Deterministic and stochastic advertising diffusion model with delay

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Abstract: This paper analysis an advertising diffusion model. A firm launches a new product and devotes a fixed proportion of sales to advertising, while customers go through a three stage adoption process with some delay, on the effect of advertisement. The mathematical model is described by a nonlinear differential system with delay. We set the condition for the existence of the delay parameter value for which the model displays a Hopf bifurcation. The direction and the stability of the Hopf bifurcation is done. The stochastic case is taken into consideration. The last part of the paper includes numerical simulations and conclusions.

Key–Words: advertising diffusion model, bifurcation theory, Hopf bifurcation, stability

1 Introduction

Innovation diffusion theory has been intensively studied since its introduction to marketing in the 1960s. Many models have been developed for explaining the spread of a new product among a population potential customers, by considering the effects of word-of-mouth, advertising, as well as other communication forms. The simplest diffusion models are described in [1], [6] and [14]. Since then, research on the modelling of diffusion of innovations has had important theoretically and practically contributions.

Some of these models are described by nonlinear dynamical systems [5], [7]. Nowadays, an increasing interest in the study of these types of systems has been noticed. Moreover, the systems with delay are ubiquitous in life and nature.

When a product is introduced into a market, it has to be advertised in order to persuade consumers to purchase it. The advertising effect is restricted by all kinds of factors, such as economy, society, culture. Therefore, the advertising effect is delayed. The time delays make system complicated and have influence to the stability.

In [7] an advertising diffusion model is presented, where a firm devotes a fixed proportion of sales to advertising, while customers go through a two-stage non adopter-adopter process. The model is described by a second order dynamical system, where the influence of word-of-mouth and advertising are introduced. It is proved that a homoclinic bifurcation to infinity exists.

In [5] an advertising diffusion model with a three stage non adopter-thinker-adopter process is considered. The thinker population is exposed to advertising and will become adopter after some delay. The dynamical behavior of the nonlinear dynamical system with delay is discussed.

Based on [5], [7], in the present paper we consider a three compartment model which consists of the non-adopter class, the thinker class and the adopter class for a firm launching a new product, a low priced, frequently repurchased one. The word-of-mouth and advertising effectiveness are highlighted. A system with three nonlinear delay differential equations describes the model. In addition of [5], we consider that the dynamics of the adopter's number at moment t is influenced by a fraction of the thinker class at moment t-τ, where τ denotes the delay of the advertising effect. The delay can lead to instability and the stochastic aspects capture the uncertainty about the environment in which the system is operating. Thus, we perturb the initial system with stochastic terms taking into consideration the equilibrium point.

The paper is organized as follows. In Section 2 the deterministic and stochastic models of the advertising diffusion process are described. The existence of the value τ_0 for which the Hopf bifurcation takes place is analyzed in Section 3. Section 4 deals with the direction and the stability of the Hopf bifurcation. Numerical simulations are carried out in Section 5. Finally, concluding remarks are given in Section 6.

2 The mathematical model

We consider a three compartment model, where a firm launches a new low price product, frequently repurchased one. Let $x_1(t)$ be the number of non adopters, $x_2(t)$ be the number of thinkers and $x_3(t)$ be the number of adopters at moment $t \in \mathbb{R}$.

Contacts between non adopters and adopters are random and through word of mouth. Then the rate of generations of adopters is $\alpha x_1(t)x_3(t)$, where α is the word of mouth effectiveness. In [20] it is assumed that $\alpha = a_2 x_3(t)$, where a_2 reflects the advertising effectiveness. The adopter purchase the product at a constant rate and at a constant price and the firm devotes a fixed proportion of sales to advertising effectiveness [5].

The dynamics of $x_1(t)$ is given by:

$$\dot{x_1}(t) = a_1 - a_2 x_1(t)x_3(t)^2 + a_3 x_3(t),$$

where a_1 represents the number of newly non-adopters who enter the market and $a_3 x_3(t)$ is the rate of adopters who become non-adopters.

In the present paper we consider delay in the thinker class, because the effect of advertisement in the non adapter class is non instantaneously. There is time delay τ between non-adopter and adopter. The proportion of thinkers who became adopters is $a_4 x_2(t-\tau)$. The dynamics of $x_2(t)$ is given by:

$$\dot{x_2}(t) = a_2 x_1(t)x_3(t)^2 - a_4 x_2(t-\tau).$$

Taking into account the above considerations and with the rate $a_5 x_3(t)$ the adopters are lost forever, the dynamics of $x_3(t)$ is given by:

$$\dot{x_3}(t) = a_4 x_2(t-\tau) - (a_3 + a_5)x_3(t).$$

Thus, the deterministic model with delay is given by the following non-linear differential system with delay:

$$\begin{aligned} \dot{x_1}(t) &= a_1 - a_2 x_1(t)x_3(t)^2 + a_3 x_3(t) \\ \dot{x_2}(t) &= a_2 x_1(t)x_3(t)^2 - a_4 x_2(t-\tau), \\ \dot{x_3}(t) &= a_4 x_2(t-\tau) - (a_3 + a_5)x_3(t), \end{aligned} \tag{1}$$

where $a_i > 0,\ i = 1,2,3,4,5$ are real positive numbers.

System (1) has only one non trivial positive equilibrium point $E(x_{10}, x_{20}, x_{30})$ obtained as:

$$x_{10} = \frac{(a_3 + a_5)a_5}{a_1 a_2}, x_{20} = \frac{(a_3 + a_5)a_1}{a_4 a_5}, x_{30} = \frac{a_1}{a_5}. \tag{2}$$

In this paper we determine the conditions that the equilibrium point E is locally asymptotically stable if $\tau = 0$. For the deterministic system (1) and $\tau \neq 0$, the Hopf bifurcation is analyzed. The direction and the stability of the Hopf bifurcation is studied, as well [17].

Moreover, the model given by (1) can be more realistic if we introduce a white noise.

Let $(\Omega,\ \mathcal{F}_t,\ P),\ t \geq 0$, be a given probability space, and $W(t) \in \mathbf{R}$ be a scalar Wiener process defined on Ω having independent stationary Gaussian increments with $W(0) = 0,\ E(W(t)-W(s)) = 0$, and $E(W(t)W(s)) = \min(t,s)$. The symbol E denotes the mathematical expectation. The sample trajectories of $W(t)$ are continuous, nowhere differentiable, and have infinite variation on any finite time interval. [11].

What we are interested in knowing is the effect of the noise perturbation on the equilibrium (x_{10}, y_{10}, z_{10}). The stochastic perturbation of (1) given by the form of a stochastic differential equation with delay is:

$$\begin{aligned} \dot{x_1}(t) &= (a_1 - a_2 x_1(t)x_3(t)^2 + a_3 x_3(t))dt+ \\ &+\sigma(x_1(t) - x_{10})dW(t) \\ \dot{x_2}(t) &= (a_2 x_1(t)x_3(t)^2 - a_4 x_2(t-\tau))dt+ \\ &+\sigma(x_2(t) - x_{20})dW(t), \\ \dot{x_3}(t) &= (a_4 x_2(t-\tau) - (a_3 + a_5)x_3(t))dt+ \\ &+\sigma(x_3(t) - x_{30})dW(t), \end{aligned} \tag{3}$$

where $\sigma > 0$ is a scalar Wiener process and we denote by $x_i(t) = x_i(t,\omega),\ i = 1,2,3,\ \omega \in \Omega$, the components of a stochastic process on the probability space [11], [13].

For (3) we have some numerical simulations.

3 Hopf bifurcation analysis for the deterministic model

By carrying out the translation $u_1(t) = x_1(t) - x_{10}$, $u_2(t) = x_2(t) - x_{20}$, $u_3(t) = x_3(t) - x_{30}$, from (1) we obtain the linearized system of (1) at the equilibrium point:

$$\begin{aligned} \dot{u}_1(t) &= a_{11}u_1(t) + a_{13}u_3(t) \\ \dot{u}_2(t) &= a_{21}u_1(t) + b_{22}u_2(t-\tau) + a_{23}u_3(t) \\ \dot{u}_3(t) &= b_{32}u_2(t-\tau) + a_{33}u_3(t) \end{aligned} \tag{4}$$

where $a_{11} = -a_2 x_{30}^2$, $a_{13} = a_3 - 2a_2 x_{10} x_{30}$, $a_{21} = a_2 x_{30}^2$, $a_{23} = 2a_2 x_{10} x_{30}$, $a_{33} = -a_3 - a_5$, $b_{22} = -a_4$, $b_{32} = a_4$.

The characteristic equation of (4) is given by:

$$h(\lambda,\tau) = \lambda^3 + m_2\lambda^2 + m_1\lambda + (n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau} = 0, \tag{5}$$

where $m_2 = a_2x_{30}^2 + a_3 + a_5$, $m_1 = x_{30}^2 a_2(a_3 + a_5)$, $n_2 = a_4$, $n_1 = a_4(2a_2x_{10}x_{30} + a_2x_{30}^2 + a_3 + a_5)$, $n_0 = a_2a_4(4a_2x_{10}x_{30} + a_5)x_{30}^2$.

Proposition 1 *If* $\tau = 0$ *and*

$$p_1p_2 - p_0 > 0 \tag{6}$$

then the equation $h(\lambda, 0) = 0$ *has the roots with negative real part. The equilibrium point E is locally asymptotically stable. The parameters* p_i, $i = 0, 1, 2$ *are given by:*

$$p_2 = m_2 + n_2, p_1 = m_1 + n_1, p_0 = n_0. \tag{7}$$

Proof. For $\tau = 0$, the equation $h(\lambda, 0) = 0$ is given by:

$$\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 = 0. \tag{8}$$

Applying Routh-Hurwitz criterion, equation (8) has the roots with negative real parts if the following conditions

$$p_1 > 0, p_2 > 0, p_1p_2 - p_0 > 0 \tag{9}$$

hold.

As $p_1 > 0$, $p_2 > 0$, equation $h(\lambda, 0) = 0$ has the roots with negative real part if (6) holds.

Now, we consider $\tau > 0$. For the occurrence of the Hopf bifurcation ([9]), a critical time delay τ_0 must exist such that $\lambda_{1,2}(\tau_0) = \pm i\omega_0$, $\omega_0 > 0$ and all other eigenvalues have negative real part at $\tau = \tau_0$ and

$$Re\left(\frac{d\lambda(\tau)}{d\tau}\right)|_{\tau=\tau_0} \neq 0. \tag{10}$$

Assume that a pair of imaginary roots exists for (5) and $\lambda = i\omega$, $\omega > 0$ is one of its roots.

Substituting in (5) and separating the real and imaginary parts, then with $\omega^2 = z$ we obtain:

$$\begin{array}{l}(n_0 - n_2\omega^2)\cos(\omega\tau) + \omega n_1 \sin(\omega\tau) = m_2\omega^2\\ (n_0 - n_2\omega^2)\sin(\omega\tau) - \omega n_1 \cos(\omega\tau) = m_1\omega - \omega^3.\end{array} \tag{11}$$

Eliminating $\sin(\omega\tau)$ and $\cos(\omega\tau)$ from (11), the following polynomial equation in ω can be obtained

$$\omega^6 + q_4\omega^4 + q_2\omega^2 + q_0 = 0 \tag{12}$$

where $q_4 = m_2^2 - 2m_1 - n_2^2$, $q_2 = m_1^2 - n_1^2 + 2n_0n_2$, $q_0 = -n_0^2$.

If $\omega^2 = z$, then (12) becomes:

$$z^3 + q_4z^2 + q_2z + q_0 = 0. \tag{13}$$

The next lemma states the condition under which equation (12) has none, at least one or more positive roots.

From [19] we have the following findings on the distribution of the roots of (13).

Lemma 1. ([8]) *If one of the following holds*

1. $q_0 \geq 0, q_4^2 - 3q_2 < 0$,
2. $q_0 \geq 0, q_4 > 0, q_2 > 0$,

then (13) has no positive root.

Therefore, the real parts of all the eigenvalues of (5) are negative for all delays $\tau \geq 0$, so that we have:

Proposition 2 *Suppose that:*

1. $p_1p_2 - p_0 > 0$,
2. *either* $q_0 \geq 0$, $q_4^2 - 3q_2 < 0$ *or* $q_0 \geq 0$, $q_4 > 0$, $q_1 > 0$.

Then, the equilibrium point E is locally asymptotically stable for all $\tau > 0$.

However, the stability of the steady state depends on the delay; if the conditions in Lemma 1 are not satisfied oscillations can occur.

Lemma 2. ([8]) *If*

$$h(z) = z^3 + q_4z^2 + q_2z + q_0, z_0 = \frac{-q_4 + \sqrt{q_4^2 - 3q_2}}{3} \tag{14}$$

and one of the followings holds

1. $q_0 < 0$,
2. $q_0 > 0, q_2 < 0$,
3. $q_0 > 0, z_0 > 0, h(z_0) \leq 0$

then (13) has at least one positive root. Thus, the characteristic equation (5) has at least a pair of purely imaginary roots.

Now, we establish condition (10).

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be a root of (5) satisfying $\alpha(\tau_0) = 0, \omega(\tau_0) = \omega_0$.

Lemma 3. ([8]) *If* $h'(\omega_0^2) \neq 0$ *then* $\frac{d}{d\tau}Re(\lambda(\tau))|_{\tau=\tau_0} > 0$.

Proof. Assume that

$$\begin{array}{l}h(\omega_0^2) = \omega_0^6 + (m_2^2 - 2m_1 - n_2^2)\omega_0^4 +\\ +(m_1^2 - n_1^2 + 2n_0n_2)\omega^2 - n_0^2\end{array} \tag{15}$$

then

$$\begin{array}{l}h'(\omega_0^2) = 3\omega_0^4 + 2(m_2^2 - 2m_1 - n_2^2)\omega_0^2 +\\ +m_1^2 - n_1^2 + 2n_0n_2.\end{array} \tag{16}$$

Because

$$\begin{array}{l}sign\left[\frac{dRe\lambda(\tau)}{d\tau}\right]|_{\tau=\tau_0} =\\ = sign[\frac{dRe\lambda(\tau)}{d\tau}]^{-1}_{\tau=\tau_0},\end{array} \tag{17}$$

$$\left[\frac{d\lambda(\tau)}{d\tau}\right]^{-1} = \\ = \frac{(3\lambda^2+2m_2\lambda+m_1)e^{\lambda\tau}}{\lambda(n_2\lambda^2+n_1\lambda+n_0)}+ \\ +\frac{2n_2\lambda+n_1}{\lambda(n_2\lambda^2+n_1\lambda+n_0)}-\frac{\tau}{\lambda}. \tag{18}$$

We have

$$sign\left[\frac{dRe\lambda(\tau)}{d\tau}\right]|_{\tau=\tau_0}^{-1} = \\ = sign[Re\left[\frac{(3\lambda^2+2m_2\lambda+m_1)e^{\lambda\tau}}{\lambda(n_2\lambda^2+n_1\lambda+n_0)}\right]_{\tau=\tau_0}+ \\ +Re\left[\frac{2n_2\lambda+n_1}{\lambda(n_2\lambda^2+n_1\lambda+n_0)}\right]|_{\tau=\tau_0} = \\ = sign\left[\frac{3\omega_0^4+2(m_2-2m_1-n_2^2)\omega_0^2+m_1^2-n_1+2n_0n_2}{n_1^2\omega_0^2+(n_0-n_2\omega_0^2)^2}\right] = \\ = sign\left[\frac{h'(\omega_0^2)}{n_1^2\omega_0^2+(n_0-n_2\omega_0^2)^2}\right] = sign(h'(\omega_0^2)).$$

If $h'(\omega_0^2) \neq 0$, then $\left[\frac{dRe\lambda(\tau)}{d\tau}\right]_{\tau=\tau_0} \neq 0$. There must be $\left[\frac{dRe\lambda(\tau)}{d\tau}\right]_{\tau=\tau_0} > 0$. This is because (5) has the positive real part roots as $\tau < \tau_0$ if $\left[\frac{dRe\lambda(\tau)}{d\tau}\right]_{\tau=\tau_0} < 0$. This contradicts to the fact when $\tau \in [0, \tau_0)$ and E is locally asymptotically stable.

By Lemma 3 we have the following theorem:

Theorem 1. *Suppose that $p_1p_2 - p_0 > 0$. If Lemma 2 holds, the equilibrium point E of (1) is locally asymptotically stable when $\tau < \tau_0$ and unstable when $\tau > \tau_0$, where τ_0 is defined by:*

$$\tau_0 = \\ = \frac{1}{\omega_0}(arccos(\frac{m_2\omega_0^2(n_0-n_2\omega_0)^2+\omega_0 n_1(\omega_0^3-m_1\omega_0)}{(n_0-n_2\omega_0^2)^2+\omega_0^2n_1^2})+ \\ +2m\pi), m = 0,1,2,..... \tag{19}$$

In addition, if $h'(\omega_0^2) \neq 0$ then a Hopf bifurcation occurs when $\tau = \tau_0$.

4 Direction and stability of Hopf bifurcation

The direction and the stability of the Hopf bifurcation is done by applying the techniques from normal form and center manifold theory introduced by Hassard et al. [10], [12].

Let $u_i(t) = x_i(t) - x_{i0}$, $i = 1, 2, 3$. System (1) becomes:

$$\dot{u_1}(t) = -a_2x_{30}^2u_1(t) - 2a_2x_{10}x_{30}u_3(t) - a_2x_{10}u_3(t)^2- \\ -2a_2x_{30}u_1(t)u_3(t) - a_2u_1(t)u_3(t)^2 \\ \dot{u_2}(t) = a_2x_{30}^2u_1(t) - a_4u_2(t-\tau) + a_2x_{10}u_3(t)^2+ \\ +2a_2x_{30}u_1(t)u_3(t) + a_2u_1(t)u_3(t)^2 \\ \dot{u_3}(t) = a_4u_2(t-\tau) - (a_3+a_5)u_3(t). \tag{20}$$

Now, we re-scale the time by $t = s\tau$, $\bar{u}_i(s) = u_i(s\tau)$, $i = 1, 2, 3$, $\tau = \tau_0 + \mu$, $\mu \in \mathbb{R}$. We denote $y_i(t) = \bar{u}_i(s)$, $i = 1, 2, 3$.

Then, system (20) can be rewritten as:

$$\dot{y_1}(t) = (\tau_0+\mu)[-a_2x_{30}^2y_1(t) - 2a_2x_{10}x_{30}y_3(t)- \\ -a_2x_{10}y_3(t)^2 - 2a_2x_{30}y_1(t)y_3(t) - a_2y_1(t)y_3(t)^2] \\ \dot{y_2}(t) = (\tau_0+\mu)[a_2x_{30}^2y_1(t) - a_4y_2(t-1)+ \\ +a_2x_{10}y_3(t)^2 - 2a_2x_{30}y_1(t)y_3(t) - a_2y_1(t)y_3(t)^2] \\ \dot{y_3}(t) = (\tau_0+\mu)[a_4y_2(t-1) - (a_3+a_5)y_3(t). \tag{21}$$

For $\phi = (\phi_1, \phi_2, \phi_3)^T \in C[-1, 0] = C([-1, 0], \mathbb{R}^3)$, we define a family of operators

$$L_\mu\phi = B_1\phi(0) + B_2\phi(-1) \tag{22}$$

where

$$B_1 = (\tau_0+\mu)\begin{pmatrix} -a_2x_{30}^2 & 0 & a_3 - 2a_2x_{10}x_{30} \\ a_2x_{30}^2 & 0 & 2a_2x_{10}x_{30} \\ 0 & 0 & -a_3 - a_5 \end{pmatrix} \tag{23}$$

$$B_2 = (\tau_0+\mu)\begin{pmatrix} 0 & 0 & 0 \\ 0 & -a_4 & 0 \\ 0 & a_4 & 0 \end{pmatrix} \tag{24}$$

and define

$$f(\mu,\phi) = \\ = (\tau_0+\mu)\begin{pmatrix} -a_2x_{10}\phi_3(0)^2- \\ -2a_2x_{30}\phi_1(0)\phi_3(0) - a_2\phi_1(0)\phi_3(0)^2 \\ a_2x_{10}\phi_3(0)^2+ \\ +2a_2x_{30}\phi_1(0)\phi_3(0) + a_2\phi_1(0)\phi_3(0)^2 \\ 0 \end{pmatrix} \tag{25}$$

By the Riesz representation theorem, there exists a matrix whose components are bounded variational functions $\eta(\theta, \mu) : [-1, 0] \to \mathbb{R}^3$ so that

$$L_\mu\phi = \int_{-1}^{0} d\eta(t,\mu)\phi(\theta). \tag{26}$$

In fact choosing

$$\eta(\theta,\mu)=(\tau_0+\mu)B_1\delta(\theta)+(\tau_0+\mu)B_2\delta(\theta+1), \quad (27)$$

where $\delta(\theta)$ is a Dirac function, then (22) is satisfied.

For $\phi=(\phi_1,\phi_2,\phi_3)\in C[-1,0]$, define

$$\mathcal{A}(\mu)\phi(\theta)=\begin{cases}\frac{d\phi(\theta)}{d\theta},\ \theta\in[-1,0)\\ \int_{-1}^{0}d\eta(s,\mu)\phi(s),\ \theta=0\end{cases} \quad (28)$$

$$\mathcal{R}(\mu)\phi(\theta)=\begin{cases}0,\ \theta\in[-1,0)\\ f(\mu,\phi),\ \theta=0\end{cases} \quad (29)$$

System (21) can be rewritten as

$$\dot{U}_t=\mathcal{A}(\mu)U_t+\mathcal{R}(\mu)U \quad (30)$$

The adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$ is defined as:

$$\mathcal{A}^*(\mu)(\psi(s))=\begin{cases}-\frac{d\psi(s)}{ds},\ s\in(0,1]\\ \int_{-1}^{0}d\eta^T(t,s)\psi(-t),\ s=0\end{cases}$$

For $\phi\in C^1([-1,0],\mathbf{C}^3)$ and $\psi\in C^1([0,1],\mathbf{C}^{3*})$, in order to normalize the eigenvector of operator $\mathcal{A}$ and its adjoint, we define the bilinear form:

$$<\psi,\phi>=\bar{\psi}(0)^\top\phi(0)- -\int_{\theta=-1}^{0}\int_{s=0}^{\theta}\bar{\psi}^\top(s-\theta)d\eta(\theta)\phi(s)ds$$

where $\eta(\theta)=\eta(\theta,0)$.

Then $<\psi,\mathcal{A}\phi>=<\mathcal{A}^*\psi,\phi>$, for $(\phi,\psi)\in D(\mathcal{A})\times D(\mathcal{A}^*)$. Normalizing q and q^* by the condition $<q^*,q>=1$, $<q^*,\bar{q}>=0$.

By the discussion in Section 3 and transformation $t=s\tau$, we know that $\pm i\tau_0\omega_0$ are eigenvalues of $\mathcal{A}$ and other eigenvalues have strictly negative real part. Thus, they are also eigenvalues of $\mathcal{A}^*$. Next, we compute the eigenvector q of $\mathcal{A}$ associated to the eigenvalue $i\tau_0\omega_0$ and the eigenvector q^* of $\mathcal{A}^*$ associated to the eigenvalue $-i\tau_0\omega_0$. Consider:

$$q(\theta)=(1,\alpha,\beta)^Te^{i\tau_0\omega_0\theta},-1<\theta\le 0.$$

From the above discussion, it is easy to know that

$$\mathcal{A}q(0)=i\tau_0\omega_0q(0),$$

i.e.

$$\tau_0\begin{pmatrix} i\omega_0+a_2x_{30}^2 & 0 & -a_3+2a_2x_{10}x_{30}\\ -a_2x_{30}^2 & i\omega_0+a_4e^{-i\omega_0\tau_0} & -2a_2x_{10}x_{30}\\ 0 & -a_4e^{-i\omega_0\tau_0} & i\omega_0+a_3+a_5\end{pmatrix}\cdot \cdot\begin{pmatrix}1\\ \alpha\\ \beta\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

We have:

$$\begin{aligned}\alpha&=\frac{(i\omega_0+a_2x_{30}^2)(i\omega_0+a_3+a_5)}{a_5(a_3-2a_2x_{10}x_{30})e^{-i\omega_0\tau_0}},\\ \beta&=\frac{i\omega_0+a_2x_{30}^2}{a_3-2a_2x_{10}x_{30}}.\end{aligned} \quad (31)$$

Similarly, assume that the eigenvector q^* of $\mathcal{A}^*$ is:

$$q^*(s)=\frac{1}{\rho}(1,\alpha^*,\beta^*)e^{i\tau_0\omega_0s},0\le s<1.$$

Then, from

$$\mathcal{A}^*q^*(0)=-i\tau_0\omega q^*(0)$$

we obtain:

$$\begin{aligned}\alpha^*&=\frac{-i\omega_0+a_2x_{30}^2}{a_2x_{30}^2},\\ \beta^*&=\frac{(-i\omega_0+a_2x_{30}^2)(-i\omega_0+a_3e^{i\omega_0\tau_0})}{a_2a_4x_{30}^2e^{-i\omega_0\tau_0}}.\end{aligned} \quad (32)$$

By $<q^*,q>=1$, we can get ρ, that is

$$\begin{aligned}&<q^*,q>=<\bar{q}^*(0),q(0)>-\\ &-\int_{\theta=-1}^{0}\int_{s=0}^{\theta}\bar{q}^\top(s-\theta)d\eta(\theta)q(s)ds=\\ &=\frac{1}{\bar{\rho}}(1+\alpha\bar{\alpha}+\beta\bar{\beta}^*)+\\ &+\tau_0\frac{1}{\bar{\rho}}a_4\alpha(\bar{\beta}^*-\bar{\alpha}^*)e^{-i\omega_0\tau_0}=1.\end{aligned}$$

Hence we have:

$$\bar{\rho}=1+\alpha\bar{\alpha}^*+\beta\bar{\beta}^*+a_4\tau_0\alpha(\bar{\beta}^*-\bar{\alpha}^*)e^{-i\omega_0\tau_0}. \quad (33)$$

By the same discussion, we can verify that $<q^*,\bar{q}>=0$. Thus, q and q^* are obtained.

Next, we study the stability of bifurcating periodic solution. As in [10], the bifurcating periodic solutions $Z(t,\mu(\varepsilon))$ have the amplitude $O(\varepsilon)$ and nonzero Floquet exponent $\beta(\varepsilon)$ with $\beta(0)=0$. Then, μ, β are given by:

$$\mu=\mu_2\varepsilon_2+\mu_4\varepsilon_4+\ldots,$$

$$\beta = \beta_2\varepsilon_2 + \beta_4\varepsilon_4 + \dots$$

The sign of μ_2 indicates the direction of bifurcation while β_2 determines the stability of $Z(t, \mu(\varepsilon))$, which is stable if $\beta_2 < 0$ and unstable if $\beta_2 > 0$. In the following, we construct the coordinates to describe a center manifold Ω_0 near $\mu = 0$, which is a local invariant, attracting a two-dimensional manifold [10].

Let U_t be the solution of equation (30) when $\mu = 0$ and define:

$$z(t) =< q^*, U_t >, W(t, \theta) = U_t(\theta) - 2Re\{z(t)q(\theta)\}. \tag{34}$$

On the center manifold Ω_0, we have $W(t, \theta) = W(z(t), \bar{z}(t), \theta)$, where

$$W(z, \bar{z}, \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \dots \tag{35}$$

and z, $\bar{z}$ are the local coordinates of the center manifold Ω_0 in the direction of q^* and $\bar{q}^*$, respectively.

Notice that W is real if U_t is real. We consider only real solutions. For the solution $U_t \in \Omega_0$ since $\mu = 0$, then

$$\begin{aligned}
&\dot{z}(t) =< q^*, \dot{U}_t >= \\
&< q^*, \mathcal{A}U_t + \mathcal{R}U_t >=< q^*, \mathcal{A}U_t > + < q^*, \mathcal{R}U_t >= \\
&< \mathcal{A}^* q^*, U_t > + < q^*, \mathcal{R}U_t >= \\
&= i\tau_0\omega_0 z(t) + \bar{q}^*(0)f(0, W(t, 0) + 2Re(z(t)q(0))).
\end{aligned} \tag{36}$$

We rewrite this equation as

$$\dot{z}(t) = i\tau_0\omega_0 z(t) + g(z, \bar{z}) \tag{37}$$

with

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \dots \tag{38}$$

In the following, the values of μ_2 and β_2 presented by Hassard et al. [10] are obtained. Substituting (30) and (36) into

$$\dot{W} = \dot{U}_t - \dot{z}q - \dot{\bar{z}}\bar{q}.$$

We have:

$$\dot{W} = \begin{cases} \mathcal{A}W - 2Re[\bar{q^*}(\theta)f(z, \bar{z})q(\theta)], & -1 \le \theta \le 0 \\ \mathcal{A}W - 2Re[\bar{q^*}(0)f(z, \bar{z})q(0)], & \theta = 0 \end{cases} \tag{39}$$

where

$$H(z, \bar{z}, \theta) = h_{20}(\theta)\frac{z^2}{2} + h_{11}(\theta)z\bar{z} + h_{02}(\theta)\frac{\bar{z}^2}{2} + \dots \tag{40}$$

Taking the derivative of W with respect to t in (35), we have:

$$\dot{W} = W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}. \tag{41}$$

From (35), (37) and (41), we obtain:

$$\begin{aligned}
&\dot{W} = (W_{20}z + W_{11}\bar{z} + \dots)(i\tau_0\omega_0 z + g) + \\
&(W_{11}z + W_{02}\bar{z} + \dots)(-i\tau_0\omega_0\bar{z} + \bar{g}).
\end{aligned} \tag{42}$$

Then, substituting (35) and (40) into (39), we obtain:

$$\begin{aligned}
&\dot{W} = (\mathcal{A}W_{20} + h_{20})\frac{z^2}{2} + (\mathcal{A}W_{11} + h_{11})z\bar{z} + \\
&+ (\mathcal{A}W_{02} + h_{02})\frac{\bar{z}^2}{2} + \dots
\end{aligned} \tag{43}$$

By comparing the coefficients of (41) and (42), we obtain:

$$\begin{aligned}
&(\mathcal{A} - 2i\tau_0\omega_0)W_{20}(\theta) = -h_{20}(\theta) \\
&\mathcal{A}W_{11}(\theta) = -h_{11}(\theta).
\end{aligned} \tag{44}$$

According to (36) and (37), we have:

$$g(z, \bar{z}) = \bar{q}^*(0)f(z, \bar{z}) = \frac{\tau_0}{\bar{\rho}}\begin{pmatrix} 1 & \bar{\alpha}^* & \bar{\beta}^* \end{pmatrix} \cdot \begin{pmatrix} -a_2x_{10}U_{3t}(0)^2 - 2a_2x_{30}U_{1t}(0)U_{3t}(0) - \\ -a_2U_{1t}(0)U_{3t}(0)^2 \\ a_2x_{10}U_{3t}(0)^2 + 2a_2x_{30}U_{1t}(0)U_{3t}(0) + \\ +a_2U_{1t}(0)U_{3t}(0)^2 \\ 0 \end{pmatrix} \tag{45}$$

where $U_t(\theta) = (U_{1t}(\theta), U_{2t}(\theta), U_{3t}(\theta))^T = W(t, \theta) + zq(\theta) + \bar{z}\bar{q}(\theta)$.

Then:

$$\begin{aligned}
&U_{1t}(0) = z + \bar{z} + W_{120}(0)\frac{z^2}{2} + W_{111}(0)z\bar{z} + \\
&+ W_{102}(0)\frac{\bar{z}^2}{2} + O(|z, \bar{z}|^3), \\
&U_{3t}(0) = \beta z + \bar{\beta}\bar{z} + W_{320}(0)\frac{z^2}{2} + W_{311}(0)z\bar{z} + \\
&+ W_{302}(0)\frac{\bar{z}^2}{2} + O(|z, \bar{z}|^3)
\end{aligned} \tag{46}$$

Comparing the coefficients of (38) with (42), it follows that:

$$
\begin{aligned}
g_{20} &= \frac{2\tau_0}{\bar{\rho}}(\bar{\alpha}^* - 1)a_2(x_{10} + 2x_{30}\beta)\\
g_{11} &= \frac{\tau_0}{\bar{\rho}}(\bar{\alpha}^* - 1)a_2(x_{10}\beta\bar{\beta} + 2x_{30}(\beta + \bar{\beta}))\\
g_{02} &= \frac{2\tau_0}{\bar{\rho}}(\bar{\alpha}^* - 1)a_2(x_{10}\bar{\beta}^2 + 2x_{30}\bar{\beta})\\
g_{21} &= \frac{2\tau_0}{\bar{\rho}}(\bar{\alpha}^* - 1)(x_{10}(2\beta W_{311}(0) + \beta W_{320}(0))) +\\
&+ x_{30}(2\beta W_{311}(0) + \beta W_{320}(0) + 2\beta W_{111}(0) +\\
&+ W_{120}(0)\bar{\beta}) - 2a_2\beta\bar{\beta})
\end{aligned}
\tag{47}
$$

Now, we compute $w_{20}(\theta)$ and $w_{11}(\theta)$. From (39), we have:

$$
\begin{aligned}
&H(z, \bar{z}, \theta) = -2Re(\bar{q}^*(\theta)f(z, \bar{z})q(\theta)) + \Re U_t =\\
&-\frac{1}{2}(g_{20}z^2 + g_{11}z\bar{z} + \frac{1}{2}g_{02}\bar{z}^2 + \dots)q(\theta) -\\
&-\frac{1}{2}(\bar{g}_{20}\bar{z}^2 + \bar{g}_{11}z\bar{z} + \frac{1}{2}\bar{g}_{02}z^2 + \dots)\bar{q}(\theta) + \Re U_t
\end{aligned}
$$

Comparing the coefficients with (40), for $\theta \in [-1, 0)$, we obtain:

$$
\begin{aligned}
h_{20}(\theta) &= -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta),\\
h_{11}(\theta) &= -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta).
\end{aligned}
\tag{48}
$$

Substituting the above equations into (44), it follows that:

$$
\begin{aligned}
\dot{W}_{20}(\theta) &= 2i\tau_0\omega_0 W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta)\\
\dot{W}_{11}(\theta) &= g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta)
\end{aligned}
\tag{49}
$$

Solving (48), we have:

$$
\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}}{\tau_0\omega_0}q(0)e^{i\tau_0\omega_0\theta} + \frac{i\bar{g}_{02}}{3\tau_0\omega_0}\bar{q}(0)e^{-i\tau_0\omega_0\theta} +\\
&+ E_2 e^{2i\tau_0\omega_0\theta}\\
W_{11}(\theta) &= -\frac{ig_{11}}{\tau_0\omega_0}q(0)e^{i\tau_0\omega_0\theta} +\\
&+ \frac{\bar{g}_{11}}{i\tau_0\omega_0}\bar{q}(0)e^{-i\tau_0\omega_0\theta} + E_1
\end{aligned}
\tag{50}
$$

In what follows, we compute E_1 and E_2. From (44), we obtain:

$$
\begin{aligned}
\mathcal{A}W_{20} &= 2i\tau_0\omega_0 W_{20}(0) - h_{20}(0)\\
\mathcal{A}W_{11}(0) &= -h_{11}(0).
\end{aligned}
\tag{51}
$$

From the definition of $\mathcal{A}$, we obtain:

$$
\begin{aligned}
&\textstyle\int_{-1}^0 d\eta(\theta)W_{20}(\theta) = 2i\tau_0\omega_0 W_{20}(0) - h_{20}(0)\\
&\textstyle\int_{-1}^0 d\eta(\theta)W_{11}(\theta) = -h_{11}(0).
\end{aligned}
\tag{52}
$$

From (48) and the definition of $f(\mu, \tau)$ we have:

$$
\begin{aligned}
&h_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + \tau_0 a_2 \cdot\\
&\cdot \begin{pmatrix} -x_{10}\beta^2 - 2x_{30}\beta \\ x_{10}\beta + 2x_{30}\beta \\ 0 \end{pmatrix},\\
&h_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \tau_0 a_2 \cdot\\
&\cdot \begin{pmatrix} -x_{10}\beta\bar{\beta} - 2x_{30}(\beta + \bar{\beta}) \\ x_{10}\beta\bar{\beta} + 2x_{30}(\beta + \bar{\beta}) \\ 0 \end{pmatrix}
\end{aligned}
\tag{53}
$$

Notice that:

$$
\begin{aligned}
&(i\tau_0\omega_0 I - \int_{-1}^0 e^{i\tau_0\omega_0\theta}d\eta(\theta))q(0) = 0\\
&(-i\tau_0\omega_0 I - \int_{-1}^0 e^{-i\tau_0\omega_0\theta}d\eta(\theta))\bar{q}(0) = 0.
\end{aligned}
$$

Substituting (50) and (53) into (52) we obtain:

$$
\begin{pmatrix} 2i\omega_0 + a_2x_{30}^2 & 0 & -a_3 + 2a_2x_{20}x_{30} \\ -a_2x_{30}^2 & 2i\omega_0 + a_4e^{-i\omega_0\tau_0} & -2a_2x_{20}x_{30} \\ 0 & -a_4e^{-i\omega_0\tau_0} & 2i\omega_0 + a_3 + a_5 \end{pmatrix} \cdot
$$

$$
\cdot E_2 = a_2(x_{10}\beta^2 + 2x_{30}\beta)\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}
$$

$$
\begin{aligned}
&\begin{pmatrix} a_2x_{30}^2 & 0 & -a_3 + 2a_2x_{10}x_{30} \\ -a_2x_{30}^2 & a_4 & -2a_2x_{10}x_{30} \\ 0 & -a_4 & a_3 + a_5 \end{pmatrix} E_1 =\\
&= a_2(x_{10}\beta\bar{\beta} + 2x_{30}(\beta + \bar{\beta}))\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}
$$

It follows that:

$$
\begin{aligned}
&E_1 = a_2(x_{10}\beta\bar{\beta} + 2x_{30}(\beta + \bar{\beta}) \cdot\\
&\cdot \begin{pmatrix} a_2x_{30}^2 & 0 & -a_3 + 2a_2x_{10}x_{30} \\ -a_2x_{30}^2 & a_4 & -2a_2x_{10}x_{30} \\ 0 & -a_4 & a_3 + a_5 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}
$$

$$E_2 = a_2(x_{10}\beta^2 + 2x_{30}\beta)\cdot$$
$$\cdot \begin{pmatrix} 2i\omega_0 + a_2x_{30}^2 & 0 & -a_3 + 2a_2x_{10}x_{30} \\ -a_2x_{30}^2 & 2i\omega_0 + a_4e^{-i\omega_0\tau_0} & -2a_2x_{10}x_{30} \\ 0 & -a_4e^{-i\omega_0\tau_0} & 2i\omega_0 + a_3 + a_5 \end{pmatrix}^{-1} \cdot$$
$$\cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

As a result, we know $W_{20}(\theta)$ and $W_{11}(\theta)$. Then, g_{ij} are determined by the parameters of the system and the time delay in (1).

Therefore, we can compute the following parameters [10]:

$$\Omega_1(0) = \frac{i}{2\omega_0}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2}$$
$$\mu_2 = -\frac{Re(\Omega_1(0))}{Re(\lambda'(\tau_0))},$$
$$T_2 = -\frac{Im(\Omega_1(0)) + \mu_2 Im(\lambda'(\tau_0))}{\tau_0\omega_0},$$
$$\beta_2 = 2Re\Omega_1(0).$$

In the above formulas, μ_2 determines the direction of the Hopf bifurcation; β_2 determines the stability of the bifurcation periodic solutions; T_2 determines the period of the bifurcating periodic solution.

Proposition 3 *([10], [12])*

1. $\mu = 0$ is the Hopf bifurcation value of system (21);

2. If $\mu_2 > 0 (< 0)$ the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solution exists for $\tau > \tau_0 (< \tau_0)$.

2. If $\beta_2 < 0 (> 0)$ the solutions are orbitally stable (unstable).

3. If $T_2 > 0 (< 0)$ the period increases (decreases).

5 Numerical simulations

We use Maple for the numerical simulations. For the parameters $a_1 = 0.5$, $a_2 = 0.15$, $a_3 = 0.12$, $a_4 = 0.13$, $a_5 = 0.1089$ all the conditions from Theorem 1 are satisfied. We obtain $x_{10} = 0.1755$, $x_{20} = 4.269$, $x_{30} = 4.59$. By formula (19) $\tau_0 = 27.676$ and $\omega_0 = 0.277$. The equilibrium point is locally asymptotically stable if $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. For $\tau = \tau_0$ the solution is periodically.

In the following figures we can see the orbits of the system for $\tau_0 = 27.676$ and $\tau_0 = 25.67$.

Fig. 1 $(t, x_1(t))$, $\tau_0 = 27.676$

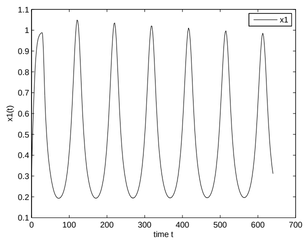


Fig. 2 $(t, x_2(t))$, $\tau_0 = 27.676$

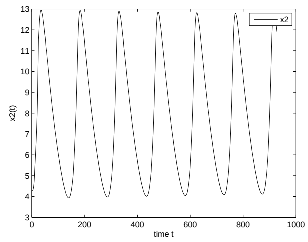
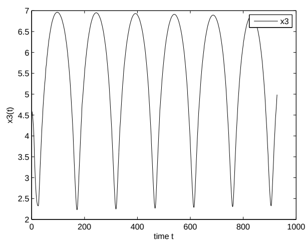


Fig. 3 $(t, x_3(t))$, $\tau_0 = 27.676$



For $\tau_0 = 27.676$, in Fig. 1, Fig. 2, Fig. 3 we can notice that the orbits corresponding to the number of non-adopters ($x_1(t)$), the number of thinkers ($x_2(t)$) and the number of adopters ($x_3(t)$) are periodically with the amplitude $\omega_0 = 0.277$.

In the following figures we can see the orbits of the system for $\tau_0 = 25.67$, $\tau_0 = 27.676$ and $\tau_0 = 28.675$.

Fig. 4 $(t, x_1(t))$, $\tau_0 = 25.67$

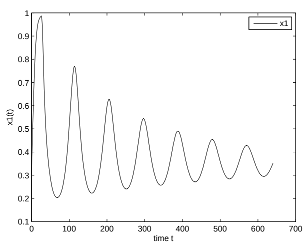


Fig. 5 $(t, x_2(t))$, $\tau_0 = 25.67$

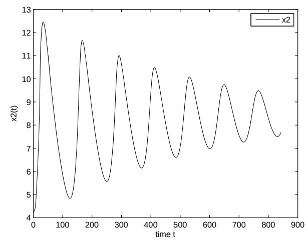
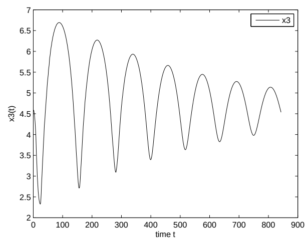


Fig. 6 $(t, x_3(t))$, $\tau_0 = 25.67$



For $\tau_0 = 25.67$, in Fig. 4, Fig. 5, Fig. 6 we can notice that the orbits corresponding to the number of non-adopters $(x_1(t))$, the number of thinkers $(x_2(t))$ and the number of adopters $(x_3(t))$ are stable.

For the stochastic model we have the following orbits:

Fig. 7 $(n, x_1(n, \omega))$, $\tau_0 = 27.676$

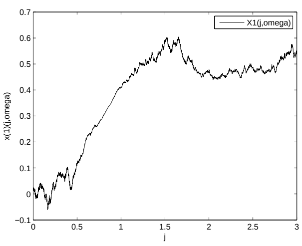


Fig. 8 $(n, x_2(n, \omega))$, $\tau_0 = 27.676$

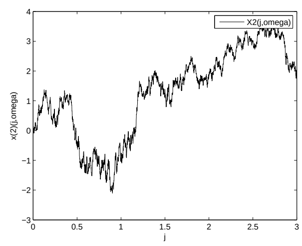
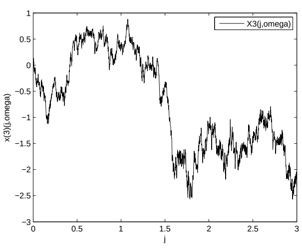


Fig. 9 $(n, x_3(n, \omega))$, $\tau_0 = 27.676$



The numerical simulations verify the theoretical results.

6 Conclusions

In the present paper an advertising diffusion model is considered, where a firm devotes a fixed proportion of sales to adverting. There are three variables corresponding to the adopter class, the thinker class and the non-adopter. It is assumed that the thinker class population becomes adopter class after some time delay τ, because the effect of the advertisement in the non-adopter class is not immediately. The model is described by a non-linear differential system with delay. The stability analysis has been carried out about the unique positive equilibrium point. The system is stable without delay under some conditions involving parameters. Introducing the delay parameter, a Hopf-bifurcation takes place due to which, the system is stable when $\tau < \tau_0$ and beyond that it unstable and it has an oscillatory character in $\tau = \tau_0$. The direction and the stability of the Hopf bifurcation is analyzed.

The delay can lead to instability and the stochastic aspects capture the uncertainty about the environment in which the system is operating. Thus, the stochastic model is introduced and using the numerical simulations the orbits can be visualized.

Using this model, new product marketing strategies with respect to advertising intensity can be developed. The present analysis can be used for other economic models or other process as in [2], [3], [4], [5], [15], [21].

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