

## Image analysis by discrete orthogonal Tchebichef Moments for 3D Object Representation

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*Abstract:* Discrete Tchebichef moments are widely used in the field of image processing application and pattern recognition. In this paper we propose a compact method of 3D Tchebichef moments computation. This new method based on Clenshaw's recurrence formula and the symmetry property produces a drastic reduction in the complexity and computational time. The recursive algorithm is then developed for fast computation of inverse Tchebichef moments transform for image reconstruction. We also extract scale and translation 3D moment invariants using a proposed direct method. The validity of the proposed algorithm is proved by simulated experiments using 3D image/Object.

*Key-Words:* Digital image processing, Pattern analysis, Image reconstruction techniques, Three-dimensional image processing, Moment methods.

### 1 Introduction

Image moments have been widely used in pattern recognition and image analysis tasks [1-5]. Geometric moments and their translation, scaling and rotation invariants were introduced by Hu [6]. Teague in [7] proposed the concept of orthogonal continuous moments such as Legendre and Zernike moments to represent image with minimum amount of information redundancy. The major disadvantage of the above moments is the discretization error, which increases by increasing the moment order. To surmount the weakness of the continuous orthogonal moments, Mukundan et al. in [8] proposed a set of discrete orthogonal Tchebichef moments.

The use of Tchebichef polynomials as basis function for image moments eliminates the discrete approximation associated with the continuous moments. Moreover, it represents an image with the minimum amount of information redundancy [8]. Based on these attractive properties, Tchebichef moments are used in many applications such as object classification [9], Analysis of noise [10], pattern recognition [11], reconstruction in medical imaging [12], texture analysis [13], Image watermarking [14], Image segmentation [15], edge detection [16], image focus measure [17] and data compression [18].

It is well known that the direct computation of

Tchebichef moments is time consuming process and the computational complexity increased by increasing the moment order [19]. Therefore, some algorithms have been developed to accelerate the computational time of Tchebichef moments. Mukundan et al. [8] analyzed some of the computational aspects of Tchebichef moments by using the well-known properties of Tchebichef polynomials, such as the symmetry, polynomial expansion, and recurrence relation of Tchebichef polynomials. But the authors have never given a detailed and complete algorithm for the computation of Tchebichef moment [19]. Then, a recursive algorithm based on Clenshaw's recurrence formula using a second order digital filter is proposed by Wang and Wang [19]. The authors developed both a direct recursive algorithm and a compact algorithm for the computation of Tchebichef moment. The effective recursive algorithm for inverse Tchebichef moment transform was also presented.

However, those approaches were developed for the case of 2D images and only few works were presented to compute 3D Tchebichef moments. In fact, 3D images require more amount of information to process. Consequently, the complexity increases with high orders more noticeably than in 2D case. The most important paper investigating the Tchebichef moments

for 3D objects is the work of Wu et al. [20] where only scale invariants computation were carried out. They proposed, to reduce the computation time, an algorithm based on matrix multiplication for computing 3D Tchebichef moments.

In this paper, we first propose a compact method of 3D Tchebichef moments computation applied for object reconstruction where the symmetry property and a recursive algorithm based on Clenshaw's recurrence formula are used to accelerate the whole computational process. Secondly, we extract via a proposed direct method scale and translation invariants for 3D objects. We present the necessary mathematical framework to derive this new set of 3D invariants. In fact, the scale invariants are achieved by eliminating the scale and translation factors contained in the transformed Tchebichef moments. With this new direct approach, we eliminate the requirement of calculating the normalization parameters of the shifted or scaled image and utilizing indirect method based on geometric moments. Traditionally, the translation and scale invariant functions of Tchebichef moments can be obtained by two methods [9]: (i) image normalization method; (ii) indirect method, where Tchebichef moments are expressed as a linear combination of the corresponding invariants of geometric moments. Unfortunately, as mentioned by Chong et al. in [21] these methods have two major drawbacks. The normalization method leads to further computational complexity due to this preprocessing normalization step. Furthermore, the normalization scheme derives moments which may differ from the true moments of the original image because of the normalization parameters of the coordinate transformation. The indirect method is time consuming process due to additional time for polynomial coefficients computation.

Summarily, in our work, we adopt the compact computation method for 3D Tchebichef moments, in spite of the straightforward method, to extract the invariants which will produce a significant reduction in the computation time through Clenshaw's and symmetry propriety. Finally, detailed complexity analysis of the proposed and straightforward methods is carried out. Complexity analysis in terms of number of additions and multiplications show the efficiency of the proposed method.

The rest of the paper is organized as follows: in Section 2, an overview of 3D Tchebichef moments and object reconstruction are given. The proposed fast and

compact recursive computation for 3D images is presented in Section 3. Section 4 is devoted to the proposed 3D Tchebichef invariants extraction. Section 5 presents the simulation results on 3D image reconstruction, translation and scale invariant and computational complexity. Finally, concluding remarks are presented in Section 6.

## 2 Three-dimensional Straightforward Tchebichef moments computation

In this section we will present the mathematical background behind the Tchebichef moment theory including polynomials, moments computation and reconstruction.

### 2.1 Tchebichef polynomials

The discrete Tchebichef polynomial of order  $n$  is defined as [8]:

$$\tilde{t}_n(x) = \frac{n!}{N^n} \sum_{k=0}^n (-1)^{n-k} \binom{N-1-k}{n-k} \binom{n+k}{n} \binom{x}{k}, \quad (1)$$

where  $n=0,1,2,3,\dots,N-1$ .

The scaled Tchebichef polynomials satisfy the following recursive relation with respect to  $n$ :

$$\tilde{t}_n(x) = \frac{(2n-1)}{n} \tilde{t}_1(x) \tilde{t}_{n-1}(x) - \frac{(n-1)}{n} \times \left(1 - \frac{(n-1)^2}{N^2}\right) \tilde{t}_{n-2}(x), \quad (2)$$

where  $n=2,3,\dots,N-1$

with the initial conditions

$$\tilde{t}_0(x) = 1 \quad \text{and} \quad \tilde{t}_1(x) = x$$

### 2.2 Three-dimensional Straightforward Tchebichef moments computation

The 3D Tchebichef moment of order  $m+n+k$  of an image intensity function  $f(x, y, z)$  are defined over the cube  $[0, M-1] \times [0, N-1] \times [0, K-1]$  as:

$$T_{mnk} = \frac{1}{\tilde{\rho}(m, M) \tilde{\rho}(n, N) \tilde{\rho}(k, K)} \times \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \sum_{z=0}^{K-1} \tilde{t}_m(x) \tilde{t}_n(y) \tilde{t}_k(z) f(x, y, z). \quad (3)$$

with

$$\tilde{\rho}(n, N) = \frac{N \left(1 - \left(\frac{1}{N^2}\right)\right) \dots \left(1 - \left(\frac{n^2}{N^2}\right)\right)}{2n+1}. \quad (4)$$

### 2.3 Object reconstruction using 3D Tchebichef moment

Since, Tchebichef polynomial  $\tilde{t}_n(x)$  forms a complete

orthogonal basis set on the interval  $[0, N - 1]$  and satisfies the orthogonal property. The 3D image/object intensity function  $f(x,y,z)$  can be expressed over cube  $[0, M - 1] \times [0, N - 1] \times [0, K - 1]$  as:

$$f_{rec}(x, y, z) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} T_{mnk} \tilde{t}_m(x) \tilde{t}_n(y) \tilde{t}_k(z). \quad (5)$$

The straightforward method needs two separate steps. We first compute the Tchebichef polynomial value and second, we evaluate the sum of eq. (3). In the present case of 3D Tchebichef moment the computation involved in Eq. (3) is highly expensive in terms of time computing. In fact, we have three sums, in spite of two in the 2D case, each of which necessitate the evaluation of three polynomials  $\tilde{t}_m$ ,  $\tilde{t}_n$  and  $\tilde{t}_k$ .

In order to speed up the total computation of Tchebichef moment which is highly expensive in the 3D case, we use in this paper a recursive implementation based on the Clenshaw's recurrence formula [22] adapted for 3D case, this method can affect the polynomials computing process simultaneously and reduce the computation cost of moment and inverse transformations especially for high order of moment.

### 3 The proposed method: compact recursive computation for 3D images

In this section we use the Clenshaw's recurrence formula in order to reduce the problem of computational complexity and for more reduction of computation time especially for 3D images we combine this recursive method with another property of Tchebichef polynomial which is the symmetry property.

Before giving the details of the proposed method, we shall present the Clenshaw's recurrence formula which may speed up the total computation compared with straightforward method described below. This recurrence computation will be useful in the case of higher order moments and therefore the related inverse transform.

#### 3.1 Clenshaw's recurrence formula

Clenshaw's recurrence formula [22] is an elegant and efficient way to evaluate a sum of coefficients times functions that obey a recurrence formula:

Suppose that

$$J(x) = \sum_{k=0}^N c_k F_k(x), \quad (6)$$

and that  $F_k$  obeys the recurrence relation:

$$F_{n+1}(x) = \alpha(n, x)F_n(x) + \beta(n, x)F_{n-1}(x), \quad (7)$$

for some functions  $\alpha(n, x)$  and  $\beta(n, x)$ .

Then Clenshaw's recurrence formula states that the sum  $J(x)$  can be evaluated by

$$J(x) = \beta(1, x)F_0(x)\psi_2 + F_1(x)\psi_1 + c_0F_0(x), \quad (8)$$

where the quantities  $\psi_n$  can be obtained from the following recurrence

$$\begin{aligned} \psi_{N+2} &= \psi_{N+1} = 0 \\ \psi_k &= \alpha(k, x)\psi_{k+1} + \beta(k+1, x)\psi_{k+2} + c_k, \quad (9) \\ &(k = N, N-1, \dots, 1) \end{aligned}$$

and solve backwards to obtain  $\psi_2$  and  $\psi_1$

Clenshaw's recurrence formula has been used to develop forward and inverse discrete moments but only for the 2D case [19]. In the next section, we propose to use Clenshaw's formula to compute 3D Tchebichef moment and its inverse function.

#### 3.2 Symmetry property

The symmetry property will be used to reduce the time and the storage required for the scaled Tchebichef polynomials, especially in the case of 3D object where a considerable amount of information to process is needed.

Orthogonal 3D Tchebichef moments are defined as the projection of the digital 3D image/object onto the Tchebichef polynomials. Since, these polynomials are defined over the closed interval  $[0, M - 1] \times [0, N - 1] \times [0, K - 1]$ , therefore, the 3D image/object must be transformed inside the cube. Orthogonal Tchebichef polynomials are defined over the cube  $[0, M - 1] \times [0, N - 1] \times [0, K - 1]$  and have the following symmetrical properties:

$$\tilde{t}_m(M - 1 - x) = (-1)^m \tilde{t}_m(x), \quad (10.1)$$

$$\tilde{t}_n(N - 1 - y) = (-1)^n \tilde{t}_n(y), \quad (10.2)$$

$$\tilde{t}_k(K - 1 - z) = (-1)^k \tilde{t}_k(z). \quad (10.3)$$

This relation suggests the subdivision of the domain of an  $M/2 * N/2 * K/2$  for 3D image/object, only one-eighth of the whole object space is required to compute the entire set of 3D Tchebichef moments as depicted in Fig.1. The implementation of this property results in 87% reduction in the computational cost. A detailed discussion of this will be found through the following subsections. The expression for Tchebichef moments in

(3) can be modified with the help of (10.1, 10.2 and 10.3) as follows. The Tchebichef polynomials  $\tilde{t}_m(M-1-x)$ ,  $\tilde{t}_n(N-1-y)$  and  $\tilde{t}_k(K-1-z)$  defined by using similar equation. Based on Eq. (3)

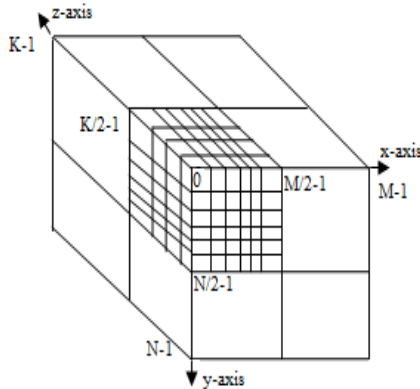


Fig.1. Owing to symmetry, Tchebichef polynomials need be computed only one-eight of the whole image

The Tchebichef moments are defined as follows:

$$T_{mnk} = \frac{1}{\tilde{\rho}(m, M)\tilde{\rho}(n, N)\tilde{\rho}(k, K)} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \sum_{z=0}^{K-1} \tilde{t}_m(x)\tilde{t}_n(y)\tilde{t}_k(z) \times \left\{ \begin{array}{l} f(x, y, z) + (-1)^m f(M-1-x, y, z) \\ + (-1)^n f(x, N-1-y, z) \\ + (-1)^k f(x, y, K-1-z) \\ + (-1)^{m+n} f(M-1-x, N-1-y, z) \\ + (-1)^{m+k} f(M-1-x, y, K-1-z) \\ + (-1)^{n+k} f(x, N-1-y, K-1-z) \\ + (-1)^{m+n+k} f(M-1-x, N-1-y, K-1-z) \end{array} \right\} \quad (11)$$

where M,N,K are even.

### 3.3 The proposed computation of 3D Tchebichef moments by Clenshaw's recurrence formula

Due to separability property of Tchebichef polynomials, the computation of 3D Tchebichef moment can be divided into computing on 1D moments.

$$T_n = \frac{1}{\tilde{\rho}(n, N)} \sum_{x=0}^{N-1} \tilde{t}_n(x) f(x) \quad (12)$$

where  $n=1,2,\dots,N$ .  $\tilde{t}_n(x)$  defined in Eq. (1) and  $\tilde{\rho}(n, N)$  defined in Eq. (4). The scaled Tchebichef polynomial  $\tilde{t}_n(x)$  can be derived from a three-term recurrence, with respect to the variable x:

$$(N-x)x\tilde{t}_n(x) = (-n(n+1) - (2x-1) \times (x-N-1) - x)\tilde{t}_n(x-1) \times (x-1)(x-N-1) - x)\tilde{t}_n(x-2), \quad (13)$$

for  $x=2,\dots,N-1$

with

$$\tilde{t}_n(0) = \frac{(1-N)(2-N)\dots(n-N)}{N^n}, \tilde{t}_n(1) = \tilde{t}_n(0) \frac{n(n-N)}{1-N} \quad (14)$$

we define

$$F_n(x) = \tilde{t}_x(n), \quad (15)$$

where

$$\alpha_a(x, n) = -\frac{n(n+1) + 2(x+1)(x-N) + x+1}{(x+1)(N-x-1)}, \quad (16)$$

and

$$\beta_a(x, n) = \beta_a(x) = -\frac{x(x-N)}{(x+1)(N-x-1)}$$

Using Eq. (9) we define

$$\psi_{N+1} = \psi_N = 0 \quad (17)$$

$$\psi_x = \alpha_a(x, n)\psi_{n+1} + \beta_a(x+1)\psi_{x+2} + f(x),$$

for  $x=N-1, N-2, \dots, 1$

with  $\alpha_a(N-1, n) = 0$  and  $\beta_a(N-1) = \beta_a(N) = 0$

using Eq. (8), we get

$$T_n = \frac{\tilde{t}_n(0)}{\tilde{\rho}(n, N)} \left[ f(0) + \frac{N-1-n(n+1)}{N-1} \psi_1 + \frac{1-N}{2(N-2)} \psi_2 \right]$$

From Eq. (19) with  $x=0$ , we have:

$$\psi_0 = \alpha_a(0, n)\psi_1 + \beta_a(1)\psi_2 + f(0) = -\frac{n(n+1) - N + 1}{N-1} \psi_1 + \frac{1-N}{2(N-2)} \psi_2 + f(0).$$

Finally, we can rewrite Eq. (12) as :

$$T_n = \psi_0 \frac{\tilde{t}_n(0)}{\tilde{\rho}(n, N)} \quad (18)$$

We can extend this algorithm to derive the 3D Tchebichef moment. According to the symmetry property, we can rewrite Eq. (3) as:

$$T_{mnk} = \frac{1}{\tilde{\rho}(m, M)} \sum_{x=0}^{M/2-1} \tilde{t}_m(x) \omega_x(n, k), \quad (19)$$

with

$$\omega_x(n, k) = \frac{1}{\tilde{\rho}(n, N)} \sum_{y=0}^{N/2-1} \tilde{t}_n(y) \omega_{xy}(k), \quad (20)$$

and

$$\omega_{xy}(k) = \frac{1}{\tilde{\rho}(k, K)} \sum_{z=0}^{K/2-1} \tilde{t}_k(z) g(x, y, z). \quad (21)$$

where

$$g(x, y, z) = \left\{ \begin{array}{l} f(x, y, z) + (-1)^m f(M-1-x, y, z) \\ + (-1)^n f(x, N-1-y, z) \\ + (-1)^k f(x, y, K-1-z) \\ + (-1)^{m+n} f(M-1-x, N-1-y, z) \\ + (-1)^{m+k} f(M-1-x, y, K-1-z) \\ + (-1)^{n+k} f(x, N-1-y, K-1-z) \\ + (-1)^{m+n+k} f(M-1-x, N-1-y, K-1-z) \end{array} \right\} \quad (22)$$

$(x = \frac{M}{2} - 1, \dots, 1, 0; y = \frac{N}{2} - 1, \dots, 1, 0; z = \frac{K}{2} - 1, \dots, 1, 0),$   
M, N and K are even.

We calculate the coefficients  $\omega_{xy}(k)$

for each  $x = \frac{M}{2} - 1, \dots, 0$  and  $y = \frac{N}{2} - 1, \dots, 0$

$$\psi_{\frac{K}{2}+1} = \psi_{\frac{K}{2}} = 0$$

$$\psi_z = \alpha_a(z, k)\psi_{z+1} + \beta_a(z+1)\psi_{z+2} + g(x, y, z)$$

for  $z = \frac{K}{2} - 1, \dots, 0,$

$$\omega_{xy}(k) = \psi_0 \frac{\tilde{t}_k(0)}{\tilde{\rho}(k, K)}, \quad (23)$$

the coefficients  $\omega_x(n, k)$  defined by Eq. (20) are first evaluated

for each  $x = \frac{M}{2} - 1, \dots, 0$

$$\phi_{\frac{N}{2}+1} = \phi_{\frac{N}{2}} = 0$$

$$\phi_y = \alpha_a(y, n)\phi_{y+1} + \beta_a(y+1)\phi_{y+2} + \omega_{xy}(k)$$

for  $(y = \frac{N}{2} - 1, \dots, 1, 0),$

$$\omega_x(n, k) = \phi_0 \frac{\tilde{t}_n(0)}{\tilde{\rho}(n, N)}, \quad (24)$$

Then  $\omega_x(n, k)$  are applied to evaluate the  $T_{mnk}$  defined by Eq. (11)

$$\chi_{\frac{M}{2}+1} = \chi_{\frac{M}{2}} = 0$$

$$\chi_x = \alpha_a(x, m)\chi_{x+1} + \beta_a(x+1)\chi_{x+2} + \omega_x(n, k)$$

for  $x = \frac{M}{2} - 1, \dots, 1, 0,$

$$T_{mnk} = \chi_0 \frac{\tilde{t}_n(0)}{\tilde{\rho}(n, N)}. \quad (25)$$

Finally, the 3D Tchebichef moment of order  $m+n+k$  is computed in three steps (Algorithm 1) by successive computation of the 1D  $m^{\text{th}}$  order moment for each row, followed by the 2D  $(m+n)^{\text{th}}$  order moment. Then, the required 3D moment is calculated as a sum of the different 2D moments. This approach significantly reduces the total number of required addition and multiplication processes.

**Algorithm 1: 3D compact Tchebichef moment computation**

For given image  $f(x, y, z)$  of size  $(M, N, K)$

Step 1: for  $z = \frac{K}{2} - 1, \dots, 0$  evaluate  $\omega_{xy}(k)$  using Eq. (22)

Step 2: for  $y = \frac{N}{2} - 1, \dots, 0$  evaluate  $\omega_x(n, k)$  using Eq. (24)

Step 3: for  $x = \frac{M}{2} - 1, \dots, 0$  finally and calculate  $T_{mnk}$  using Eq. (25)

**3.4 Computation of inverse 3D Tchebichef moment transform**

For the 3D case, the inverse moment transformation (5) can be re-expressed as :

$$f(x, y, z) = \sum_{m=0}^{M-1} \tilde{t}_m(x) \omega_m(y, z), \quad (26)$$

where

$$\omega_m(y, z) = \sum_{n=0}^{N-1} \tilde{t}_n(y) \omega_{nm}(z), \quad (27)$$

and

$$\omega_{nm}(z) = \sum_{k=0}^{K-1} \tilde{t}_k(z) T_{mnk}. \quad (28)$$

we define

$$F_n(x) = \tilde{t}_n(x), \quad (29)$$

Then compare Eq. (2) with Eq. (7), we obtain:

$$\alpha_b(n, x) = -\frac{(2n+1)(2x-N+1)}{(x+1)N}, \quad (30)$$

and

$$\beta_b(n, x) = \beta_b(x) = -\frac{n(N^2 - n^2)}{(x+1)N^2}.$$

In the first step we evaluate the coefficients  $\omega_{nm}(z)$

defined by Eq. (28), for each  $m = M-1, M-2, \dots, 0$  and each

$n = N-1, N-2, \dots, 0 :$

$$\psi_{K+1} = \psi_K = 0$$

$$\psi_k = \alpha_b(k, z)\psi_{k+1} + \beta_b(k+1)\psi_{k+2} + T_{mnk}$$

for  $k = K-1, K-2, \dots, 0,$

$$\omega_{nm}(z) = \psi_0, \quad (31)$$

In the second step, we use Eq. (31) to calculate the coefficients  $\omega_m(y, z)$  defined by Eq. (27) for each  $m = M-1, M-2, \dots, 0$

$$\phi_{N+1} = \phi_N = 0$$

$$\phi_n = \alpha_b(n, y)\phi_{n+1} + \beta_b(n+1)\phi_{n+2} + \omega_{nm}(z)$$

for  $n = N-1, N-2, \dots, 0,$

$$\omega_m(y, z) = \phi_0, \quad (32)$$

Finally,  $\omega_m(y, z)$  are applied to evaluate recursively  $f(x, y, z)$  defined by Eq. (26)

$$\begin{aligned} \chi_{M+1} &= \chi_M = 0 \\ \chi_m &= \alpha_b(m, x)\chi_{m+1} + \beta_b(m+1)\chi_{m+2} + \omega_m(y, z) \\ \text{for } m &= M-1, M-2, \dots, 0, \\ f(x, y, z) &= \chi_0 \end{aligned} \tag{33}$$

**Algorithm 2: Image reconstruction using 3D compact Tchebichef moment**  
 For given order  $M, N, K$   
 Step 1: for  $k = K-1, \dots, 0$  evaluate  $\omega_{mm}(z)$  using Eq. (31)  
 Step 2: for  $n = N-1, \dots, 0$  evaluate  $\omega_m(y, z)$  using Eq.(32)  
 Step 3: for  $m = M-1, \dots, 0$  finally and calculate  $f(x, y, z)$  using Eq. (33)

### 3.6 Computational Complexity: Number of additions and multiplications

In this section, the total number of addition and multiplication operations required to compute 3D Tchebichef moment of 3D digital image/object with size  $M * N * K$  in the proposed method is explicitly compared with the straightforward method.

Table 1 shows the computational complexity for computing 3D Tchebichef moment of order  $m + n + k$ . For the proposed method in comparison with the methods that we will refer as straightforward Method (1) and (2).

We can conclude for the result that the proposed method is more efficient than both the Straightforward Method with respect to order and to  $x$ . For example for

a given order  $(M-1) + (N-1) + (K-1)$ , the proposed method reduces

$$\frac{1}{8}MNK - NK - 2K - 5 \text{ multiplications and } \frac{3}{4}NK + \frac{3}{2}K + 3 \text{ additions.}$$

The proposed method reduced  $\frac{1}{8}MNK - NK - 2K - 5$  multiplication compared with Straightforward Method respect to  $x$  and  $\frac{1}{8}MNK + (2M-5) + \frac{NK}{4}(2N-6) + \frac{K}{2} \times (2K-6) - 1$  additions compared to the straightforward Method respect to order and an important reduction in terms of additions of  $\frac{1}{8}MNK(M-17) + \frac{NK}{2}(N-3) + \frac{K}{2}(K-3)$  compared with the straightforward Method respect to order with a Straightforward increase of  $\frac{3}{4}NK + \frac{3}{2}K + 3$  compared with Straightforward method respect to  $x$ .

## 4 The proposed 3D Tchebichef Invariants Extraction

Inspired from the work of Zhu et al. [9] who developed the demonstration in the case of 2D, we generalize the translation invariance for the case of 3D.

### 4.1 Translation invariance of 3D Tchebichef moment

Central 3D Tchebichef moments are translation invariants. These moment invariants of order  $m + n + k$  are defined as

Table 1. Comparison of computational complexity for computing a single 3D Tchebichef moment between our proposed method (recursive compact method), Straightforward Method (2), and Wu's method [20].

Method	Additions	Multiplications
Straightforward Method using Eq. (2) (Straightforward Method (1) : Wu's method [20])	$(m-7)\frac{MNK}{8} + (n-1)\frac{NK}{4} + (k-1)\frac{K}{2}$	$(2m-1)\frac{MNK}{8} + (2n-1)\frac{NK}{4} + (2k-1)\frac{K}{2}$
Straightforward Method using Eq.(12) (Straightforward Method (2))	$\frac{9}{8}MNK - \frac{NK}{4} - \frac{K}{2} - 3$	$\frac{3}{8}MNK - \frac{NK}{4} - \frac{K}{2} - 4$
Proposed method : recursive compact method	$\frac{9}{8}MNK + \frac{NK}{2} + K$	$\frac{1}{4}MNK + \frac{3NK}{4} + \frac{3K}{2} + 1$

$$T'_{mnk} = \frac{1}{\tilde{\rho}(m, M)\tilde{\rho}(n, N)\tilde{\rho}(k, K)} \times \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \sum_{z=0}^{K-1} \tilde{t}_m(x+x_0)\tilde{t}_n(y+y_0) \times \tilde{t}_k(z+z_0)f(x, y, z) . \tag{34}$$

where  $(x_0, y_0, z_0)$  denote the coordinates of centroid of the image given by

$$x_0 = \frac{m_{100}}{m_{000}}, y_0 = \frac{m_{010}}{m_{000}}, z_0 = \frac{m_{001}}{m_{000}} . \tag{35}$$

and  $m_{ijk}$  is the geometric moment of order  $i + j + k$ , defined as:

$$m_{ijk} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \sum_{z=0}^{K-1} x^i y^j z^k f(x, y, z) . \tag{36}$$

To compute these central 3D moments, we can prove that the translated Tchebichef polynomials can be expressed in terms of the original Tchebichef polynomials according to the relation:

$$\tilde{t}_m(x+x_0) = \sum_{p=0}^m \tilde{v}_{m,m-p}(x_0)\tilde{t}_{m-p}(x) , \tag{37.1}$$

$$\tilde{t}_n(y+y_0) = \sum_{q=0}^n \tilde{\tau}_{n,n-q}(y_0)\tilde{t}_{n-q}(y) , \tag{37.2}$$

$$\tilde{t}_k(z+z_0) = \sum_{r=0}^k \tilde{\eta}_{k,k-r}(z_0)\tilde{t}_{k-r}(z) . \tag{37.3}$$

The matrix  $\tilde{v}_{m,m-p}$  is defined by Zhu et al [9]:

$$\tilde{v}_{m,m-p}(x_0) = \sum_{i=0}^{k-1} \tilde{f}_i(n, k) \prec x_0 \succ_{k-i} , \tag{38}$$

where  $\tilde{f}_i(n, k)$  is expressed for a given integer  $k$  less than or equal to  $n$ , and for  $0 \leq i \leq k - 1$  by:

$$\tilde{f}_i(n, k) = \frac{(n-k)!}{(2n-2k)!} \left\{ \frac{(2n-i)!}{i!(k-i)!(n-i)!} \frac{\beta(n-k, N)}{\beta(n, N)} \times \prec n-N \succ_i - \frac{1}{(n-k)!} \sum_{m=0}^{i-1} \frac{(2n-2k-m+i)!}{(i-m)!} \times \frac{\beta(n-k, N)}{\beta(n-m-k+i, N)} \times \prec n-m-k+i-N \succ_{i-m} \times \tilde{f}_m(n, k+m-i) \right\} . \tag{39}$$

where

$$\beta(m, M) = (M-1)(M-2)...(M-m) .$$

and

$$\prec a \succ_k = (-1)^k (a)_k ,$$

with

$$(a)_k = a(a+1)(a+2)...(a+k-1) . \tag{40}$$

for  $k > 0$  and  $(a)_0 = 1$

The matrices  $\tilde{\tau}_{n,n-q}$  and  $\tilde{\eta}_{k,k-r}$  are determined by using similar equations. Eq. (38) show that the matrices  $\tilde{v}_{m,m-p}$ ,  $\tilde{\tau}_{n,n-q}$  and  $\tilde{\eta}_{k,k-r}$  are image-dependent. Substituting Eqs. (37) in (34), we can rewrite the central 3D Tchebichef moment as:

$$T'_{mnk} = \sum_{p=0}^m \sum_{q=0}^n \sum_{r=0}^k \sum_{a=0}^{p-1} \sum_{b=0}^{q-1} \sum_{c=0}^{r-1} \tilde{f}_a(m, p) \times \tilde{f}_b(n, q)\tilde{f}_c(k, r) \prec -x_0 \succ_{p-a} \times \prec -y_0 \succ_{q-b} \prec -z_0 \succ_{r-c} T_{m-p, n-q, k-r} . \tag{41}$$

Where  $T_{mnk}$  is defined by Eq. (25).

Eq. (41) shows that the 3D Tchebichef central moments  $T'_{mnk}$  can be expressed as linear combination of normal Tchebichef moments  $T_{mnk}$  with  $0 \leq p \leq m$ ,  $0 \leq q \leq n$  and  $0 \leq r \leq k$ , so that the translation invariants of Tchebichef moments can be directly derived from the normal Tchebichef moments.

Note that Eq. (41) deals with both non-symmetrical and symmetrical images when the Legendre and Zernike moments do not. As indicated by Chong et al. [21], both Legendre central moments and Zernike central moments give zero values for odd order moments when they are used for images with symmetry along x and/or y directions, and symmetry at centroid. These limitations may cause difficulties in pattern classification. A solution was proposed by Chong et al. to surmount this shortcoming. Since the Tchebichef central moments do not encounter this problem, they should be more suitable for use as pattern feature descriptors compared to the Legendre and Zernike moments.

#### 4.2 Scale invariance of 3D Tchebichef moments

The scale invariant property of 3D image/object moments has a high significance in pattern recognition. Scaling can be either uniform or non-uniform in the x, y and z-directions. As indicated in the introduction, the scale invariants of Tchebichef moments can be usually achieved by image normalization method or in Straightforward method. This subsection presents a new approach to derive the scale invariants of Tchebichef moments when an image is scaled. Let us assume that the original image is scaled with factors a, b and c, along x, y and z-directions, respectively. The scaled Tchebichef moments can be defined as follows:

$$T''_{mnk} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \sum_{z=0}^{K-1} \tilde{t}_m(ax) \tilde{t}_n(by) \tilde{t}_k(cz) f(x, y, z). \quad (42)$$

Where a, b and c are unequal non-zero real numbers representing the scaling factors in the three direction. Zhu et all in [9] expressed the scaled Tchebichef polynomials in terms of the original Tchebichef polynomials as follows:

$$\sum_{p=0}^m \lambda_{m,p} \tilde{t}_p(ax) = a^m \sum_{p=0}^m \lambda_{m,p} \tilde{t}_p(x), \quad (43.1)$$

$$\sum_{q=0}^n \lambda_{n,q} \tilde{t}_q(by) = b^n \sum_{q=0}^n \lambda_{n,q} \tilde{t}_q(y), \quad (43.2)$$

$$\sum_{r=0}^k \lambda_{k,r} \tilde{t}_r(cz) = c^k \sum_{r=0}^k \lambda_{k,r} \tilde{t}_r(z). \quad (43.3)$$

The matrix  $\lambda_{m,p}$  is defined using the following equations

$$\begin{aligned} \lambda_{m,m} &= 1 \\ \lambda_{m,p} &= \sum_{r=0}^{m-p-1} \frac{-c_{m-r,p} \lambda_{m,r}}{C_{p,p}}, \end{aligned} \quad (44)$$

with  $c_{m,i}$  defined by [9]

$$c_{m,i} = \sum_{k=0}^{m-i} \tilde{B}_{m,m-k} s(m-k, i) = \sum_{k=0}^{m-i} c_k(m-i), \quad (45)$$

with

$$c_k(m-k) = \tilde{B}_{m,m-k} s(m-k, i),$$

and

$$\tilde{B}_{m,k} = \frac{B_{m,k}}{\beta(m, M)},$$

where

$$B_{m,k} = \frac{(m+k)!}{(m-k)!(k!)^2} \prec m-N \succ_{m-k}.$$

In order to reduce the computational complexity in the calculation of  $c_{m,i}$  defined by Eq. (45) we use the following recurrence relations to compute  $c_k(m, i)$  with

$$c_k(m-k) = \tilde{B}_{m,m-k} s(m-k, i),$$

$$c_k(m-i) = 0 \quad \text{for } i > m-k \quad \text{and}$$

$$\begin{aligned} c_k(m, i) &= \frac{k(2m-k+1)}{(m-n-k+1)(m-k+1)^2} \\ &\times \frac{s(m-k+1, i)}{s(m-k, i)} c_k(m-1, i), \end{aligned} \quad (46)$$

for  $i \leq m-k, 0 \leq k \leq m-1$

with

$$c_m(m, 0) = \tilde{B}_{m,0} \quad (47)$$

and  $s(k, i)$  are the stirling numbers of the first kind satisfying the following recurrence relations:

$$\begin{aligned} s(k, i) &= s(k-1, i-1) - (k-1)s(k-, i). \\ k &\geq 1, i \geq 1 \end{aligned} \quad (48)$$

with

$$s(k, 0) = s(0, i) = 0, k \geq 1, i \geq 1 \quad \text{and} \quad s(0, 0) = 1.$$

The relationship between the original and scaled Tchebichef moments can then be established as

$$\begin{aligned} \varphi_{mnk} &= \sum_{p=0}^m \sum_{q=0}^n \sum_{r=0}^k \lambda_{mp} \lambda_{nq} \lambda_{kr} T''_{pqr} \\ &= a^{m+1} b^{n+1} c^{k+1} \sum_{p=0}^m \sum_{q=0}^n \sum_{r=0}^k \lambda_{mp} \lambda_{nq} \lambda_{kr} T_{pqr}. \end{aligned} \quad (49)$$

Finally By eliminating the scale factor a, b and c, we can constructed the following scale invariant of 3D Tchebichef moment.

$$\begin{aligned} \psi_{mnk} &= \frac{\varphi_{mnk} \varphi_{000}^{\zeta+2}}{\varphi_{(m+\zeta)00} \varphi_{0(n+\zeta)0} \varphi_{00(k+\zeta)}}. \\ m, n, k &= 0, 1, \dots \text{and } \zeta \geq 0 \end{aligned} \quad (50)$$

## 5 Numerical Simulation

In this section, experimental results are provided to validate the ability of the algorithms developed in the previous sections. The following section is divided into two subsections. In the first one, we address the 3D image reconstruction capability of 3D Tchebichef moments using the proposed compact recursive computation. Then we compare the performances of the proposed method with the Wu's method [20] referred in this paper as straightforward method (1) using Eq. (2) and straightforward method (2) using Eq. (12) (cited in table (1)) in terms of elapsed CPU time under the same computing environment.

Due to the importance of 3D features in the context of pattern recognition, the second subsection is devoted to numerical experiments on translation and scale invariance of 3D Tchebichef moment, for this we use 3D characters and simulated object.

### 5.1 Three-dimensional image reconstructions and computation time of Tchebichef moment

The performance of the 3D image reconstruction using Tchebichef moment will be explored using the Mean-Square Error  $MSE$  which is defined as:

$$MSE = \frac{1}{MNK} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \sum_{z=0}^{K-1} (f_{rec}(x, y, z) - f(x, y, z))^2. \quad (51)$$

Where  $M * N * K$  is image size and  $f_{rec}$  is the reconstructed image function for different orders.

Figure 2 shows how images reconstructed using Tchebichef moment for different orders. The size of each of the gray images is  $(32 * 32 * 32)$  representing the



characters “E” and “A”. We illustrate different orders from 6 to 30. As it can be seen from the figures, the reconstructed images using Tchebichef moment show progressively more visual resemblance with the original image in the early orders. The edges of the reconstructed images are also better defined with less jaggedness.

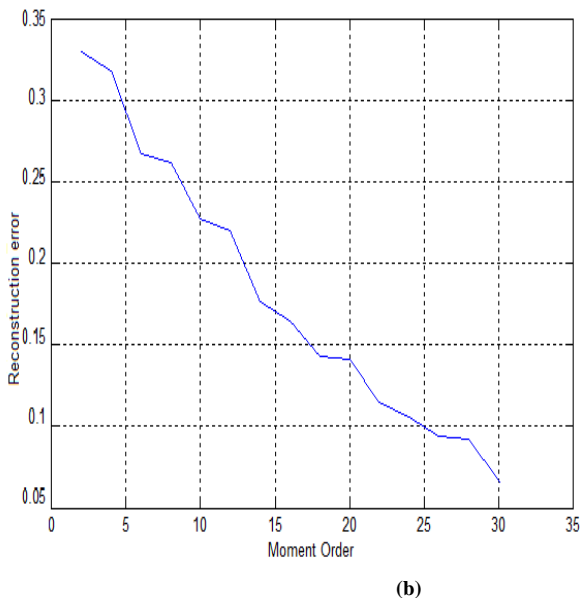
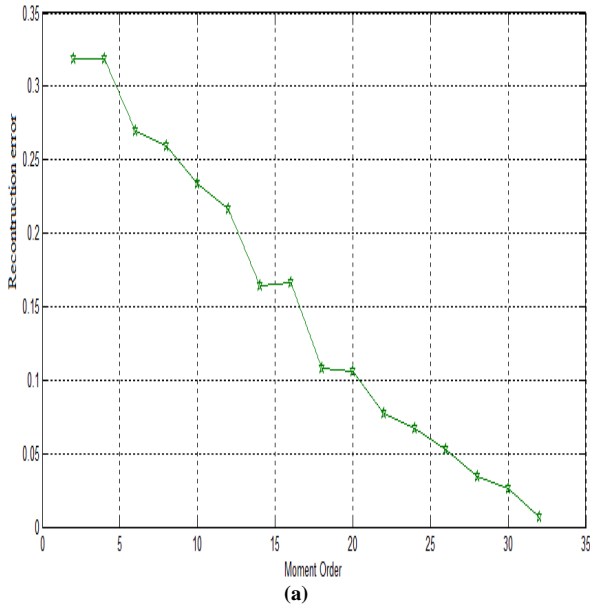


Fig.2. Image Reconstruction error of 3D letter ‘A’ and ‘E’ in (a) and (b) respectively

Fig 2 presents the plot of the reconstruction error for different moment orders up to 30. We can see that the convergence is faster in terms of reconstruction error, which means that the reconstructed images will be

closer to the original one when the maximum moment order increases

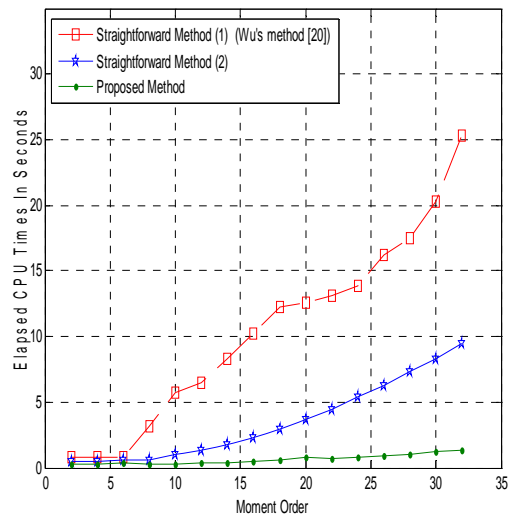


Fig.3. Elapsed CPU time in seconds for 3D Tchebichef moment computation

Fig. 3 shows CPU elapsed times using our compact recursive method in comparison with Wu’s method [20] referred as straightforward method (1) using Eq. (2) and straightforward method (2) using Eq. (25) (cited in table (1)). The CPU times is obtained using Matlab 9 and implemented on HP 630 Laptop machine equipped with a processor of 3.4 Ghz and 4 GB RAM. Fig. 6 implies that our moment computation method requires a noticeable less computation time even with moments of high orders compared with straightforward methods.

### 5.2 Translation and Scale Invariance

Invariance of orthogonal moments is very useful for pattern recognition applications. In this subsection, a series of numerical experiments are achieved to evaluate the invariance of our 3D Tchebichef moments with respect to translation, scaling and reflection. In the first set of experiments, we explore the translation invariance of characters “E” and “A” under different translation parameters (table (2) and (3)). The other set of experiments show the scale invariance for the 3D character “E” (table (4)) and the scale and reflection for the 3D simulated object (table (5)).

The obtained results show that factors the 3 D Tchebichef moment invariants has the same values whatever the translation, scaling and reflections. Those results show the invariance properties of the proposed 3D Tchebichef moment features efficiency of the proposed method.

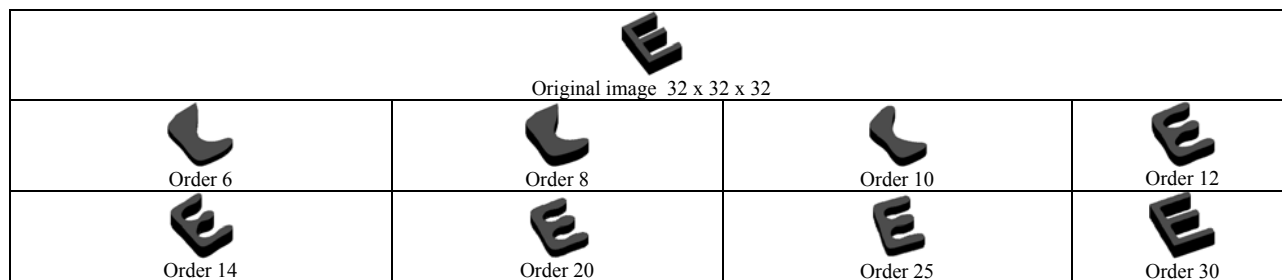


Fig.4. Image reconstruction of 3D letter ‘E’ of size (32 x 32 x 32)

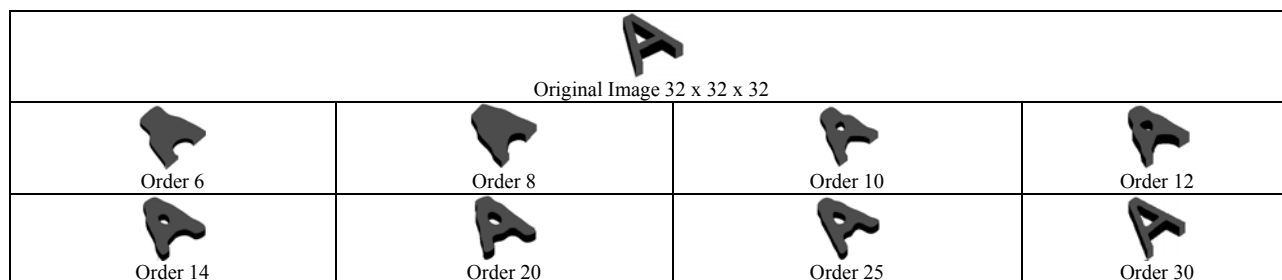


Fig.5 Image reconstruction of 3D letter ‘A’ of size (32 x 32 x 32)

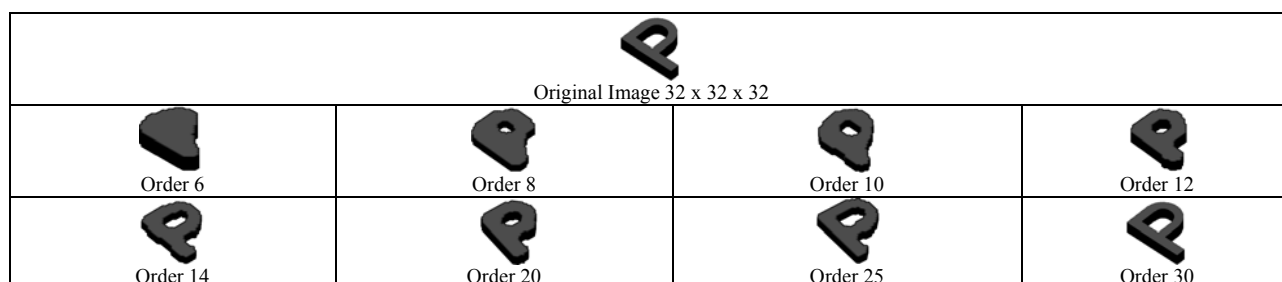


Fig.6. Image reconstruction of 3D letter ‘P’ of size (32 x 32 x 32)

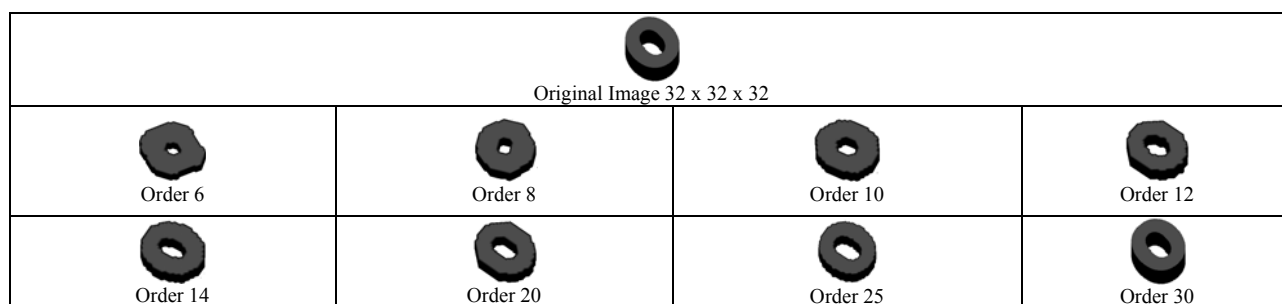


Fig.7. Image reconstruction of 3D letter ‘O’ of size (32 x 32 x 32)

Table 2. Translation Invariance for selected orders for the 3D character “E” being symmetric along x-axis

Translation	$T'_{100}$	$T'_{010}$	$T'_{001}$	$T'_{200}$	$T'_{110}$	$T'_{300}$	$T'_{030}$
$\Delta i = 1, \Delta j = 1, \Delta k = 1$	2.0296	4.0700	3.7207	-1.1472	-1.6619	4.9957	5.1983
$\Delta i = 1, \Delta j = 1, \Delta k = -1$	2.0296	4.0700	3.7207	-1.1472	-1.6619	4.9957	5.1983
$\Delta i = 1, \Delta j = -1, \Delta k = 1$	2.0296	4.0700	3.7207	-1.1472	-1.6619	4.9957	5.1983
$\Delta i = 1, \Delta j = -1, \Delta k = -1$	2.0296	4.0700	3.7207	-1.1472	-1.6619	4.9957	5.1983
$\Delta i = -1, \Delta j = 1, \Delta k = 1$	2.0296	4.0700	3.7207	-1.1472	-1.6619	4.9957	5.1983
$\Delta i = -1, \Delta j = 1, \Delta k = -1$	2.0296	4.0700	3.7207	-1.1472	-1.6619	4.9957	5.1983

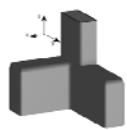
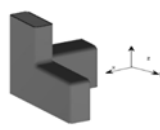

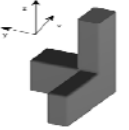
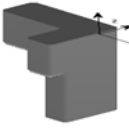

Table 3. Translation Invariance for selected orders for the 3D character “A”

Translation	$T'_{100}$	$T'_{010}$	$T'_{001}$	$T'_{200}$	$T'_{110}$	$T'_{300}$	$T'_{300}$
$\Delta i = 2, \Delta j = 1, \Delta k = 1$	2.1697	4.0740	3.5230	-1.6627	-1.9528	6.5843	8.4240
$\Delta i = 1, \Delta j = 1, \Delta k = -2$	2.1697	4.0740	3.5230	-1.6627	-1.9528	6.5843	8.4240
$\Delta i = 2, \Delta j = -1, \Delta k = 1$	2.1697	4.0740	3.5230	-1.6627	-1.9528	6.5843	8.4240
$\Delta i = 1, \Delta j = -1, \Delta k = -1$	2.1697	4.0740	3.5230	-1.6627	-1.9528	6.5843	8.4240
$\Delta i = -1, \Delta j = 1, \Delta k = 1$	2.1697	4.0740	3.5230	-1.6627	-1.9528	6.5843	8.4240
$\Delta i = -1, \Delta j = 1, \Delta k = -1$	2.1697	4.0740	3.5230	-1.6627	-1.9528	6.5843	8.4240

Table 4. Scale invariants for the 3D letter “E”

Scale (*1.0e-05)	Original shape a=1,b=1,c=1	Transf shape 1 a=0.5,b=1,c=1	Transf shape 2 a=1,b=0.5,c=1	Transf shape 3 a=1,b=1,c=0.5	Transf shape 4 a=2,b=1,c=1	Transf shape 5 a=1,b=2,c=1	Transf shape 6 a=1,b=1,c=2
$\psi_{100}$	-0.6682	-0.6682	-0.6682	-0.6682	-0.6682	-0.6682	-0.6682
$\psi_{010}$	-0.6733	-0.6733	-0.6733	-0.6733	-0.6733	-0.6733	-0.6733
$\psi_{200}$	0.0289	0.0289	0.0289	0.0289	0.0289	0.0289	0.0289
$\psi_{110}$	0.7460	0.7460	0.7460	0.7460	0.7460	0.7460	0.7460
$\psi_{300}$	-1.3764	-1.3764	-1.3764	-1.3764	-1.3764	-1.3764	-1.3764
$\psi_{030}$	-1.3864	-1.3864	-1.3864	-1.3864	-1.3864	-1.3864	-1.3864
$\psi_{102}$	-9.7441	-9.7441	-9.7441	-9.7441	-9.7441	-9.7441	-9.7441

Table 5. Scale and reflection invariants for the 3D simulated object and its elongated, contracted and reflected versions.

Scale and reflection	 a=1,b=1,c=1	 a=-1,b=1,c=1	 a=0.5,b=-2,c=0.6	 a=-1,b=0.8,c=1.5	 a=-1,b=-1,c=-2	 a=1,b=1,c=-2
*1.0e-02						
$\psi_{100}$	-0.0030	-0.0030	-0.0030	-0.0030	-0.0030	-0.0030
$\psi_{010}$	-0.0017	-0.0017	-0.0017	-0.0017	-0.0017	-0.0017
$\psi_{200}$	0.0023	0.0023	0.0023	0.0023	0.0023	0.0023
$\psi_{110}$	0.0029	0.0029	0.0029	0.0029	0.0029	0.0029
$\psi_{300}$	-0.0022	-0.0022	-0.0022	-0.0022	-0.0022	-0.0022
$\psi_{030}$	-0.0035	-0.0035	-0.0035	-0.0035	-0.00354	-0.0035
$\psi_{102}$	-0.0042	-0.0042	-0.0042	-0.0042	-0.0042	-0.0042

## 6 Conclusion

This paper proposed a new method for 3D Tchebichef moments computation and its inverse transform by using Clenshaw's recurrence formula combined with symmetry property. Our 3D compact algorithm reduces significantly the computational complexity compared with the straightforward method. The second contribution of this work pointed out a new method for direct invariant scale and translation extraction. Simulated result clearly showed that the elapsed CPU time is significantly reduced and the proposed 3D descriptors remain invariant under any image translating, scaling, and reflecting.

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