_1 p_2 \dots p_r \end{cases}$$

The modified Möbius function, $\mu_1(n)$, is defined exactly as above, except that the number -1 is replaced by i. That is:

$$\mu_1(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n \text{ has a square factor (except 1)} \\ i^r & \text{if} \quad n = p_1 p_2 \dots p_r \end{cases}$$

It is obvious that $\mu_1(n)$ is multiplicative, but not completely multiplicative.

As usual, for $n > 1$, we write $\omega(n)$ for the number r defined above, $\Omega(n)$ for $\sum_{j=1}^{r} \alpha_j$ and set $\omega(1) = \Omega(1) = 0$.

We define the arithmetic functions:

- $u(n) = 1$ for all n,
- $I(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$

Also, as usual we define the *Dirichlet Product*, $f * g$ of two arithmetic functions, f, g by

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right) = \sum_{d|n} f\left(\frac{n}{d}\right) g(d).$$

The function g is called the *Dirichlet Inverse* of f if $f * g = I$.

Since $\mu_1(n)^2 = \mu(n)$, it immediately follows from the well-known result for μ that

$$\sum_{d|n} \mu_1^2(d) = 0.$$

Note also that $\mu(n)\mu_1(n) = \overline{\mu_1(n)}$.

2 Arithmetic Properties of μ_1.

Theorem 1 *For $n > 1$,*

$$\sum_{d|n} \mu_1(d) = (1+i)^{\omega(n)}.$$

Proof: See [1], p. 245. □

The function $h(n) = (1+i)^{\omega(n)}$ will arise in several places further in this paper. The Dirichlet inverses of μ_1 and of h are easy to compute.

Lemma 1 a. The Dirichlet inverse for μ_1 is given by

$$\mu_1^{-1}(n) = (-i)^{\Omega(n)};$$

b. With $h(n)$ defined above, the Dirichlet inverse, $h^{-1}(n)$ is given by

$$h^{-1}(n) = (1-i)^{\omega(n)}(-i)^{\Omega(n)} = \overline{h(n)}\mu_1^{-1}(n).$$

Proof: a. Since μ_1 and its supposed inverse are multiplicative, it suffices to show that $\sum_{d|n} \mu_1\left(\frac{n}{d}\right)(-i)^{\Omega(d)}$ is zero when $n = p^\alpha$, for p a prime and $\alpha > 0$ and 1 when $n = 1$. This is an easy exercise.

b. Since h and its supposed inverse are multiplicative, it suffices to show that
$\sum_{d|n} h\left(\frac{n}{d}\right)(1-i)^{\omega(d)}(-i)^{\Omega(d)}$ is zero when $n = p^\alpha$, for p a prime and $\alpha > 0$ and 1 when $n = 1$. This is an easy exercise. □

2.1 Analogs of the Arithmetic Functions:

The arithmetic functions,

$$\tau(n) = \sum_{d|n} 1, \qquad \sigma(n) = \sum_{d|n} d, \qquad \phi(n) = \sum_{(d,n)=1} 1$$

are all intimately connected with the Möbius function. Hence, we would expect the modified Möbius function to produce a new suite of analogous arithmetic functions.

2.1.1 Analogs of u and τ.

Since $u = \mu^{-1}$, we define $u_1 = \mu_1^{-1} = (-i)^\Omega$ and note that $|u_1(n)| = u(n)$.

Also, since $\tau = u * u$, we define $\tau_1 = \mu_1^{-1} * \mu_1^{-1}$, then, since τ_1 is multiplicative, and

$$\tau_1(p^\alpha) = \sum_{d|p^\alpha} \mu_1^{-1}(d)\mu_1^{-1}\left(\frac{p^\alpha}{d}\right)$$

$$= \sum_{d|p^\alpha} (-i)^{\Omega(d)+\Omega(\frac{p^\alpha}{d})} = (\alpha+1)(-i)^\alpha,$$

we have

$$\tau_1(n) = \prod_j (\alpha_j+1)(-i)^{\alpha_j} = (-i)^{\Omega(n)}\tau(n).$$

Thus $|\tau_1(n)| = \tau(n)$.

In consequence of our definition, we have $\tau_1 * \mu_1 = (-i)^\Omega = u_1$.

2.1.2 Analogs of N and σ.

We replace the (completely multiplicative) function N, defined by $N(n) = n$, by $N_1(n) = (-i)^{\Omega(n)}n$. Since $\sigma = N * u$, we define $\sigma_1 = N_1 * u_1$. Since σ_1 is multiplicative, and

$$\sigma_1(p^\alpha) = \sum_{d|p^\alpha} N_1(d)\mu_1^{-1}\left(\frac{p^\alpha}{d}\right) = \sum_{d|p^\alpha} d(-i)^{\Omega(d)+\Omega(\frac{p^\alpha}{d})}$$

$$= (-i)^\alpha(1+p+p^2+\cdots+p^\alpha) = (-i)^\alpha\left(\frac{p^{\alpha+1}-1}{p-1}\right),$$

we have

$$\sigma_1(n) = \prod_j (-i)^{\alpha_j}\left(\frac{p_j^{\alpha_j+1}-1}{p_j-1}\right) = (-i)^{\Omega(n)}\sigma(n).$$

Note again that $|\sigma_1(n)| = \sigma(n)$.

In consequence of our definition, we have $\sigma_1 * \mu_1 = N_1$.

2.1.3 Analog of ϕ.

Since $\phi = N * \mu$, we define $\phi_1 = N_1 * \mu_1$. Since ϕ_1 is multiplicative, and

$$\phi_1(p^\alpha) = \sum_{d|p^\alpha} N_1(d)\mu_1\left(\frac{p^\alpha}{d}\right)$$

$$= \sum_{d|p^\alpha} \mu_1(1)N_1(p^\alpha) + \mu_1(p)N_1(p^{\alpha-1})$$

$$= (-i)^\alpha p^\alpha \left(1 - \frac{1}{p}\right),$$

we have

$$\phi_1(n) = n \prod_j (-i)^{\alpha_j} \left(1 - \frac{1}{p_j}\right)$$

$$= (-i)^{\Omega(n)} \phi(n).$$

Note again that $|\phi_1(n)| = \phi(n)$.

2.1.4 Analog of λ.

The classical Liouville function λ is defined by

$$\lambda(n) = (-1)^{\Omega(n)}.$$

It is not immediately clear what the *correct* analogue should be. In order to obtain results analogous to the classical results, it seemed best to define:

$$\lambda_1(n) = i^r \qquad \text{where } n = M^2(p_1p_2...p_r).$$

When n is square, we take $\lambda(n) = 1$.
It is immediately clear that λ_1 is multiplicative.

Theorem 2 *For $n \geq 1$,*

$$\lambda_1(n) = \sum_{d^2|n} \mu_1\left(\frac{n}{d^2}\right).$$

Proof: It suffices to prove that these expressions agree at prime powers.

$$\sum_{d^2|n} \mu_1\left(\frac{n}{d^2}\right) = \mu_1(p^\alpha) + \mu_1(p^{\alpha-2}) + ...$$

This is $\mu_1(1) = 1 = \lambda_1(p^\alpha)$ in the case when α is even and $\mu_1(p) = i = \lambda_1(p^\alpha)$ in the case when α is odd. □

By analogy with the classical case, we have

Theorem 3 *For $n \geq 1$,*

$$(\lambda_1 * u_1)(n) = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise} \end{cases}$$

Proof: Since the left-hand side is a multiplicative function, it again suffices to check the prime powers.

$$(\lambda_1 * u_1)(p^\alpha) = \sum_{d|p^\alpha} \lambda_1(d)u_1(\frac{p^\alpha}{d})$$

$$= \sum_{d^2|p^\alpha} \lambda_1(d^2)u_1(\frac{p^\alpha}{d^2}) + \sum_{\substack{d|p^\alpha \\ d \text{ not square}}} \lambda_1(d)u_1(\frac{p^\alpha}{d}).$$

In the case that α is even, these sums become:

$$u_1(p^\alpha) + u_1(p^{\alpha-2}) + ...$$

$$+u_1(1) + \lambda_1(p)u_1(p^{\alpha-1}) + \lambda_1(p^3)u_1(p^{\alpha-3}) + ...$$

$$+\lambda_1(p^{\alpha-1})u_1(p)$$

$$= 1.$$

since, when evaluated, all the terms cancel except $u(1) = 1$. Similarly, the case α odd gives a sum of zero. □

Theorem 4 *For $n > 1$,*

$$\sum_{d|n} \lambda_1(d)i^{\Omega(d)} = \begin{cases} -1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise.} \end{cases}$$

Proof: This is simply checked by again considering the sum at prime powers. □

2.2 Möbius Inversion Analog:

The analog of Möbius inversion is provided by:

Theorem 5 *Suppose the arithmetic functions F and f are connected by*

$$F(n) = \sum_{d|n} f(d)(-i)^{\Omega(\frac{n}{d})},$$

then for $n > 1$,

$$f(n) = (F * \mu_1)(n), \text{ that is,}$$

$$f(n) = \sum_{d|n} F(\frac{n}{d})\mu_1(d) = \sum_{d|n} F(d)\mu_1\left(\frac{n}{d}\right).$$

Proof: Since $F(n) = \sum_{d|n} f(d)(-i)^{\Omega(\frac{n}{d})}$, can be written as $F = f * u_1$, we have

$$F * \mu_1 = (f * u) * \mu_1 = f * (u * \mu_1) = f.$$

Translating back, the result follows. □

2.3 Connections with the classical Arithmetic Functions.

There are many relations between these analogs and the classical functions. Most of these are just exercises, so I have only included a small number of the more important connections.

Theorem 6 *Suppose the arithmetic functions F and f are connected by*

$$F(n) = \sum_{d|n} f(d),$$

then for $n > 1$,

$$(F * \mu_1)(n) = (f * h)(n), \text{ that is,}$$

$$\sum_{d|n} F(\frac{n}{d})\mu_1(d) = \sum_{d|n} f(\frac{n}{d})h(d).$$

Proof: We can write Theorem 1 as
$u * \mu_1(n) = (1+i)^{\omega(n)} = h(n)$.
Also, $F(n) = \sum_{d|n} f(d)$, can be written as $F = f * u$ and so

$$F * \mu_1 = (f * u) * \mu_1 = f * (u * \mu_1) = f * h.$$

Translating back, the result follows. □

Theorem 7 *For $n > 1$,*

$$\mu_1(n) = \sum_{d|n} \mu(\frac{n}{d})h(d) = (\mu * h)(n).$$

Proof:
Use ordinary Möbius inversion on the result of Theorem 1. □

Theorem 8 *For $n > 1$,*

$$(\mu_1 * \tau)(n) = \sum_{d|n} \mu_1(d)\tau(\frac{n}{d}) = \sum_{d|n} (1+i)^{\omega(d)}$$

$$= \sum_{d|n} h(d).$$

Proof: This follows from a direct application of Theorem 6 □

Theorem 9 *For $n > 1$,*

$$\sum_{d|n} \frac{\mu_1(d)}{d} = \frac{1}{n}\sum_{n|d} \phi\left(\frac{n}{d}\right)(1+i)^{\omega(d)} = \frac{1}{n}(\phi * h)(n)$$

$$= \prod_k \left(1 + \frac{i}{p_k}\right).$$

Proof: From the well-known result, $\sum_{d|n} \phi(d) = n$, a direct application of Theorem 6 yields,

$$\sum_{d|n} \frac{n}{d}\mu_1(d) = \sum_{n|d} \phi\left(\frac{n}{d}\right)(1+i)^{\omega(d)}.$$

The second equality can be obtained easily by evaluating the first sum at prime powers. □

3 Analytic Theorems regarding μ_1.

We now consider the partial sums related to μ_1 and try to obtain some preliminary asymptotics.

Theorem 10 *For $x > 1$*

$$\sum_{n \le x} \mu_1(n)\left[\frac{x}{n}\right] = \sum_{n \le x} (1+i)^{\omega(n)} = \sum_{n \le x} h(n).$$

Proof:

$$\sum_{n \le x} \mu_1(n)\left[\frac{x}{n}\right] = \sum_{n \le x} \mu_1(n) \sum_{j \le \frac{x}{n}} 1 = \sum_{n \le x} \sum_{d|n} \mu_1(d)$$

$$= \sum_{n \le x} (1+i)^{\omega(n)}$$

from Theorem 1. □

(Note: The penultimate equality uses the so-called *sum-divisor identity*:
viz $\sum_{n \le x} g(n) \sum_{j \le \frac{x}{n}} f(nj) = \sum_{n \le x} f(n) \sum_{d|n} g(d)$.)

The following result gives the Dirichlet series for $h(n)$.

Corollary For $x > 1$,

$$\sum_{n \le x} \mu_1(n)\left[\frac{x}{n}\right]^2 = \frac{6x^2}{\pi^2}\sum_{n=1}^{\infty} \frac{h(n)}{n^2} + O(x \log x).$$

Proof:

$$\sum_{n \le x} \mu_1(n)\left[\frac{x}{n}\right]^2 = \sum_{n \le x} \mu_1(n)\left(\frac{x}{n} + O(1)\right)^2$$

$$= \sum_{n \le x} \mu_1(n)\frac{x^2}{n^2} + O\left(\sum_{n \le x} \frac{x}{n}\right)$$

$$= x^2 \sum_{n \le x} \frac{\mu_1(n)}{n^2} + O\left(x \sum_{n \le x} \frac{1}{n}\right)$$

$$= \frac{6x^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(1+i)^{\omega(n)}}{n^2} + O(x \log x)$$

using the previous theorem and standard asymptotic results. □

3.1 The Sum $\sum_{n\le x} h(n)$.

Theorem 11 *For $x > 1$,*

$$\left|\sum_{n\le x} h(n)\right| = O(x\log x).$$

Proof: From Theorem 10,

$$\left|\sum_{n\le x}(1+i)^{\omega(n)}\right| = \left|\sum_{n\le x}\mu_1(n)\left[\frac{x}{n}\right]\right| \le \sum_{n\le x}\left[\frac{x}{n}\right]$$

$$\le x\sum_{n\le x}\frac{1}{n} = O(x\log x).$$

□

Note: It is not clear that this is the best estimate. Despite this, numerical evidence suggests that the result of Theorem 11 has an implied constant about 0.2.

3.2 Series involving λ_1

Using Theorem 3 and the fact that $\sum_{n\le x}\sum_{d|n} f(d) = \sum_{n\le x} f(n)\lfloor\frac{x}{n}\rfloor$ for any arithmetic function f and for $x > 0$, we can prove:

Theorem 12 *For $x > 4$*

$$\left|\sum_{n\le x}\frac{\lambda_1(n)i^{\Omega(n)}}{n}\right| \le \frac{3}{2}.$$

Proof: Applying the result above, we have

$$\sum_{n\le x}\lambda_1(n)i^{\Omega(n)}\lfloor\frac{x}{n}\rfloor = \sum_{n\le x}\sum_{d|n}\lambda_1(n)i^{\Omega(n)}.$$

By Theorem 3, the inner sum is zero whenever n is not a square and -1 when it is, so we can replace the double sum by $-\lceil\sqrt{x}\rceil$. Also, writing $\lfloor\frac{x}{n}\rfloor$ as $\frac{x}{n} - \left\{\frac{x}{n}\right\}$ and re-arranging we have

$$\left|x\sum_{n\le x}\frac{\lambda_1(n)i^{\Omega(n)}}{n}\right| = \left|\sum_{n\le x}\lambda_1(n)i^{\Omega(n)}\left\{\frac{x}{n}\right\} - \lceil\sqrt{x}\rceil\right|$$

$$\le x + \lceil\sqrt{x}\rceil \le \frac{3}{2}x$$

since for $x \ge 4$, $\sqrt{x} \le \frac{1}{2}x$. Dividing by x gives the result. □

4 The sum $\sum_{n=1}^{\infty}\frac{\mu_1(n)}{n^s}$ and the analog of the Zeta Function.

Suppose $s > 1$.

We define the *modified Zeta function*, $\zeta_1(s)$, by:

$$\zeta_1(s) = \sum_{n=1}^{\infty}\frac{\mu_1^{-1}(n)}{n^s} = \sum_{n=1}^{\infty}\frac{(-i)^{\Omega(n)}}{n^s},\ \text{for } s > 1,$$

with Euler product $\prod_p\left(1 - \frac{1}{ip^s}\right)^{-1}$.

Also, $\frac{1}{\zeta_1(s)} = \sum_{n=1}^{\infty}\frac{\mu_1(n)}{n^s}$.

These series and products only converge absolutely for $s > 1$.

The following result is useful for numerical calculations

Lemma

For $s > 1$,

$$\sum_{n=1}^{\infty}\frac{\mu_1(n)}{n^s} = (1 - 2^s i)\sum_{n\equiv 2 \bmod 4}\frac{\mu_1(n)}{n^s}.$$

Proof: We note that for k odd, $\mu_1(2k) = i\mu_1(k)$ and that by definition $\mu_1(4k) = 0$. Splitting the series according as n is even or odd, we have

$$\sum_{n=1}^{\infty}\frac{\mu_1(n)}{n^s} = \sum_{\substack{n=1\\ n\,\text{even}}}^{\infty}\frac{\mu_1(n)}{n^s} + \sum_{\substack{n=1\\ n\,\text{odd}}}^{\infty}\frac{\mu_1(n)}{n^s}$$

$$= \left\{\frac{\mu_1(1)}{1^s} + \frac{\mu_1(3)}{3^s} + \frac{\mu_1(5)}{5^s} + ...\right\}$$

$$+ \left\{\frac{\mu_1(2)}{2^s} + \frac{\mu_1(4)}{4^s} + \frac{\mu_1(6)}{6^s} + ...\right\}$$

$$= \frac{2^s}{i}\left\{\frac{\mu_1(2)}{2^s} + \frac{\mu_1(6)}{6^s} + \frac{\mu_1(10)}{10^s} + ...\right\}$$

$$+ \left\{\frac{\mu_1(2)}{2^s} + \frac{\mu_1(6)}{6^s} + \frac{\mu_1(10)}{10^s} ...\right\}$$

$$= (1 - 2^s i)\sum_{n\equiv 2 \bmod 4}\frac{\mu_1(n)}{n^s}.$$

□

Using exactly the same idea, given **any** prime p, we can split the sums into terms with $n \equiv 1, 2, 3, ..., p \bmod p$ and deduce that for $s > 1$

$$\sum_{n=1}^{\infty}\frac{\mu_1(n)}{n^s} = (1 - p^s i)\sum_{\substack{n\equiv 0 \bmod p\\ n>0}}\frac{\mu_1(n)}{n^s}.$$

4.1 Dirichlet Series for $h(s)$.

Theorem 13 *For $s > 1$,*

$$\sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{\mu_1(n)}{n^s}.$$

Proof: See [1], p. 247. □

Theorem 13 can now be written as

$$\left| \sum_{n=1}^{\infty} \frac{h(n)}{n^s} \right|^2 = \frac{\zeta^2(s)\zeta(2s)}{\zeta(4s)},$$

for $s > 1$.

For $s > 1$, we can split the series into even and odd values of n and a simple re-arrangement leads to the identity

$$\sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \left(\frac{2^s + i}{2^s - 1} \right) \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{h(n)}{n^s},$$

valid for $s > 1$.

More generally, if p is prime, we can write

$$\sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \left(\frac{p^s + i}{p^s - 1} \right) \sum_{\substack{n \geq 1 \\ n \not\equiv 0 \bmod p}} \frac{h(n)}{n^s},$$

valid for $s > 1$. From this we have

$$\sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \left(\frac{p^s + 1}{1 + i} \right) \sum_{\substack{n \geq 1 \\ n \equiv 0 \bmod p}} \frac{h(n)}{n^s},$$

valid for $s > 1$.

Note: The Dirichlet series for $\dfrac{\zeta_1(2s)}{\zeta_1(s)}$ does not appear to be analogous to the classical $\dfrac{\zeta(2s)}{\zeta(s)} = \displaystyle\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}$.[1]

Corollary

For $s > 1$,

$$\sum_{n=1}^{\infty} \frac{(1-i)^{\omega(n)}(-i)^{\Omega(n)}}{n^s} = \frac{\zeta(4s)}{\zeta(2s)\zeta(s)}.$$

[1] My Vacation Scholar, Mr. Trent Merbach, discovered and proved the following result: Writing each integer n uniquely in the form $n = N^2M$, with M square-free, we have, for $s > 1$,

$$\frac{\zeta_1(2s)}{\zeta_1(s)} = \sum_{n=1}^{\infty} \frac{i^{\Omega(M)}(-i)^{\Omega(N)}}{n^s}$$

Using the well-known formula for $\zeta(2t)$, t an integer, we also have

Corollary

$$|\zeta_1(t)|^2 = \frac{(-1)^t(2\pi)^{2t}(2t)!}{(4t)!} \frac{B_{4t}}{B_{2t}}$$

where t is an integer ≥ 1, and B_t are the Bernoulli numbers.[2]

Corollary

For t an integer greater than 1, $\left| \dfrac{\zeta_1^2(t)}{\zeta_1(2t)} \right|^2$ is rational.

4.2 Dirichlet series for modified arithmetic functions.

Since $\sigma_1(n) = (-i)^{\Omega(n)}\sigma(n)$, we can use the standard Euler product method to find the Dirichlet series for $\sigma_1(n)$ in terms of the modified Zeta function as follows.

Theorem 14 *For $s > 2$,*

$$\sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n^s} = \zeta_1(s-1)\zeta_1(s).$$

Proof:

$$\prod_p 1 + \frac{\sigma_1(p)}{p^s} + \frac{\sigma_1(p^2)}{p^{2s}} + ...$$

$$= \prod_p 1 - i\frac{\sigma(p)}{p^s} - \frac{\sigma(p^2)}{p^{2s}} + i\frac{\sigma(p^2)}{p^{2s}} + ...$$

$$= \prod_p \frac{p-1}{p-1} - \frac{i}{p^s}\left(\frac{p^2-1}{p-1}\right)$$

$$-\frac{1}{p^{2s}}\left(\frac{p^3-1}{p-1}\right) + \frac{i}{p^{3s}}\left(\frac{p^4-1}{p-1}\right) + ...$$

$$= \prod_p \frac{1}{p-1}\left[p - \frac{i}{p^{s-2}} - \frac{1}{p^{2s-3}} + \frac{i}{p^{3s-4}} + ...\right]$$

$$-\frac{1}{p-1}\left[1 - \frac{i}{p^s} - \frac{1}{p^{2s}} + \frac{i}{p^{3s}} + ...\right]$$

$$= \prod_p \frac{1}{p-1}\left[\frac{p}{1+ip^{1-s}} - \frac{1}{1+ip^{-s}}\right]$$

$$= \prod_p \frac{1}{(1+ip^{1-s})(1+ip^{-s})} = \zeta_1(s-1)\zeta_1(s).$$

□

A similar calculation gives

[2] There are several (inconsistent) definitions of these. Here they are defined by the power series $\dfrac{t}{e^t - 1} = \displaystyle\sum_{n=0}^{\infty} \frac{B_n t^n}{n!}$.

Theorem 15 *For $s > 2$,*

$$\sum_{n=1}^{\infty} \frac{\phi_1(n)}{n^s} = \frac{\zeta_1(s-1)}{\zeta_1(s)}.$$

Hence $\displaystyle\sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n^s} = \zeta_1^2(s) \sum_{n=1}^{\infty} \frac{\phi_1(n)}{n^s}$.

Theorem 16 *For $s > 1$,*

$$\sum_{n=1}^{\infty} \frac{\tau_1(n)}{n^s} = \zeta_1^2(s).$$

Proof:

$$\zeta_1^2(s) = \sum_{n=1}^{\infty} \frac{(-i)^{\Omega(n)}}{n^2} . \sum_{n=1}^{\infty} \frac{(-i)^{\Omega(n)}}{n^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} (-i)^{\Omega(d)} . (-i)^{\Omega(\frac{n}{d})}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} (-i)^{\Omega(n)} = \sum_{n=1}^{\infty} \frac{1}{n^s} (-i)^{\Omega(n)} \tau(n)$$

$$= \sum_{n=1}^{\infty} \frac{\tau_1(n)}{n^s}.$$

□

References:

[1] Brown, P.G., *Modifying Möbius*, The Australian Mathematical Society Gazette, Vol 37, No.4, 2010, pp. 244-249.

[2] Jameson, G.J.O. *The Prime Number Theorem*, 2003.

[3] McCarthy, P.J. *Introduction to Arithmetical Functions*, 1986.