Reduced-order Observer-based Containment Control of General Linear Multi-agent Systems With Multiple Interaction Leaders

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Abstract: This paper investigates the reduced-order observer-based containment control problems of general linear multi-agent systems (MASs) with multiple interaction leaders. Assuming that (A, B, C) is stabilizable and detectable and the directed graph is weakly connected, we establish the necessary and sufficient containment control criteria for continuous-time MASs and discrete-time MASs, respectively. Our main results show that the eigenvalues of the Laplacian matrix, the communication topology graph, the selection of protocol parameters and gain matrices play an important role in the achievement of containment control. Finally, numerical simulations are given to illustrate our main results.

Key–Words: Containment control; Consensus; Reduced-order observer; General linear system.

1 Introduction

In recent years, cooperative control of multi-agent systems has made great progress due to the rapid and sustainable developments of computer science and communication technologies. It has attracted more and more attention in a wide range including system control theory, applied mathematics, biology, communication, computer science and so on. This is partly due to its challenging features and many applications in multiple spacecraft alignment, rendezvous of multiple vehicles, cooperative surveillance and so on. For multi-agent systems, consensus is an important and fundamental problem, which generally means to develop distributed control protocols such that the agents reach an agreement on a state of interest. The consistent value might represent physical quantities such as altitude, position, voltage and so on. Consensus algorithms were studied in [1-5] for a group of single, double and high-order integrators under fixed/switching network topologies. The relatively complete coverage on consensus can be found in [6-7].

The earlier mentioned references mainly focus on consensus for a group of agents without any leader or with only a leader. However, in some practical applications, there may exist multiple stationary or dynamic leaders in the agent network. In the case of multiple leaders, the agents can fulfill more complicated tasks and the containment control problem arises, where the followers are driven into a given geometric space spanned by the leaders. As a kind of cooperative control, containment control has several potential applications. For instance, a flock of autonomous agents (designated as leaders) equipped with necessary sensors to detect the obstacles can be used to safely maneuver another flock of agents (designated as followers) from one target to another, such that the followers are contained within the safety area formed by the leaders. Containment control of multi-agent systems was put forward firstly in [8]. In [9], containment control of general linear multi-agent systems under weakly connected topologies was studied, where it assumed that the system state can be obtained directely. Suppose that distributed dynamic containment controllers were based on the relative outputs of the neighboring, [10] obtained the sufficient conditions to achieve containment control with full-order observer-based protocol, but there existed no interaction between leaders. Based on [10], the authors in [11] studied the containment control problems with full-order observe-based protocol, whereas, there existed interaction between leaders. The full-order observer constructs an estimate of the entire state, while part of the state information is already reflected in the system output. Because of this, the author in [12] proposed a reduced-order observer-based protocol and studied the containment control of multi-agent systems. However, [12] neglected the interaction between leaders.

Motivated by the above discussions, this paper will focus on investigating the reduced-order observer-based containment control problems of general linear multi-agent systems with multiple interaction leaders. In addition, this paper investigates the containment control with continuous-time and discrete-time protocol under weakly connected topology, respectively. The **main contributions** of this paper are as follows. Firstly, compared with previous papers that study the multi-agent systems with first-order or second-order dynamics, this paper investigates the containment control with general linear system. Secondly, compared with [11], we study the reduced-order observer-based protocols, which can eliminate the redundancy of the system. Thirdly, compared with [12], we study the containment control with multiple interaction leaders. Consequently, we get the necessary and sufficient conditions of achieving containment control and we select the parameters to guarantee the achievement of containment control.

This paper is organized as follows. In Section 2, we introduce some graph knowledge and basic definitions. Our main results are given in Section 3. Several examples are given in Section 4 to illustrate the effectiveness of our results. Conclusions are finally drawn in section 5.

1.1 Notations

We use standard notations throughout this paper. Given a complex number $\lambda \in \mathcal{C}$, $Re(\lambda), Im(\lambda)$ and $|\lambda|$ are the real part, the imaginary part and the modulus of λ, respectively. $\mathbb{i}$ is the imaginary unit. The superscript $\top$ means transpose for real matrices and H means conjugate transpose for complex matrices. Let $\mathbb{R}^{n\times n}$ be the set of $n \times n$ real matrix. $M > 0 (M < 0)$ means that matrix M is positive definite (negative definite). I_n represents the identity matrix of dimension n, and I denotes the identity matrix of an appropriate dimension. $det(\cdot)$ represents the determinant of a matrix. $A \otimes B$ denotes the Kronecker product. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2 Preliminaries

In this section, some graph theory knowledge and definitions are presented.

2.1 Graph theory

It is natural to model information exchange among agents by weighted directed graphs. $\mathscr{G} = \{\mathscr{V}, \mathscr{E}, \mathscr{A}\}$ is a weighted directed graph, where $\mathscr{V} = \{v_1, \cdots, v_n\}$ is a finite nonempty node set, $\mathscr{E} \in \mathscr{V} \times \mathscr{V}$ is an edge set, and $\mathscr{A} = [a_{ij}] \in R^{n\times n}$ is a weighted adjacency matrix with nonnegative elements. In this paper, $a_{ii} = 0$ for $i = 1, \cdots, N$, and $a_{ij} > 0$ implies that there is an edge from v_j to v_i. Let $\mathcal{D} = [d_{ij}] \in R^{n\times n}$ be a row-stochastic matrix with the additional assumption that $d_{ii} > 0, d_{ij} > 0$ if $(v_j, v_i) \in \mathscr{E}$ and $d_{ij} = 0$ otherwise. An edge (v_i, v_j) in a weighted directed graph denotes that agent j can obtain information from agent i, but not necessarily vice versa. The set of neighbors of node i is denoted by $\mathscr{N}_i = \{v_j \in \mathscr{V} : (v_j, v_i) \in \mathscr{E}, j \neq i\}$.

A directed path in a directed graph $\mathscr{G}$ is a sequence $(v_{i_1}, \cdots, v_{i_k})$ of vertices such that for $s = 1, \cdots, k-1, (v_{i_s}, v_{i_{s+1}}) \in \mathscr{E}$ and a weak path, with either $(v_{i_s}, v_{i_{s+1}}) \in \mathscr{E}$ or $(v_{i_{s+1}}, v_{i_s}) \in \mathscr{E}$. A weak path in a directed graph $\mathscr{G}$ is a sequence $(v_{i_1}, \cdots, v_{i_k})$ of vertices such that for $s = 1, \cdots, k-1$, either $(v_{i_s}, v_{i_{s+1}}) \in \mathscr{E}$ or $(v_{i_{s+1}}, v_{i_s}) \in \mathscr{E}$. A directed tree is a directed graph, where every node, except for the root, has exactly one parent. A directed spanning tree is a directed tree, which consist of all the nodes and some edges in $\mathscr{G}$. A directed graph $\mathscr{G}$ is strongly connected if between every pair of distinct vertices v_i, v_j in $\mathscr{G}$, there is a directed path that begins at v_i and ends at v_j, and is weakly connected if any two vertices can be jointed by a weak path. A strong component of a directed graph is an induced subgraph that is maximal, subjected to being strongly connected.

The Laplacian matrix $\mathcal{L} = [l_{ij}] \in R^{n\times n}$ of $\mathscr{G}$ is defined as

$$l_{ij} = \begin{cases} \sum\limits_{k=1,k\neq i}^{n} a_{ik}, i = j; \\ -a_{ij} \qquad ,i \neq j. \end{cases}$$

The in-degree and out-degree of node i are, respectively, defined as $\deg_{in}(i) = \sum\limits_{j\in\mathscr{N}_i} a_{ij}$, $\deg_{out}(i) = \sum\limits_{j\in\mathscr{N}_i} a_{ji}$. The degree matrix is an $n \times n$ matrix defined as $D = [\tilde{d}_{ij}]$, where $\tilde{d}_{ij} = \deg_{in}(i)$ for $i = j$, otherwise, $\tilde{d}_{ij} = 0$. Then the Laplancian matrix of the graph $\mathscr{G}$ can be written as $\mathcal{L} = D - \mathscr{A}$.

In this paper, the definition of condensation directed graph of $\mathscr{G}(\mathscr{A})$ is the same as [9]. Let $\bar{\mathscr{G}}(\bar{\mathscr{A}})$ be the condensation directed graph of $\mathscr{G}(\mathscr{A})$, where the vertex set consists of all the strong components of $\mathscr{G}(\mathscr{A})$, i.e., denote $\bar{v}_i$ as the strong component $\mathscr{G}_i$. There is an edge between a pair of distinct vertices $\bar{v}_i$ and $\bar{v}_j$, where $i \neq j$, in $\bar{\mathscr{G}}(\bar{\mathscr{A}})$ if and only if there exists $v_i \in \mathscr{V}(\mathscr{G}_i)$ and $v_j \in \mathscr{V}(\mathscr{G}_j)$, such that $(v_i, v_j) \in \mathscr{E}(\mathscr{G}(\mathscr{A}))$.

2.2 Definitions

In this subsection, we give some basic definitions below.

Definition 1. *For a directed graph $\bar{\mathscr{G}}(\bar{\mathscr{A}})$. We define the vertices with 0 in-degree as leaders, and the others as followers.*

In this paper, we assume that there are M leaders and $S-M$ followers.

Definition 2. *For a directed graph $\mathscr{G}(\mathscr{A})$. We define the vertices in the leaders set as boundary vertices, and the others as internal vertices.*

Definition 3. *[13] A set $K \in R^m$ is said to be convex if $(1-\gamma)x+\gamma y \in K$ for arbitrary $x \in K, y \in K$ and $0 < \gamma < 1$. The convex hull of a finite set of points $x_1, \cdots, x_n \in R^m$ is the minimal convex set containing all points $x_i, i = 1, \cdots, n$, denoted by $co\{x_1, \cdots, x_n\}$. Particularly, $co\{x_1, \cdots, x_n\} = \{\sum_{i=1}^{n} \alpha_i x_i | \alpha_i \in R, \alpha_i > 0, \sum_{i=1}^{n} \alpha_i = 1\}$.*

Definition 4. *If the leaders in the same strong component achieve consensus asymptotically, and all the followers converge to the convex hull spanned by the leaders, then the containment control achieved.*

3 Containment Control Analysis

In this section, we assume that the directed graph is weakly connected and each agent has access to the relative output measurements with respect to its neighbors. Next, we will analyze the containment control of multi-agent systems with reduced-order observer-based continuous-time protocol and discrete-time protocol, respectively.

3.1 Continuous-time case

Consider N agents with general linear dynamics as follows

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t), t \in \mathbb{R}^+, \\ y_i(t) = Cx_i(t), i = 1, \cdots, N, \end{cases} \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ is the state, $u_i(t) \in \mathbb{R}^p$ is the control input, and $y_i(t) \in \mathbb{R}^q$ is the measured output. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$ are the system matrix, input matrix and the output matrix, respectively. Without loss of generality, C is assumed to have full row rank.

Here, we adopt the same reduced-order observer-based continuous-time protocol as shown in [12]

$$\begin{aligned} \dot{v}_i &= Fv_i + Gy_i + TBu_i, \\ u_i &= cKQ_1 \sum_{j\in\mathcal{F}\cup\mathcal{R}} a_{ij}(y_i - y_j) \\ &\quad + cKQ_2 \sum_{j\in\mathcal{F}\cup\mathcal{R}} a_{ij}(v_i - v_j), \end{aligned} \tag{2}$$

where $i \in \mathcal{F} \cup \mathcal{R}, v_i \in R^{n-q}$ is the protocol state, $c > 0$ is the coupling strength, $F \in R^{(n-q)\times(n-q)}$ is hurwitz and has no eigenvalues in common with those of A, $G \in R^{(n-q)\times q}, T \in R^{(n-q)\times n}$ is the unique solution to the following Sylvester equation:

$$TA - FT = GC, \tag{3}$$

which further satisfies that $\begin{bmatrix} C \\ T \end{bmatrix}$ is nonsingular, $Q_1 \in R^{n\times q}$ and $Q_2 \in R^{n\times(n-q)}$ are given by $\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1}$, and $K \in R^{p\times n}$ is the feedback gain matrix to be designed.

Remark 5. *The full-order observer-based protocols give an estimate of all the state, in practice, part of the information has already reflected in the system output. The reduced-order observer-based protocol proposed here eliminate this redundancy compared with full-order observer and thereby can considerably reduce the dimension of the system.*

Let $z_i = [x_i^\top, v_i^\top]^\top$ and $z = [z_1^\top, \cdots, z_N^\top]$. Then, system (1) with protocol (2) can be written as

$$\dot{z}(t) = (I_N \otimes \mathcal{M} + c\mathcal{L} \otimes \mathcal{H})z(t), \tag{4}$$

where $\mathcal{L}$ is the Laplacian matrix of $\mathscr{G}$, $\mathcal{M} = \begin{bmatrix} A & 0 \\ GC & F \end{bmatrix}, \mathcal{H} = \begin{bmatrix} BKQ_1C & BKQ_2 \\ TBKQ_1C & TBKQ_2 \end{bmatrix}$.

Let $\varepsilon_i(t) = \sum_{j\in\mathcal{F}\cup\mathcal{R}} a_{ij}(z_i(t) - z_j(t)), i \in \mathcal{F} \cup \mathcal{R}$. Then, we obtain

$$\begin{aligned} \varepsilon(k+1) &= (\mathcal{L} \otimes I_{2n-q})z(k) \\ &= (\mathcal{L} \otimes I_{2n-q})(I_N \otimes \mathcal{M} + c\mathcal{L} \otimes \mathcal{H}) \\ &\quad (\mathcal{L}^{-1} \otimes I_{2n-q})\varepsilon(k) \\ &= (I_N \otimes \mathcal{M} + c\mathcal{L} \otimes \mathcal{H})\varepsilon(k). \end{aligned} \tag{5}$$

Suppose that the directed graph is weakly connected and contains $M (M \geq 1)$ zero in-degree strong components, then the Laplacian matrix $\mathcal{L}$ can be written as

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \mathcal{L}_{22} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \mathcal{L}_{MM} & 0 & \cdots & 0 \\ \mathcal{L}_{M+1,1} & \mathcal{L}_{M+2,1} & \cdots & \mathcal{L}_{M+1,M} & \mathcal{L}_{M+1,M+1} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{S1} & \mathcal{L}_{S2} & \cdots & \mathcal{L}_{S,M} & \mathcal{L}_{S,M+1} & \cdots & \mathcal{L}_{SS} \end{bmatrix},$$

where $\mathcal{L}_{ii} \in R^{n_i \times n_i}$ is the Laplacian matrix corresponding to the strong component $\mathscr{G}_i, 1 \le i \le M$.

Lemma 6. *[9] Suppose that directed graph $\mathscr{G}(\mathscr{A})$ is weakly connected and $\mathcal{L}$ is the Laplacian matrix of $\mathscr{G}$. Then $Rank(\mathcal{L}) = N - M$ if and only if $\mathscr{G}(\mathscr{A})$ contains M zero in-degree strong components.*

Let $\mathscr{V}(\mathscr{G}_1) = \{v_1, v_2, \cdots, v_{k_1}\}$,
$\mathscr{V}(\mathscr{G}_2) = \{v_{k_1+1}, v_{k_1+2}, \cdots, v_{k_2}\}, \cdots,$
$\mathscr{V}(\mathscr{G}_M) = \{v_{k_{M-1}+1}, v_{k_{M-1}+2}, \cdots, v_{k_M}\}$,
$\mathscr{V}(\mathscr{G})\backslash\mathscr{V}(\cup_{i=1}^{M}) = \{v_{k_M+1}, v_{k_M+2}, \cdots, v_{k_{M+1}}\}$, where $k_1 = n_1, k_2 = k_1 + n_2, \cdots, k_M = k_{M-1} + n_M, k_{M+1} = k_M + n_{M+1} + \cdots + n_s$. Then, the Laplacian matrix becomes

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{11} & 0 & \cdots & 0 & 0 \\ 0 & \mathcal{L}_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathcal{L}_{MM} & 0 \\ \mathcal{L}_{M+1,1} & \mathcal{L}_{M+1,2} & \cdots & \mathcal{L}_{M+1,M} & \mathcal{L}_{M+1,M+1} \end{bmatrix}$$

Remark 7. *From the notations above, we know n_i is the number of agents in $\mathscr{G}_i, i = 1, \cdots, M$.*

From the Definition 2, we can obtain that $v_i, i = 1, 2, \cdots, k_M$ are the boundary vertices and $v_i, i = k_M + 1, \cdots, k_{M+1}$ are the internal vertices.

Let $\bar{z}_1 = [z_1^\top, \cdots, z_{k_1}^\top]^\top$,
$\bar{z}_2 = [z_{k_1+1}^\top, \cdots, z_{k_2}^\top]^\top, \cdots,$
$\bar{z}_{M+1} = [z_{k_M+1}^\top, \cdots, z_{k_{M+1}}^\top]^\top$, then (4) can be rewritten as follows

$$\begin{bmatrix} \dot{\bar{z}}_1(t) \\ \dot{\bar{z}}_2(t) \\ \vdots \\ \dot{\bar{z}}_M(t) \\ \dot{\bar{z}}_{M+1}(t) \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{L}}_{11} & 0 & \cdots & 0 & 0 \\ 0 & \bar{\mathcal{L}}_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \bar{\mathcal{L}}_{MM} & 0 \\ \bar{\mathcal{L}}_{M+1,1} & \bar{\mathcal{L}}_{M+1,2} & \cdots & \bar{\mathcal{L}}_{M+1,M} & \bar{\mathcal{L}}_{M+1,M+1} \end{bmatrix} \times \begin{bmatrix} \bar{z}_1(t) \\ \bar{z}_2(t) \\ \vdots \\ \bar{z}_M(t) \\ \bar{z}_{M+1}(t) \end{bmatrix}, \tag{6}$$

where $\bar{\mathcal{L}}_{ii} = I_{n_i} \otimes \mathcal{M} + c\mathcal{L}_{ii} \otimes \mathcal{H}, \bar{\mathcal{L}}_{M+1,i} = c\mathcal{L}_{M+1,i} \otimes \mathcal{H}, , i = 1, \cdots, M$, and $\bar{\mathcal{L}}_{M+1,M+1} = I_{n_{k_{M+1}-k_M}} \otimes \mathcal{M} + c\mathcal{L}_{M+1,M+1} \otimes \mathcal{H}$.

Next, we will give the necessary and sufficient containment control criteria for MASs with continuous-time reduced-order observer-based protocol as follows.

Theorem 8. *Assume that the directed graph $\mathscr{G}(\mathscr{A})$ is weakly connected and contains M zero in-degree strong components. System (1) under observer-based protocol (2) achieves containment control if and only if $A + c\lambda_i BK$ is Hurwitz stable, $\lambda_i \in \wedge^+(\mathcal{L})$. Specifically, the final states of the leaders and followers are given as*

$$\begin{aligned} \lim_{t\to\infty} \quad \bar{z}_i(t) &= \lim_{t\to\infty} 1_{n_i} \bar{f}_i^\top \otimes e^{\mathcal{M}t} \bar{z}_i(0), i = 1, \cdots, M, \\ \lim_{t\to\infty} \quad \bar{z}_{M+1}(t) &= -\lim_{t\to\infty} \mathcal{L}_{M+1,M+1}^{-1} \\ &\quad \times [\mathcal{L}_{M+1,1}, \mathcal{L}_{M+1,2}, \cdots, \mathcal{L}_{M+1,M}] \begin{bmatrix} \bar{z}_1(t) \\ \vdots \\ \bar{z}_M(t) \end{bmatrix} \end{aligned} \tag{7}$$

Proof. (Sufficiency.) From Lemma 6, we know that the algebraic multiplicity of eigenvalue 0 of the Laplacian matrix $\mathcal{L}$ is M. Let

$$\omega_{r1} = \begin{bmatrix} 1_{n_1} \\ 0_{n_2 \times 1} \\ \vdots \\ 0_{n_M \times 1} \\ -\mathcal{L}_{M+1,M+1}^{-1}\mathcal{L}_{M+1,1}1_{n_1} \end{bmatrix}, \cdots,$$

$$\omega_{rM} = \begin{bmatrix} 0_{n_1 \times 1} \\ 0_{n_2 \times 1} \\ \vdots \\ 1_{n_M} \\ -\mathcal{L}_{M+1,M+1}^{-1}\mathcal{L}_{M+1,M}1_{n_M} \end{bmatrix}.$$

It's easy to get that $\mathcal{L}\omega_{ri} = 0 \cdot \omega_{ri}, i = 1, \cdots, M$. Moreover, ω_{ri} are linearly independent vectors, and we can get that the geometric multiplicity of eigenvalue 0 is also M. Let $\omega_{l1} = [\bar{f}_1^\top, 0, \cdots, 0, 0], \cdots, \omega_{lM} = [0, 0, \cdots, \bar{f}_M^\top, 0]$. We can get that $\omega_{li}^\top \mathcal{L} = 0 \cdot \omega_{li}^\top, i = 1, \cdots, M$.

Then the Laplacian matrix $\mathcal{L}$ can be written in Jordan canonical form as follows

$$\begin{aligned} \mathcal{J} &= \mathcal{W}^{-1}\mathcal{L}\mathcal{W} \\ &= \begin{bmatrix} \omega_{l1}^\top \\ \vdots \\ \omega_{lM}^\top \\ \omega_{l,M+1}^\top \\ \vdots \\ \omega_{lN}^\top \end{bmatrix} \mathcal{L} \begin{bmatrix} \omega_{r1}^\top & \cdots & \omega_{rM}^\top & \omega_{r,M+1}^\top & \cdots & \omega_{rN}^\top \end{bmatrix} \\ &= \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \mathcal{J}_1 \end{bmatrix}, \end{aligned}$$

where $\mathcal{J}_1 \in \mathbb{C}^{(N-M)\times(N-M)}$ is the Jordan upper diagonal block matrix whose diagonal entries are the positive eigenvalues of $\mathcal{L}$. Then, we can get that

$$\begin{aligned} & (\mathcal{W}^{-1}\otimes I_{2N})(I_N\otimes\mathcal{M}+c\mathcal{L}\otimes\mathcal{H})(\mathcal{W}\otimes I_{2N}) \\ = \; & I_N\otimes\mathcal{M}+c\mathcal{W}^{-1}\mathcal{L}\mathcal{W}\otimes\mathcal{H} \\ = \; & \begin{bmatrix} \mathcal{M} & & & \\ & \mathcal{M} & & \\ & & \mathcal{M} & \\ & & & I_{N-M}\otimes\mathcal{M}+c\mathcal{J}_1\otimes\mathcal{H} \end{bmatrix} \triangleq \Delta. \end{aligned}$$

Set $\mathcal{Q}=\begin{bmatrix} I & 0 \\ -T & I \end{bmatrix}$. One obtains that

$$\begin{aligned} & \mathcal{Q}(\mathcal{M}+c\lambda_i\mathcal{H})\mathcal{Q}^{-1} \\ = \; & \begin{bmatrix} A+c\lambda_iBK(Q_1C+Q_2T) & c\lambda_iBKQ_2 \\ -AT+GC+FT & F \end{bmatrix} \\ = \; & \begin{bmatrix} A+c\lambda_iBK & c\lambda_iBKQ_2 \\ 0 & F \end{bmatrix}. \end{aligned}$$

Since $A+c\lambda_iBK$ and F are Hurwitz stable, then

$$\lim_{t\to\infty} e^{(I_{N-M}\otimes\mathcal{M}+c\mathcal{J}_1\otimes\mathcal{H})t}=0,$$

where λ_i is the positive eigenvalue of $\mathcal{L}$, $\mathcal{M},\mathcal{H},\mathcal{J}_1$ are defined above.

Next, we will consider the solution of system (4).

$$\begin{aligned} & \lim_{t\to\infty} z(t) \\ = \; & \lim_{t\to\infty}\begin{bmatrix} \bar{z}_1(t) \\ \bar{z}_2(t) \\ \vdots \\ \bar{z}_M(t) \\ \bar{z}_{M+1}(t) \end{bmatrix} \\ = \; & \lim_{t\to\infty} e^{(I_N\otimes\mathcal{M}+c\mathcal{L}\otimes\mathcal{H})t}z(0) \\ = \; & \lim_{t\to\infty}(\mathcal{W}\otimes I_{2N})e^{\Delta t}(\mathcal{W}^{-1}\otimes I_{2N})z(0) \\ = \; & \lim_{t\to\infty}\begin{bmatrix} 1_{n_1}\bar{f}_1^\top\otimes e^{\mathcal{M}t} & 0 & \cdots & 0 & 0 \\ 0 & 1_{n_2}\bar{f}_2^\top\otimes e^{\mathcal{M}t} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1_{n_M}\bar{f}_M^\top\otimes e^{\mathcal{M}t} & 0 \\ Q_{M+1,1} & Q_{M+1,2} & \cdots & Q_{M+1,M} & 0 \end{bmatrix} z(0), \end{aligned}$$

where $Q_{M+1,i} = -(\mathcal{L}_{M+1,M+1}^{-1}\mathcal{L}_{M+1,i}1_{n_i}\bar{f}_i^\top)\otimes e^{\mathcal{M}t}, i=1,\cdots,M$.

Then, we can get that the final states of boundary agents in $\mathscr{G}_i$ are as follows

$$\lim_{t\to\infty}\bar{z}_i(t)=\lim_{t\to\infty}1_{n_i}\bar{f}_i^\top\otimes e^{\mathcal{M}t}\bar{z}_i(0), i=1,\cdots,M,$$

the final states of the followers are given as follows

$$\begin{aligned} & \lim_{t\to\infty}\bar{z}_{M+1}(t) \\ = \; & -\lim_{t\to\infty}\mathcal{L}_{M+1,M+1}^{-1}\times \\ & [\mathcal{L}_{M+1,1},\mathcal{L}_{M+1,2},\cdots,\mathcal{L}_{M+1,M}]\begin{bmatrix} \bar{z}_1(t) \\ \vdots \\ \bar{z}_M(t) \end{bmatrix} \end{aligned}$$

As proved in [9], we can get that $-\mathcal{L}_{M+1,M+1}^{-1}[\mathcal{L}_{M+1,1},\mathcal{L}_{M+1,2},\cdots,\mathcal{L}_{M+1,M}]$ is a nonnegative matrix with row sums being $\mathbb{1}$, i.e.,
(1) $-\mathcal{L}_{M+1,M+1}^{-1}[\mathcal{L}_{M+1,1},\mathcal{L}_{M+1,2},\cdots,\mathcal{L}_{M+1,M}]\geq 0$,
(2) $-\mathcal{L}_{M+1,M+1}^{-1}[\mathcal{L}_{M+1,1},\mathcal{L}_{M+1,2},\cdots,\mathcal{L}_{M+1,M}]1_{k_M}=1_{k_{M+1}-k_M}$.
Moreover, the agent in the same strong component can achieve consensus, then from Definition 4, system (1) under reduced-order observer-based protocol (2) achieves containment control.

(Necessity.) Suppose that there exists at least one $\lambda_i\in\wedge^+(\mathcal{L})$ such that $A+c\lambda_iBK$ is not Hurwitz stable, when $\lambda_i\in\wedge^+(\mathcal{L}_{ii}), i=1,\cdots,M$, which means the agents in the same strong component will not achieve consensus, when $\lambda_i\in\wedge^+(\mathcal{L}_{M+1,M+1}^+)$, which means the internal agents will not converge to the convex hull of boundary agents. Therefore, for all $\lambda_i\in\wedge^+(\mathcal{L})$, $A+c\lambda_iBK$ must be Hurwitz stable.

From Theorem 8, we can see even there exists interaction between leaders, our containment control criteria is consistent with those shown in [12]. [12] proposes an algorithm to guarantee the achievement control of system (1). Specially, the containment control can be ascribed to the Hurwitz stable of $A+c\lambda_iBK$. Then, we can choose the parameters as follows.

- Let $c=\dfrac{1}{\min\{Re(\lambda_i)\}}$, $K=-B^\top P^{-1}$, where λ_i is the i-th non-zero eigenvalue of $\mathcal{L}$, $P>0$ is the solution of linear matrix inequality(LMI):
$$AP+PA^\top-2BB^\top<0.$$
- Let $K=-B^\top Q^{-1}$, where $Q>0$ is the solution of LMI:
$$AQ+QA^\top-2BB^\top+2\alpha Q<0.$$
c is the same as above. Then, system (1) achieves containment control with a convergence rate larger than α.

3.2 Discrete-time case

In this subsection, we adopt the following discrete-time dynamics below

$$\begin{aligned} x_i(k+1) &= Ax_i(k) + Bu_i(k), \\ y_i(k+1) &= Cx_i(k), i = 1, \cdots, N, \end{aligned} \quad (8)$$

where $x_i(k), u_i(k), y_i(k)$ are the state, control input and measured output at the current time instant, respectively. A, B, C are the same as above.

Similar to the continuous-time case, we adopt the following discrete-time observer-based protocol

$$\begin{aligned} v_i(k+1) &= Fv_i(k) + Gy_i(k) + TBu_i(k), \\ u_i(k) &= KQ_1 \sum_{j\in\mathcal{F}\cup\mathcal{R}} d_{ij}(y_i(k) - y_j(k)) \\ &\quad + KQ_2 \sum_{j\in\mathcal{F}\cup\mathcal{R}} d_{ij}(v_i(k) - v_j(k)), \end{aligned} \quad (9)$$

where $i \in \mathcal{F} \cup \mathcal{R}$, $v_i \in R^{n-q}$ is the state of the protocol, $F \in R^{(n-q)\times(n-q)}$ is Schur stable and have no eigenvalues in common with those of A, $G \in R^{(n-q)\times q}$, $T \in R^{(n-q)\times n}$ is the unique solution of equation (3), which further satisfies that $\begin{bmatrix} C \\ T \end{bmatrix}$ is nonsingular, $Q_1 \in R^{n\times q}$ and $Q_2 \in R^{n\times(n-q)}$ are given by $\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1}$, and $K \in R^{p\times n}$ is the feedback gain matrix to be designed, d_{ij} is the (i, j)-th entry of the row-stochastic matrix $\mathcal{D}$.

Let $z_i(k) = [x_i^\top(k), v_i^\top(k)]$, $z(k) = [z_1^\top(k), \cdots, z_N^\top(k)]$, we can get

$$z(k+1) = (I_N \otimes \mathcal{M} + (I_N - \mathcal{D}) \otimes \mathcal{H})z(k), \quad (10)$$

where $\mathcal{M}, \mathcal{H}, \mathcal{D}$ are defined above.

Let $\varepsilon_i(k) = \sum_{j\in\mathcal{F}\cup\mathcal{R}} a_{ij}(z_i(k) - z_j(k)), i \in \mathcal{F}\cup\mathcal{R}$. Then, we obtain

$$\begin{aligned} \varepsilon(k+1) &= (\mathcal{L} \otimes I_{2n-q})z(k+1) \\ &= (\mathcal{L} \otimes I_{2n-q})(I_N \otimes \mathcal{M} + (I_N - \mathcal{D}) \otimes \mathcal{H}) \\ &\quad (\mathcal{L}^{-1} \otimes I_{2n-q})\varepsilon(k) \\ &= (I_N \otimes \mathcal{M} + (I_N - \mathcal{D}) \otimes \mathcal{H})\varepsilon(k). \end{aligned} \quad (11)$$

Similar to the continuous-time case, $\mathcal{D}$ can be written as

$$\mathcal{D} = \begin{bmatrix} \mathcal{D}_{11} & 0 & \cdots & 0 & 0 \\ 0 & \mathcal{D}_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathcal{D}_{MM} & 0 \\ \mathcal{D}_{M+1,1} & \mathcal{D}_{M+1,2} & \cdots & \mathcal{D}_{M+1,M} & \mathcal{D}_{M+1,M+1} \end{bmatrix}. \quad (12)$$

Let $\bar{z}_1 = [z_1^\top, \cdots, z_{k_1}^\top]^\top$,
$\bar{z}_2 = [z_{k_1+1}^\top, \cdots, z_{k_2}^\top]^\top, \cdots,$
$\bar{z}_{M+1} = [z_{k_M+1}^\top, \cdots, z_{k_{M+1}}^\top]^\top$, we obtain

$$\begin{bmatrix} \bar{z}_1(k+1) \\ \bar{z}_2(k+1) \\ \vdots \\ \bar{z}_M(k+1) \\ \bar{z}_{M+1}(k+1) \end{bmatrix} = \bar{\mathcal{D}} \begin{bmatrix} \bar{z}_1(k) \\ \bar{z}_2(k) \\ \vdots \\ \bar{z}_M(k) \\ \bar{z}_{M+1}(k) \end{bmatrix},$$

where $\bar{\mathcal{D}}_{ii} = I_{n_i} \otimes \mathcal{M} + (I_{n_i} - \mathcal{D}_{ii}) \otimes \mathcal{H}, \bar{\mathcal{L}}_{M+1,i} = (I_{n_i} - \mathcal{D}_{M+1,i}) \otimes \mathcal{H}, i = 1, \cdots, M$, and $\bar{\mathcal{D}}_{M+1,M+1} = I_{n_{k_{M+1}-k_M}} \otimes \mathcal{M} + (I_{n_{k_{M+1}-k_M}} - \mathcal{D}_{M+1,M+1}) \otimes \mathcal{H}$, $\bar{\mathcal{D}} = \begin{bmatrix} \bar{\mathcal{D}}_{11} & 0 & \cdots & 0 & 0 \\ 0 & \bar{\mathcal{D}}_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \bar{\mathcal{D}}_{MM} & 0 \\ \bar{\mathcal{D}}_{M+1,1} & \bar{\mathcal{D}}_{M+1,2} & \cdots & \bar{\mathcal{D}}_{M+1,M} & \bar{\mathcal{D}}_{M+1,M+1} \end{bmatrix}$,

Then, we will give the necessary and sufficient conditions of achieving containment control with discrete-time reduced-order observer-based protocol (9) below.

Theorem 9. *Assume that $\mathscr{G}(\mathscr{A})$ is weakly connected and contains M zero in-degree strong components. System (8) under observer-based protocol (9) achieves containment control if and only if $A + (1 - \tilde{\lambda}_i)BK$ is Schur stable, $\tilde{\lambda}_i \in \wedge(\mathcal{D})$. Specifically, the final states of the leaders and followers are given as*

$$\begin{aligned} \lim_{t\to\infty} \bar{z}_i(k) &= \lim_{t\to\infty} 1_{n_i} \bar{f}_i^\top \otimes \mathcal{M}^k \bar{z}_i(0), i = 1, \cdots, M, \\ \lim_{t\to\infty} \bar{z}_{M+1}(k) &= \lim_{t\to\infty} (I - \mathcal{D}_{M+1,M+1})^{-1} \\ &\quad \times \ [\mathcal{D}_{M+1,1}, \mathcal{D}_{M+1,2}, \cdots, \mathcal{D}_{M+1,M}] \begin{bmatrix} \bar{z}_1(k) \\ \vdots \\ \bar{z}_M(k) \end{bmatrix}. \end{aligned}$$

Proof. (Sufficiency.) Similar with the prove of Theorem 1, there exists $\mathcal{W}$ such that

$$\begin{aligned} &(\mathcal{W}^{-1} \otimes I_{2N})(I_N \otimes \mathcal{M} + (I_N - \mathcal{D}) \otimes \mathcal{H})(\mathcal{W} \otimes I_{2N}) \\ &= I_N \otimes \mathcal{M} + \mathcal{W}^{-1}(I_N - \mathcal{D})\mathcal{W} \otimes \mathcal{H} \\ &= \begin{bmatrix} \mathcal{M} & & & \\ & \mathcal{M} & & \\ & & \mathcal{M} & \\ & & & I_{N-M} \otimes \mathcal{M} + (I_{N-M} - \wedge) \otimes \mathcal{H} \end{bmatrix}, \end{aligned}$$

where $\wedge$ is an upper-triangular matrix with $1 - \lambda_i, \lambda_i$ is the eigenvalue of $\mathcal{D}$ except 1.

Then, we can get

$$\begin{aligned} &\mathcal{Q}(\mathcal{M} + c\lambda_i\mathcal{H})\mathcal{Q}^{-1} \\ &= \begin{bmatrix} A + c\lambda_i BK & c\lambda_i BKQ_2 \\ 0 & F \end{bmatrix}. \end{aligned}$$

where $\mathcal{Q}$ is defined above.

By system (10),we get

$$\begin{aligned}
&\lim_{k\to\infty} z(k) \\
= &\lim_{t\to\infty} \begin{bmatrix} \bar{z}_1(t) \\ \bar{z}_2(t) \\ \vdots \\ \bar{z}_M(t) \\ \bar{z}_{M+1}(t) \end{bmatrix} \\
= &\lim_{k\to\infty} (I_N \otimes \mathcal{M} + (I_N - \mathcal{D}) \otimes \mathcal{H})^k z(0) \\
= &\lim_{k\to\infty} (\mathcal{W} \otimes I_{2N}) (\begin{bmatrix} \mathcal{M} & 0 & \cdots & 0 & 0 \\ 0 & \mathcal{M} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathcal{M} & 0 \\ 0 & 0 & 0 & 0 & \Theta \end{bmatrix})^k \\
&(\mathcal{W}^{-1} \otimes I_{2N}) z(0) \\
= &\begin{bmatrix} 1_{n_1} \bar{f}_1^\top \otimes \mathcal{M}^k & 0 & \cdots & 0 & 0 \\ 0 & 1_{n_1} \bar{f}_2^\top \otimes \mathcal{M}^k & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1_{n_1} \bar{f}_M^\top \otimes \mathcal{M}^k & 0 \\ S_{M+1,1} & S_{M+1,2} & \cdots & S_{M+1,M} & 0 \end{bmatrix} z(0),
\end{aligned}$$

where $S_{M+1,i} = (I - \mathcal{D}_{M+1,M+1})^{-1} \mathcal{D}_{M+1,i} 1_{n_i} \bar{f}_i^\top$, $i = 1, \cdots, M, \Theta = I_{N-M} \otimes \mathcal{M} + (I_{N-M} - \Omega) \otimes \mathcal{H}$, $\mathcal{W}$ and $\mathcal{W}^{-1}$ is defined above.

Then, we can get that the final states of agents in $\mathscr{G}_i$ are as follows

$$\bar{z}_i(k) \to 1_{n_i} \bar{f}_i^\top \otimes \mathcal{M}^k \bar{z}_i(0), i = 1, \cdots, M,$$

the final states of the followers are given as follows

$$\begin{aligned}
&\lim_{t\to\infty} \bar{z}_{M+1}(k) \\
= &\lim_{t\to\infty} (I - \mathcal{D}_{M+1,M+1})^{-1} \\
\times \quad &[\mathcal{D}_{M+1,1}, \mathcal{D}_{M+1,2}, \cdots, \mathcal{D}_{M+1,M}] \begin{bmatrix} \bar{z}_1(k) \\ \vdots \\ \bar{z}_M(k) \end{bmatrix}.
\end{aligned}$$

The rest prove is similar to Theorem 8, and it's omitted here for lack of space.

(Necessity.) Similar to Theorem 8, and it's omitted here.

From Theorem 9, we can see even there exists interaction between leaders, our containment control criteria is consistent with those shown in [12]. [12] proposes an algorithm to guarantee the achievement control of system (8). Specially, the containment control can be ascribed to the Schur stable of $A + (1 - \tilde{\lambda}_i) BK$. Then, we can choose the parameters as follow.

- Choose $K = -(B^\top PB + I)^{-1} B^\top PA$, where $P > 0$ is the solution of the modified algebraic Riccati equation (MARE):

$$P = A^\top PA - (1 - \max_{|\tilde{\lambda}_i|<1} |\tilde{\lambda}_i|^2) A^\top PB (B^\top PB + I)^{-1} B^\top PA + Q,$$

$Q > 0$, $\tilde{\lambda}_i$ is the i-th eigenvalue of $\mathcal{D}$.

4 Simulations

In this section, several examples are given to illustrate our main results about continuous-time case in this paper.

Example 1. Assume that the directed graph is given by Fig.1, then the Laplacian matrix is shown as follows, where agent 1, agent 2, agent3 and agent 8, agent 9 are boundary agents, the others are internal agents.

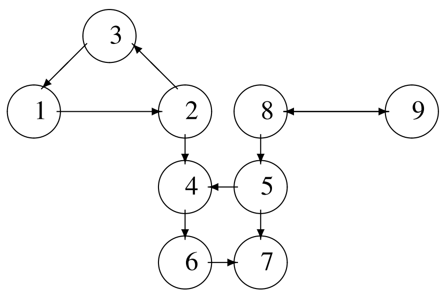

Fig. 1 A weakly connected directed graph.

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

The dynamics in (1) are given as

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

In addition, choose $F = -2, G = -1$. From equation (3), we can obtain $T = [-0.50.25]$.

Then, we can get $Q_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, Q_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$. Choose $K = [-0.8543 - 2.5628], c \geq 1$. Let initial state be $[-5, -1, 2, 5, 4, 1, -5, 2, 0, -1, 2, 1, 2, 1, 2, 3, 2, -3, 4, 2, 4, 6, 3, 5, -6, 0, 4]$. From Fig.2- Fig.4, we get that the containment control can be achieved.

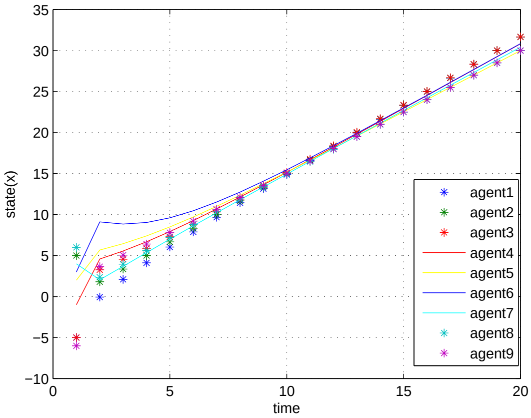

Fig.2 Trajectories of x_1 under continuous-time protocols.

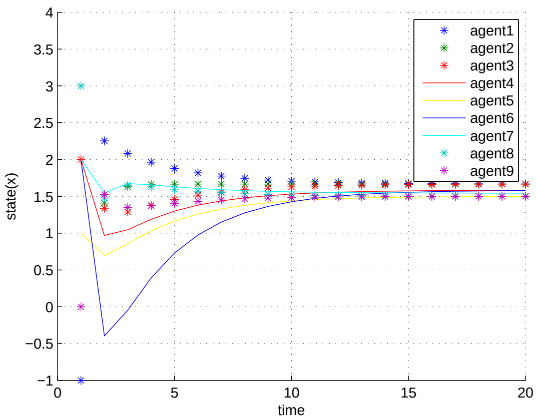

Fig.3 Trajectories of x_2 under continuous-time protocols.

Example 2. Still see Fig.1, choose $A, B, C, F,$, G, T, Q_1, Q_2 and the initial state the same as the above example, select $\alpha = 1, K = [-5.1121 - 3.6481]$. From Fig.5 - Fig.7, we obtain that the containment control can be achieved with a convergence rate larger than 1.

5 Conclusion

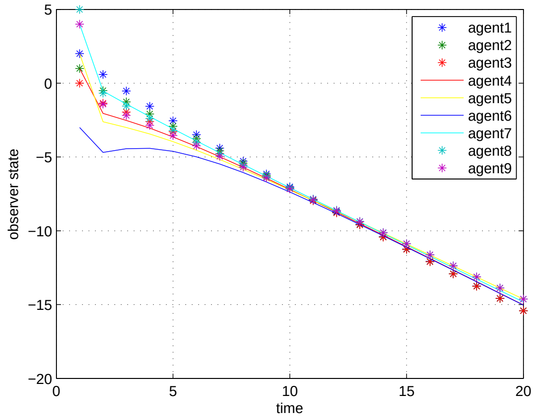

Fig.4 Trajectories of v under continuous-time protocols.

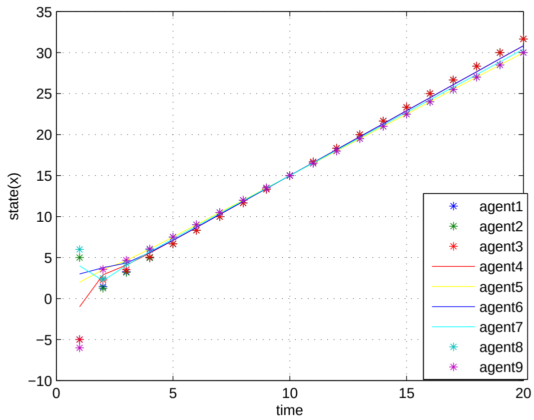

Fig.5 Trajectories of x_1 under continuous-time protocols.

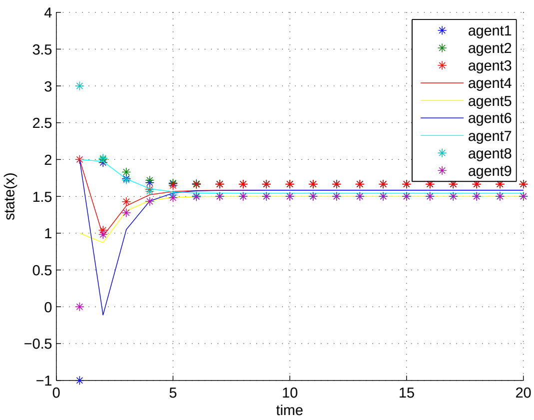

Fig.6 Trajectories of x_2 under continuous-time protocols.

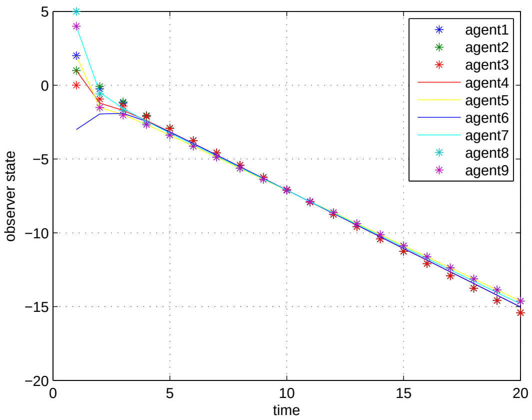

Fig.7 Trajectories of v under continuous-time protocols.

This paper studies the reduced-order observer-based containment control problems for multi-agent systems with multiple interaction leaders, where the interaction topology is weakly connected. Necessary and sufficient containment control criteria are established for both continuous-time and discrete-time multi-agent systems, respectively. Our main results show that the eigenvalues of the Laplacian matrix, the communication topology graph, the selection of protocol parameters and gain matrices play an important role in the achievement of containment control. Our future work includes (observer-based) containment control for multi-agent systems with interaction leaders and time-delays, since time-delays are ubiquitous. This is an interesting and challenging research problems deserving further investigation.

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