Invariant Sets in Sliding Mode Control Theory with Application to Servo Actuator System with Friction

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Abstract: - The mismatched perturbations and system chattering are the two main challenging problems in sliding mode control. This paper tries to solve these problems by deriving the invariant sets created by the sliding mode controller where the present work is devoted to a second order nonlinear affine system. If the state started in these sets it will not leave it for all future time. The first invariant set is found function to the initial condition only. Accordingly, the state bound is estimated and used when determining the gain of the sliding mode controller. This step overcomes an arithmetic difficulty that consists of calculating suitable controller gain value that ensures the attractiveness of the switching manifold with lower chattering behavior. Moreover to eliminate system chattering and to attenuate the effects of the mismatched perturbations, the signum function is replaced by an approximate form which yields a differentiable sliding mode controller. Therefore, the state will converge to a second positively invariant set rather than the origin. The size of this set, as derived here, is function to the parameters that can be chosen by the designer. This result enables us to control the size of the steady state error which means also that the effect of mismatched perturbation is attenuated. The sliding mode controller is then applied to the servo actuator system with friction based on the derived invariant sets. The friction model is represented by the major friction components; Coulomb friction, the Stiction friction, and the viscous friction. The simulation results demonstrate the rightness of the derived sets and the ability of the differentiable sliding mode controller to attenuate the friction effect and regulate the state to the positively invariant set with a prescribed steady state error.

Key-Words: -Positively Invariant Set, Sliding Mode Control, Servo Actuator, Friction Model.

1 Introduction

The Sliding mode control (SMC) is a well known robust technique, for its ability to reject the uncertainties in system model and to the external disturbances that satisfying the matching condition [1]. When the perturbations satisfy the matching condition, it enters the state equation at the same point as the control input [1]. The core idea of designing SMC algorithms consists in two steps; the first is the selecting a manifold in state space such that when the state is confined to it the state reaches asymptotically (slide) the origin unaffecting by the matched perturbations. This type of behavior is known as sliding motion. While the second is to designing a discontinuous control to enforce the state to the manifold and stay there for all future time. [2]. Due to its robustness and ease of implementation, the sliding mode control algorithm has been applied to many engineering application in the recent decade. Stephen J. Doddes et al. [3] design a SMC for the permanent magnet synchronous motor drives. The SMC law was derived for a variable speed wind turbine by Oscar Barambones et al. [4]. Also the SMC was applied to active vehicle suspensions by Milad Geravand et al. [5]. The fuzzy logic is utilized in [5] to adjust the gains for the sliding mode control when applying it to the physical model of semi-active quarter-car suspension.

In spite of the robustness of SMC against the matched disturbances and ease of implementation, but it have two main disadvantages. The first is in the case of mismatched disturbances. F. Castaños et al. [1] suggests use the integral sliding mode to reject the matched disturbances and the H_∞ techniques to attenuate the unmatched one. The Integral Sliding Mode Control (ISMC) is also applied for the nonlinear Systems with matched and mismatched perturbations by Matteo Rubagotti et al. [6]. The ISMC was also applied to the hydraulically actuated active suspension system, by Y. M. Sam et al. [7] in the presence of mismatched uncertainties. The second problem is the chattering behavior which is frequently appears in sliding mode control system for many reasons such as the non ideality of the switching process [2]. Methods for eliminating the chattering in sliding mode control system are reported in [2,8], but the simplest method is introduced by J. J. Sloten [9], where the signum function is replaced by a saturation function. By using saturation function the sliding mode controller will introduce a positively invariant set around the origin with a size determined by the design parameters [10]. H. K. Khalil [10], derives the invariant set created by the sliding mode controller that uses the saturation function as suggested by J. J. Sloten in reference [10]. The formulation of the disadvantages in sliding mode control, as mentioned above, is presented in the following section in terms of the interconnected systems.

2 Problem Statement

Consider a dynamical system described by:

$$\dot{x}_1 = x_2 \tag{1-a}$$

$$\dot{x}_2 = f_1(x_1,x_2) + g_1(x_1,x_2)x_3 + d_1(x_1,x_2,t) \tag{1-b}$$

$$\dot{x}_3 = f_2(x_1,x_2,x_3) + g_2(x_1,x_2,x_3)u + d_2(x_1,x_2,x_3,t) \tag{1-c}$$

$$y = x_1$$

where $d_1(x_1,x_2,t)$ and $d_2(x_1,x_2,x_3,t)$ are the mismatched and matched disturbances respectively. Many dynamical engineering systems have models similar to Eq. (1), like in electromechanical system where the upper subsystem (the model in Eqs. (1-a) and (1-b)) is the mechanical system and Eq. (1-c) is the electrical system model where $d_1(x_1,x_2,t)$ may represents the uncertainty in system model, the friction force and external disturbances. This type of system is also known as interconnected system [11]. The servo actuator system is an example of this type of system, where the torque that actuates the mechanical system is not the actual input (for a D.C. motor the voltage is the signal which represents the actual input). Later, the servo actuator system, as an application example, will be selected to validate the results of the present work.

From a control design point of view the system cannot be linearized via successive differentiation of the output y due to the unknown and may be discontinuous disturbance $d_1(x_1,x_2,t)$. The Backstepping method, which regards x_3 as a virtual controller, cannot also be used since the mismatched disturbance, generally, does not satisfy the linear growth condition [10,11]. This type of disturbance is known as a nonvanishing disturbance because $d_1(0,0,t) \neq 0$. The best that x_3 can do it, if it's regarded as a virtual controller, is to attenuate the effects of $d_1(x_1,x_2,t)$ [10]. This task may be accomplished via $H\infty$ controller. In all cases, and since that the virtual controller must be a differentiable function to the state, the best that we can do it is to regulate the state to a positively invariant set including the origin and stay there for all future time. Controlling the size of the invariant set becomes the challenging task in this respect.

In the present work we suggest the use of the sliding mode control theory in designing the virtual controller to control the size of the positively invariant set. The signum function that causes the discontinuity in sliding mode controller is replaced here by the arc tan function as an approximation. Approximating the signum function will eliminate the chattering and thus solve the chattering problem; also will attenuate the effect of the mismatched disturbance and that by regulating the state to a specified region including the origin. That means we can control the steady state error but of course cannot eliminate it. Consequently, but with a prescribed steady state error, the mismatched disturbance problem is solved. To this end, it is required to derive the invariant and the positively invariant sets for the system uses the sliding mode controller in order to implement the solutions suggested above to the chattering and to the mismatched disturbance problems. This will represents the task of the present work.

The organization of this paper is as follows; the concept of the invariant set and the positively invariant set is introduced in section three. The positively invariant sets created by the sliding mode controller are derived in sections four and five, while in section six the results of the present work are demonstrated where the servo actuator system is used as an application example.

3 Invariant Set

The invariant and positively invariant sets are defined in this section, where we refer mainly to the excellent reference [10].

Conceptually any successful controller try to regulate the state to the origin or to a positively invariant set includes the origin. Moreover the controller will creates a region includes the origin known as the area of attraction [10]. If the state initiate inside the area of attraction the controller will be able to regulate it. If the area of attraction is the whole state space the controller is global, otherwise the control system is local. As an example the linear state feedback for a linearized nonlinear system create an area of attraction around the equilibrium point where the controller is generally local (see p. 138 & 139 in [10]).

The area of attraction forms the so called the positively invariant set. The set notion appears in control theory when we considered three aspects, which are crucial in control systems design, these are: constraints, uncertainties, and design specifications [12].

So, consider the second order autonomous system

$$\dot{x} = f(x)$$

where $x \in \mathcal{R}^2$ and $f(x)$ is a locally Lipschitz map from a domain $D \subset \mathcal{R}^2$ into $\mathcal{R}^2$. Let $x(t)$ be a solution to the second order autonomous system in Eq. (1) and also let $x = 0$ be an equilibrium point; that is $f(0) = 0$. Now, the set M, with respect to the system in Eq. (1), is said to be invariant set if

$$x(0) \in M \Rightarrow x(t) \in M,\ \ \forall t \in \mathcal{R}$$

It means that: if $x(t)$ belongs to M at some time instant, then it belongs to M for all past and future time, i.e., it will never come from a region outside it or leave it for all future time. A set M is said to be a positively invariant set if

$$x(0) \in M \Rightarrow x(t) \in M,\ \ \forall t \geq 0$$

In this case the state may be come from outside the positively invariant set but will never leave for all future time. We also say that $x(t)$ approaches a set M as t approaches infinity, if for each $\varepsilon > 0$there is $T > 0$ such that

$$\text{dist}(x(t), M) < \varepsilon, \forall t > T$$

where $\text{dist}(x(t), M)$ denotes the distance from a point $x(t)$ to a set M.The positive limit point is defined as the limit for the solution $x(t)$ when the time approaches infinity. The set of all positive limit points of $x(t)$ is called the positive limit set of $x(t)$. Accordingly, the asymptotically stable equilibrium is the positive limit set of every solution starting sufficiently near the equilibrium point, while the stable limit cycle is the positive limit set of every solution starting sufficiently near the limit cycle. The solution approaches the limit cycle as $t \to \infty$. The equilibrium point and the limit cycle are invariant sets, since any solution starting in either set remains in the set for $\forall t \in \mathcal{R}$. Moreover, let the set of positively limit set for a point p denoted by the ω limit set of p, namely $\omega(p)$, then some properties of it are stated in the following fact [13]:
Let M be a compact, positively invariant set and $p \in M$, then $\omega(p)$ satisfies the following properties:

1. $\omega(p) \neq \emptyset$, that is, the ω limit set of a point is not empty.
2. $\omega(p)$ is closed.
3. $\omega(p)$ in a positively invariant set.
4. $\omega(p)$ is connected.

This fact, in later sections, will be helpful in determining the behavior of the state trajectory when it is initiated in a positively invariant set.

4 The First Positively Invariant Set

In the following analysis, the first invariant set for a second order system using a sliding mode controller is estimated. Consider the following second order affine system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x) + g(x)u\,,\ g(x) > 0 \end{aligned} \qquad (2)$$

Let the controller in Eq. (2) is the sliding mode controller

$$u = -k\text{sgn}(s)\,, s = x_2 + \lambda x_1\,,\ \lambda > 0 \qquad (3)$$

where s is the switching function which it is selected such that the system at the switching manifold ($s = 0$) is attractive. The attractiveness of the switching manifold is the main idea behind the selection of the sliding mode controller gain k. To calculate k we use the following nonsmooth Lyapunov function

$$V = |s| \qquad (4)$$

The switching manifold is guaranteed to be attractive if the derivative of the Lyapunov function is negative. Consequently,

$$\begin{aligned} \dot{V} &= \dot{s} * \text{sgn}(s) \\ &= \{f(x) - g(x)k * \text{sgn}(s) + \lambda x_2\} * \text{sgn}(s) \\ &= -\{g(x)k - (f(x) + \lambda x_2) * \text{sgn}(s)\} \end{aligned} \qquad (5)$$

Now if k is chosen such that $\dot{V} < 0$, then the switching manifold is attractive. Thus,

$$k > \max\left|\frac{f(x) + \lambda x_2}{g(x)}\right| = h \qquad (6)$$

When k satisfies the inequality (6), then the state reaches $s = 0$ in a finite time. In fact, satisfying inequality (6) is the main calculation problem during design process. Generally, we may use a large gain value to ensure satisfying inequality (6), and consequently the area of attraction becomes large. But the gain cannot be chosen freely without limit due to control saturation. As a result, the size of the area of attraction is determined directly by the gain value.

In this work, we aim to find the invariant set for a second order system that uses the sliding mode controller as given in Eq. (3). When the state initiated in it will never leave it for all future time. Hence, the gain is calculated depending on the invariant set size and the region of attraction will include at least the invariant set. In the literatures, the existence of the invariant set is assumed (by assign the maximum state value) and accordingly the sliding mode controller gain is calculated. In this case the sliding controller will be able to force the state toward the switching manifold at least when it initiated in this invariant set. However, the gain value may be large and again the saturation problem arises. Other designer, uses a certain gain value in the design of sliding controller and, may be, by doing extensive simulations they prove that the area of attraction will include the nominal initial conditions for a certain application [14].

To find the invariant set, we need to derive its bounds. The first bound on the invariant set is derived by using the Lyapunov function given in Eq. (4). Suppose that we use a certain value for the gain k, then there is a certain basin of attraction such that the time rate of change of the Lyapunov function is less than zero, namely

$$\dot{V} < 0 \Rightarrow V(t) - V(t_o) < 0$$

or

$$|s(t)| - |s(t_o)| < 0$$

Therefore the switching function level is bounded by:

$$|s(t)| < |s(t_o)|, \forall t > t_o \tag{7}$$

Of course the inequality (7) holds due to the action of the sliding mode controller with gain k. Hence, the inequality (7) shows that the state will lie in a region bounded by

$$-s(t_o) < s(t) < s(t_o), \forall t > t_o$$

but without assign the equilibrium point with respect to the switching function. So we need to show that, as it is known, that the switching manifold is attractive manifold due to the sliding mode controller. To prove the attractiveness of $s = 0$, the time derivative of the switching function is found first when k satisfy inequality (6), as follows:

$$\dot{s} = \dot{x}_2 + \lambda\dot{x}_1 = f(x) - g(x)ksgn(s) + \lambda x_2$$

$$\Rightarrow \dot{s} = -\beta(x)sgn(s) \;\;,\;\; 0 \le \beta(x)$$

Now, we return to the Lyapunov function, Eq. (4), to find its derivative

$$\dot{V}(s) = \dot{s} * sgn(s)$$

$$\Rightarrow \dot{V}(s) = -\beta(x) < 0$$

Since $V(0) = 0$ and $\dot{V}(s) < 0$ in the set $\{x \in \mathcal{R}^2 : s \neq 0\}$, then $s = 0$ is a stable manifold (theorem 4-1 in reference [10]). Moreover, we must note that the solution of the dynamical system at the switching manifold does not exist [15]. This is due to the discontinuity in sliding mode controller formula. Indeed the state will reach $s = 0$ in a finite time. Ideally the state will slide along the switching manifold to the origin, i.e., the state trajectory will identify the switching manifold until it reaches the origin. Therefore, the bound given in the inequality (7) becomes:

$$0 \le |s(t)| < |s(t_o)|$$

$$\Rightarrow 0 \le s(t) * sgn(s) < s(t_o) * sgn(s_o)$$

But in sliding mode control

$$sgn(s) = sgn(s_o), \forall t > t_o$$

thus,

$$0 \le s(t) * sgn(s) < s(t_o) * sgn(s) \tag{8}$$

Accordingly we have

$$\left.\begin{array}{l} 0 \le s(t) < s(t_o) \text{ for } s > 0 \\ 0 \ge s(t) > s(t_o) \text{ for } s < 0 \end{array}\right\} \tag{9}$$

In words, inequality (9) shows that if the state initiated in the positive side of the switching manifold, then the state will stay in an open region bounded by $s = s(t_o)$ and $s = 0, \forall t > t_o$. The same thing is happened if the state initiated with negative switching function level. Inequality (9) is the first bound; while the second is derived here for x_1 as follows:

$$\dot{x}_1 + \lambda x_1 = s(t)$$

$$\Rightarrow d\{e^{\lambda t}x_1(t)\} = e^{\lambda t}s(t)dt$$

or

$$e^{\lambda t}x_1(t) - x_1(t_o) = \int_{t_o}^{t} s(\tau)e^{\lambda\tau}\,d\tau$$

By taking the absolute for both sides and considering the inequality (7), we obtain

$$\left|e^{\lambda t}x_1(t)\right| - \left|e^{\lambda t_o}x_1(t_o)\right| \le \left|e^{\lambda t}x_1(t) - e^{\lambda t_o}x_1(t_o)\right|$$

$$= \left|\int_{t_o}^{t} s(\tau)e^{\lambda\tau}\, d\tau\right| \le \int_{t_o}^{t}|s(\tau)|e^{\lambda\tau}\, d\tau$$

$$\le |s(t_o)|\int_{t_o}^{t} e^{\lambda\tau}\, d\tau = \frac{|s(t_o)|}{\lambda}\left(e^{\lambda t} - e^{\lambda t_o}\right)$$

$$\Rightarrow \left|e^{\lambda t}x_1(t)\right| \le \left|e^{\lambda t_o}x_1(t_o)\right| + \frac{|s(t_o)|}{\lambda}\left(e^{\lambda t} - e^{\lambda t_o}\right)$$

$$\Rightarrow |x_1(t)| \le |x_1(t_o)|e^{-\lambda(t-t_o)} + \frac{|s(t_o)|}{\lambda}\left(1 - e^{-\lambda(t-t_o)}\right)$$

$$\therefore |x_1(t)| \le max\left\{|x_1(t_o)|\,, \frac{|s(t_o)|}{\lambda}\right\} \qquad (10)$$

The result in the inequality (10) is a consequence of the convexity of the set

$$\Psi = \left\{x_1(t) : x_1(t) = \mu|x_1(t_o)| + (1-\mu)\frac{|s(t_o)|}{\lambda}, 0 \le \mu \le 1\right\}$$

In this case the maximum element of the set is at $\mu = 0$ or at $\mu = 1$. Therefore the invariant set is bounded by the inequalities (9) and (10) in terms of the initial condition only. Accordingly, the invariant set is given by:

$$\Theta = \left\{x \in \mathcal{R}^2 : 0 \le s(t)sgn(s) < s(t_o)sgn(s), |x_1(t)| \le max\left(|x_1(t_o)|, \frac{|s(t_o)|}{\lambda}\right)\right\} \quad (11)$$

The figure below plot the invariant set in the phase plane and one can find geometrically that the bound for $x_2(t)$ inside Ψ is

$$|x_2(t)| \le max\{|x_2(t_o)|\,, |s(t_o)|\} \qquad (12)$$

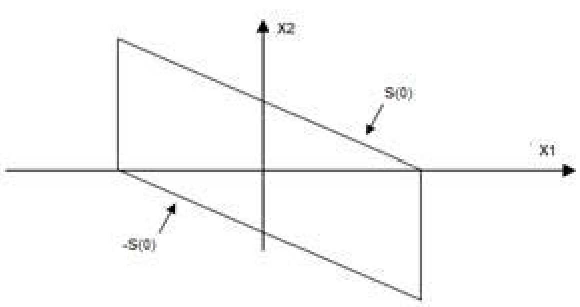


Fig.1: Positively Invariant Set.

5 The Second Positively Invariant Set

In classical sliding mode control theory, there exist a trivial invariant set. This set is the origin of the state space where the controller regulates the state to it and kept the state there for all future time. The sliding mode control that does the above task is a discontinuous control and it may cause the chattering problem. There are many solutions to the chattering problem in the literatures (see references [2], [8] and [16]). A simplest method to remove chattering is by replacing the signum function, which it used in sliding mode controller, by an approximate form. This idea is first introduced by J.J. Sloten in [9] using the saturation function instead of the signum function. Later, many other approximate signum functions are used to remove chattering as found in reference [17]. However, when replacing the signum function the state will not be regulated to the origin, instead it will regulated to a certain set around the origin known as positively invariant set. The size of this set is determined by the design parameters and the approximation form. In the present work the signum function is replaced by the arc tan function, namely

$$sgn_{approx.}(s) = \frac{2}{\pi}tan^{-1}(\gamma s) \qquad (13)$$

where $tan^{-1}(\gamma s)$ is a continuously differentiable, odd, monotonically increasing function with the properties:

$$tan^{-1}(0) = 0$$

$$\lim_{|s|\to\infty} tan^{-1}(\gamma s) = \lim_{\gamma\to\infty} tan^{-1}(\gamma s) = \frac{\pi}{2}sgn(s)$$

and

$$sgn(s) * tan^{-1}(\gamma s) = tan^{-1}(\gamma|s|) \ge 0$$

The sliding mode controller (Eq. (3)), using the approximation in Eq.(13), becomes

$$u_{approx.} = -\frac{2k}{\pi}tan^{-1}(\gamma s) \qquad (14)$$

Now, let us state the following:

When the sliding mode controller uses the approximate signum function as given in Eq. (13), and the controller gain satisfy the inequality (6), then the state will be regulated to a positively invariant set defined by

$$\Delta_\delta = \left\{x \in \mathcal{R}^2 : |x_1| < \frac{\delta}{\lambda}, |s| \le \delta\right\} \qquad (15)$$

To prove that Δ_δ is positively invariant set for a second order affine system (Eq. (2)), we return to use the Lyapunov function as given in Eq. (4) which has the time rate of change

$$\dot{V} = \left\{f(x) - g(x)\frac{2k}{\pi}tan^{-1}(\gamma s) + \lambda x_2\right\}sgn(s)$$

$$= -\left\{g(x)\frac{2k}{\pi}tan^{-1}(\gamma|s|) - (f(x) + \lambda x_2) * \mathrm{sgn}(s)\right\}$$

For the switching manifold to be attractive $\dot{V}$ must be less than zero, namely

$$-\left\{g(x)\frac{2k}{\pi}tan^{-1}(\gamma|s|) - (f(x) + \lambda x_2) * \mathrm{sgn}(s)\right\} < 0$$

$$\Rightarrow \frac{2k}{\pi}tan^{-1}(\gamma|s|) > max\left|\frac{f(x) + \lambda x_2}{g(x)}\right| = h$$

or

$$k > \frac{\pi h}{2tan^{-1}(\gamma|s|)} \quad (16)$$

Now, let $|s| = \delta$ be the chosen boundary layer, then inequality (16) for a certain γ reveals that: *for any* δ *there is* k*, such that the state will be regulated to an open region* Γ *given by*

$$\Gamma = \{x \in \mathcal{R}^2 : |s| < \delta\} \quad (17)$$

Accordingly, the gain k will be

$$k = \frac{\alpha\pi h}{2tan^{-1}(\gamma\delta)} \quad , \alpha > 1 \quad (18)$$

In addition, to determine γ, Eq. (18) may be written as:

$$k = \alpha h\beta, \quad \beta > 1 \quad (19)$$

provided that;

$$\gamma\delta = tan\frac{\pi}{2\beta} \quad (20)$$

The next step in the determination of the invariant set Δ_δ is to found the boundary with respect to x_1 inside Γ. This is done by using the following Lyapunov function

$$V = \frac{1}{2}x_1^2 \quad (21)$$

with the x_1 dynamics, from Eqs. (2) and (3):

$$\dot{x}_1 = -\lambda x_1 + s(t) \quad (22)$$

Therefore the time rate of change for the Lyapunov function is

$$\dot{V} = x_1\dot{x}_1 = x_1\left(-\lambda x_1 + s(t)\right) = -\lambda|x_1|^2 + x_1 s(t)$$

$$\leq -\lambda|x_1|^2 + |x_1||s(t)| \leq -\lambda|x_1|^2 + |x_1|\delta$$

$$= -|x_1|(\lambda|x_1| - \delta)$$

Thus, $\dot{V} \leq 0$ for the following unbounded interval:

$$|x_1| > \frac{\delta}{\lambda} \quad (23)$$

Inequality (23) proves that the state x_1 will reach and stay within the interval $-\frac{\delta}{\lambda} \leq x_1 \leq \frac{\delta}{\lambda}$. This ends the proof that the set $\left\{x \in \mathcal{R}^2 : |x_1| < \frac{\delta}{\lambda}, |s| \leq \delta\right\}$ is positively invariant for the system in Eq. (2) where the sliding mode controller uses an approximate signum function (Eq. (14)).

Note that the state inside Δ_δ may or may not reaches an equilibrium point; the situation depends on system dynamics, i.e., the state, instead of that, will reach a limit cycle inside Δ_δ. Consequently, and for the design purpose, δ may be determined according to a desired permissible steady state deviation of the state x_1 and for a selected λ, as a design parameter, as follows:

$$\delta = \lambda * x_{1per.} \quad (24)$$

The set Δ_δ is now written as:

$$\Delta_\delta = \left\{x \in \mathcal{R}^2 : |x_1| < x_{1per.}, |s| \leq \delta\right\} \quad (25)$$

It is also noted that for arbitrary small $x_{1per.}$ the positively invariant set Δ_δ becomes arbitrary small and it may lead, again, to the state chattering. This situation may explain the chattering phenomena as the state oscillation with a very small width, i.e., the interval $|x_1| < x_{1per.}$ is very small.

6 Sliding Mode Controller Design for Servo Actuator with Friction

In this section the servo actuator system is adapted as an application example used to validate the results derived in this work. Consider the model for the servo actuator with friction:

$$J\ddot{x} = u - F - T_L \quad (26)$$

where x is the actuator position, J is the moment of inertia, u is the control input torque, F is the friction torque, including the static and dynamic components, and T_L is the load torque. The friction model taken here is a combination of Coulomb friction F_c, Stiction friction F_s, and the viscous friction (for more details one can refer to the survey papers [18] &[19]), namely

$$F = F_s e^{-\left(\frac{\dot{x}}{\dot{x}_s}\right)^2} * sgn(\dot{x}) + F_c\left(1 - e^{-\left(\frac{\dot{x}}{\dot{x}_s}\right)^2}\right) * sgn(\dot{x}) + \sigma\dot{x}$$

or

$$F = \left\{F_s e^{-\left(\frac{\dot{x}}{\dot{x}_s}\right)^2} + F_c\left(1 - e^{-\left(\frac{\dot{x}}{\dot{x}_s}\right)^2}\right) + \sigma|\dot{x}|\right\} * sgn(\dot{x}) \quad (27)$$

where $\dot{x}_s$ is called the Stribeck velocity and σ is the viscous friction coefficient. In addition, the servo actuator model in Eq. (26) is considered uncertain with a bounded load torque. The uncertainty in the model parameters is assumed here to reach 20% of their nominal values. Note that the control input u in Eq. (26) can be regarded as the virtual control when compared with Eq. (1).

Now, define $e_1 = x - x_d$ and $e_2 = \dot{x} - \dot{x}_d$, then the system model in Eq. (26) in state space form (in (e_1, e_2) plane) is written as:

$$\left.\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= \left(\frac{1}{J}\right)(u - F - T_L) - \ddot{x}_d \end{aligned}\right\} \quad (28)$$

In this work the desired position and velocity are taken as in reference [14]:

$$\left.\begin{aligned} x_d &= \frac{1}{16\pi}sin(8\pi t) - \frac{1}{24\pi}sin(12\pi t) \;\Rightarrow\; |x_d| \le \frac{5}{48\pi} \\ \dot{x}_d &= sin(10\pi t) * sin(2\pi t) \;\Rightarrow\; |\dot{x}_d| \le 1 \end{aligned}\right\} \quad (29)$$

Also, the switching function and its derivative are

$$\left.\begin{aligned} s &= e_2 + \lambda e_1 \\ \dot{s} &= \left(\frac{1}{J}\right)(u - F - T_L) - \ddot{x}_d + \lambda e_2 \end{aligned}\right\} \quad (30)$$

where

$$\ddot{x}_d = 10\pi * cos(10\pi t)sin(2\pi t) - 2\pi * sin(10\pi t)cos(2\pi t)$$

and

$$|\ddot{x}_d| \le 12\pi.$$

Since the calculation of k as given in Eq. (19) depends on the initial condition, so we will design the sliding mode controller for two different loci of initial condition (the position and the velocity at time $t = 0$). The first initial condition lies in the second positively invariant set (see (15)). While in the second case the initial condition is taken in the first positively invariant set and outside the second positively invariant set. The controller parameters are calculated for each case in appendices (A) and (B) using the following nominal parameters and external load values [20]

Table 1: Nominal Servo Actuator Parameters and the External Load values

Par.	Definition	Value	Units
J_o	Moment of inertia.	0.2	kgm^2
F_{so}	Stiction friction.	2.19	Nm
F_{co}	Coulomb friction.	16.69	Nm
$\dot{x}_{so}$	Stribeck velocity.	0.01	rad/sec
σ_o	viscous friction coefficient	0.65	$Nm \cdot sec/rad$
T_{Lo}	External Torque	2	Nm

The simulation results and discussions are presented in the following section.

7 Simulations Result and Discussions

For the first case the state is started from the rest, which means $e(0) = (0,0)$ (this is because $x_d(0) = \dot{x}_d(0) = 0$). In this case the state is initiated inside the positively invariant set Δ_δ, and accordingly the state will not leave it $\forall t \ge 0$. The state after that reaches an invariant set (it stills inside Δ_δ) , namely the ω limit set of the point $e(0)$. For the servo actuator with non-smooth disturbance (the friction), this set is a limit cycle lies inside the positively invariant set Δ_δ (the fact in section 2). Indeed, the state will reach the ω limit set if it is started at any point in Δ_δ. This behavior is confirmed by the simulation results presented below.

The approximate sliding mode controller in this case is (the details of the calculations is found in Appendix (A))

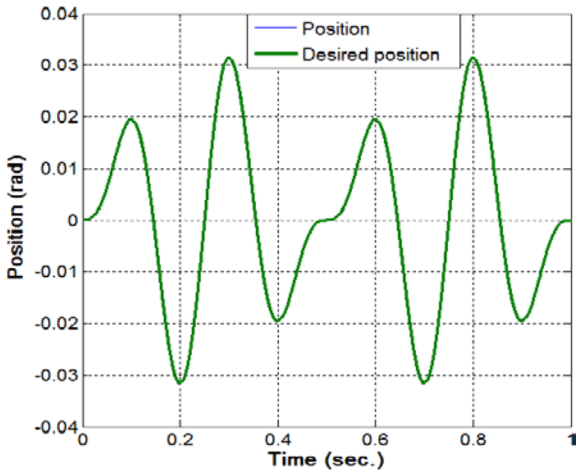
$$\left.\begin{aligned} u_{approx.} &= -(84/\pi) * tan^{-1}(141 * s) \\ s &= (\dot{x} - \dot{x}_d) + 25 * (x - x_d) \end{aligned}\right\} \quad (31)$$

This controller will be able to maintain the state in the following invariant set:

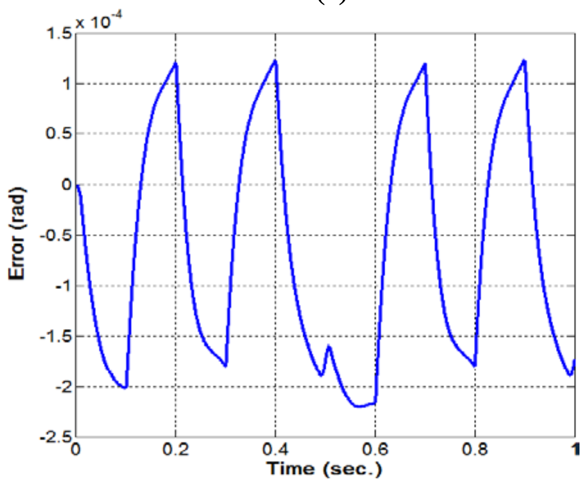
$$\Delta_\delta = \left\{x \in \mathcal{R}^2 : |x - x_d| < \frac{\pi}{3600}, |s| \le \frac{\pi}{144}\right\} \quad (32)$$

The response of the servo actuator system when started at the origin is shown in Fig. 2. In this figure the position response is plotted with time and it appears very close to the desired position. This result is verified when plotting the error and the maximum error shown in the plot, where it does not exceed 1.5×10^{-4} radian. For the velocity, Fig. 3 plot the time response and again the maximum error, which it does not exceed 6.5×10^{-3} radian per second. This shows the closeness between the velocity response and the desired velocity. The error phase plot is found in Fig. 4 where the state reaches the ω limit set of the origin point. The ω limit set forms here a non-smooth limit cycle and accordingly, the error state will oscillate for all future time within certain amplitude. The oscillation amplitude has an upper bound according to the earlier choice of the permissible error.

The positively invariant set formed by the sliding mode controller, as it is given by (32), enables the same controller to regulate the state when it is started within this set. This situation is verified in Fig. 5 for two starting points where the state reaches the ωlimit set corresponding to each point.

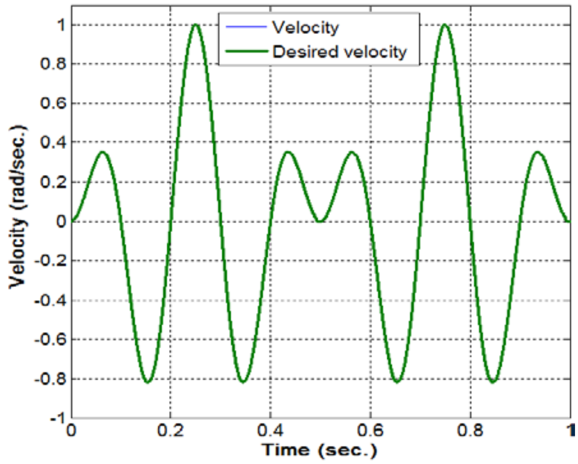


(a)

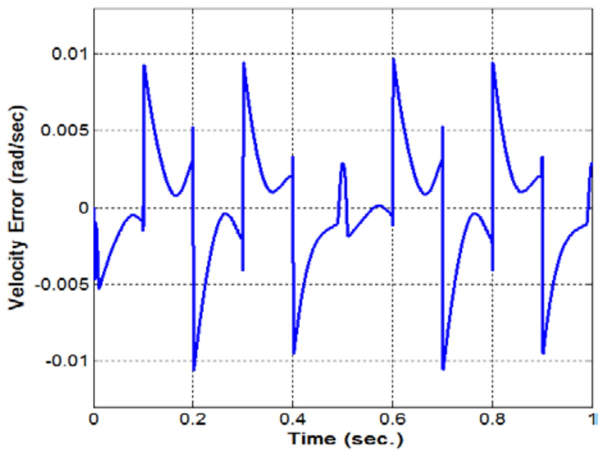


(b)

Fig. 2: a) Position and the desired position vs. time (equation (29)). b) The position error for 5 second.



(a)



(b)

Fig. 3: a) Velocity and the desired velocity vs. time (Eq. (29)). b) The velocity error for 5 second.

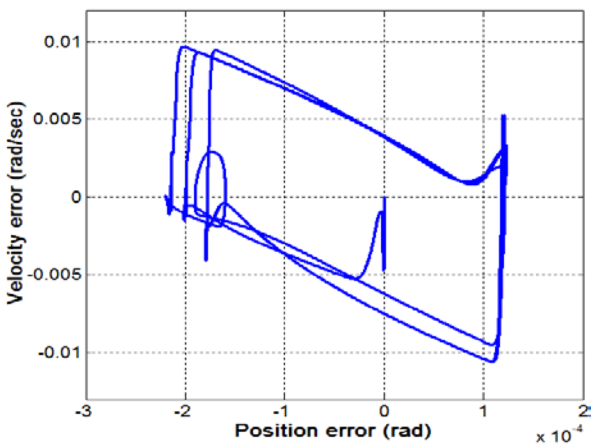
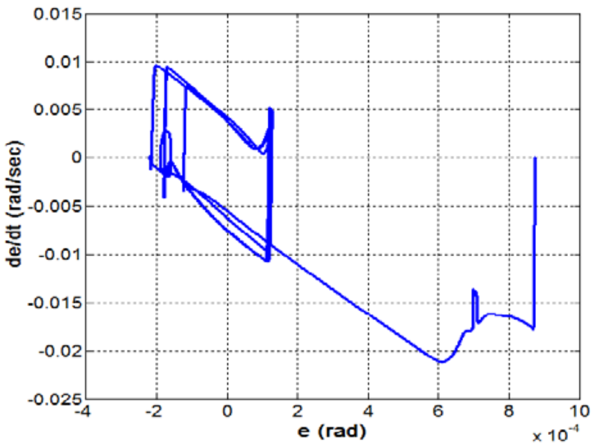
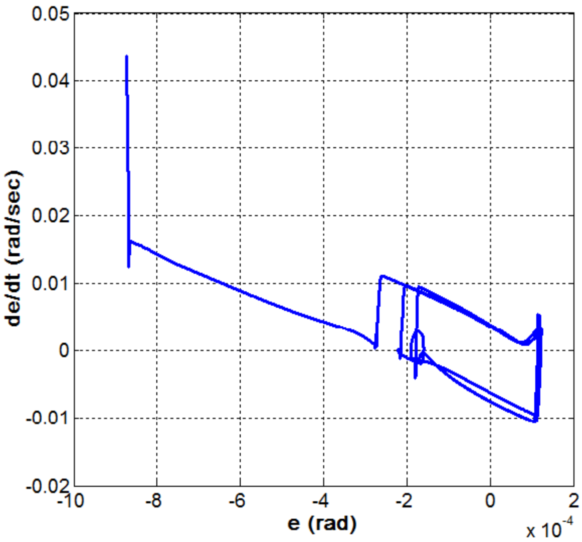


Fig. 4: The phase plane plot when the error started at the origin.



(a)



(b)

Fig. 5: The phase plane plot a) when the error started at $\left(e,\frac{de}{dt}\right)=\left(\frac{\pi}{3600},0\right)$ b) when the error started at $\left(e,\frac{de}{dt}\right)=\left(-\frac{\pi}{3600},2\frac{\pi}{144}\right)$.

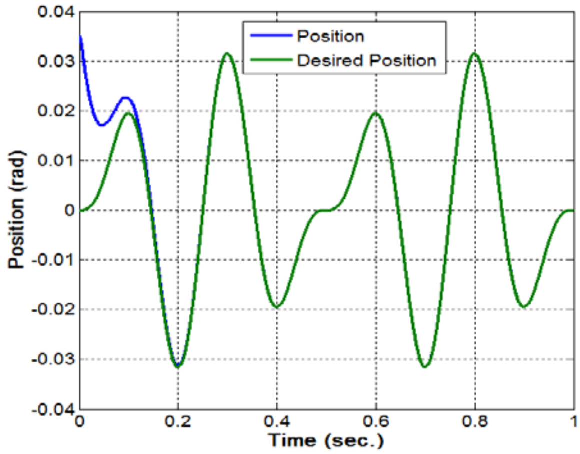
For the second case the sliding mode controller, as calculated in appendix (B), is

$$\left.\begin{aligned} u &= -45 * \mathrm{sgn}(s) \\ s &= (\dot{x}-\dot{x}_d) + 25 * (x - x_d) \end{aligned}\right\} \quad (33)$$

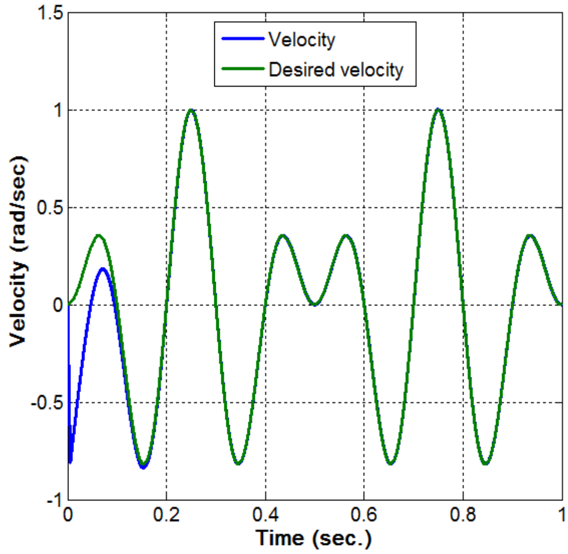
The controller will be able to regulate the error to the origin if it initiated in the following positively invariant set:

$$\Omega = \{x \in \mathcal{R}^2 : |s(t)| < 0.875 \, , |x - x_d| \le 0.035\} \quad (34)$$

The simulations result for the position and the velocity when the state starting at $(x,\dot{x}) = (0.035,0)$ are shown in Fig. 6. In this figure the position and the velocity track the desired response after a period of time not exceeding 0.12 second.



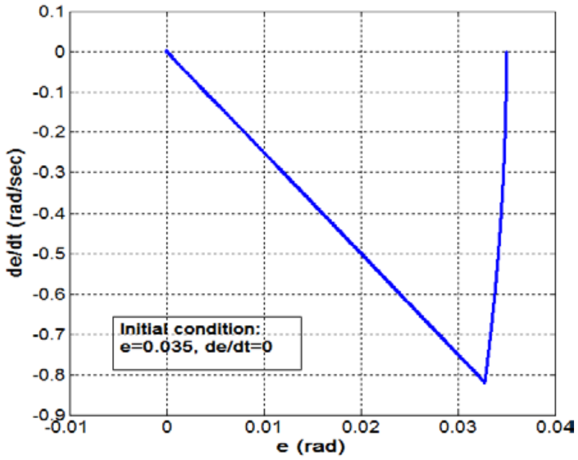
(a)



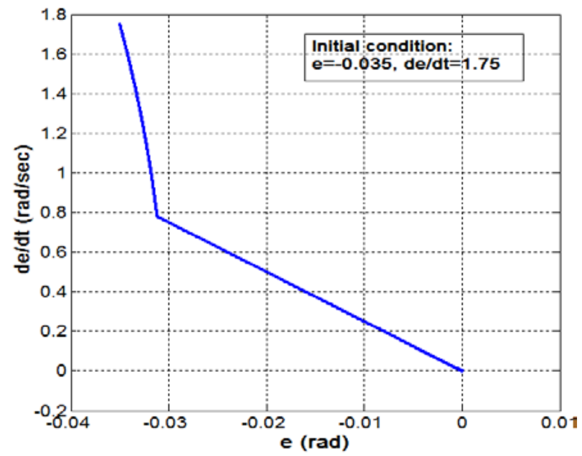
(b)

Fig. 6: Servo actuator response for the initial condition $\left(e,\frac{de}{dt}\right)=(0.035,0)$ a) The position vs. time b) velocity vs. time.

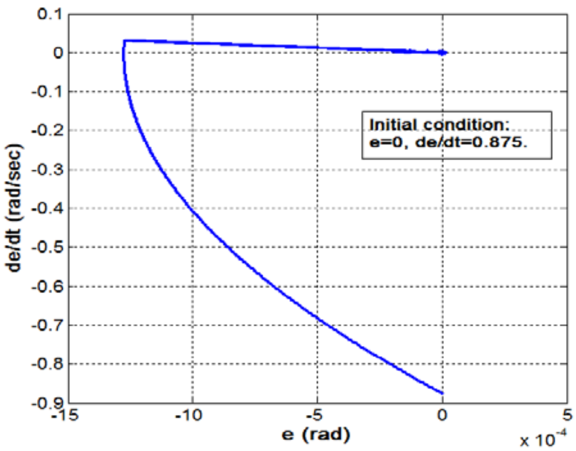
As for the sliding mode controller in Eq. (31), the sliding mode controller in Eq. (33) will create a positively invariant region (34) such that if the state initiated inside this set, it will be regulated to the origin. This situation is confirmed in Fig. 7 for three different starting points including the case of Fig. 6.



(a)



(b)



(c)

Fig. 7: Error phase plot for different initial conditions a) $\left(e, \frac{de}{dt}\right) = (0.035,0)$ b) $\left(e, \frac{de}{dt}\right) = (-0.035,1.75)$ c) $\left(e, \frac{de}{dt}\right) = (0,-0.875)$.

If it is required to remove the chattering that exists in the system response for the second case, we again replace the signum function by the arc tan function. In this case we replace the gain $k = 45$ by the following quantity:

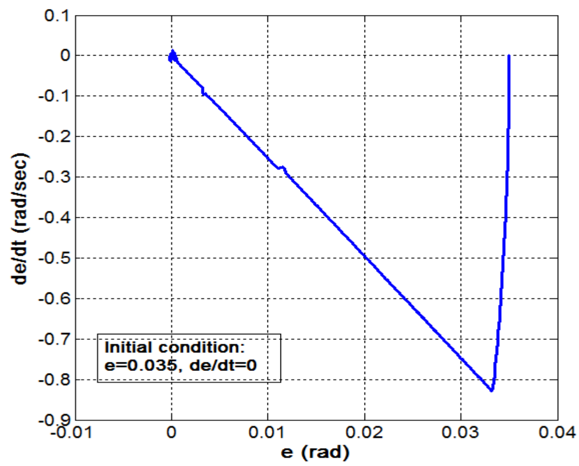
$$k = 45 * 1.25 = 57, \quad \beta = 1.25$$

Then, we get

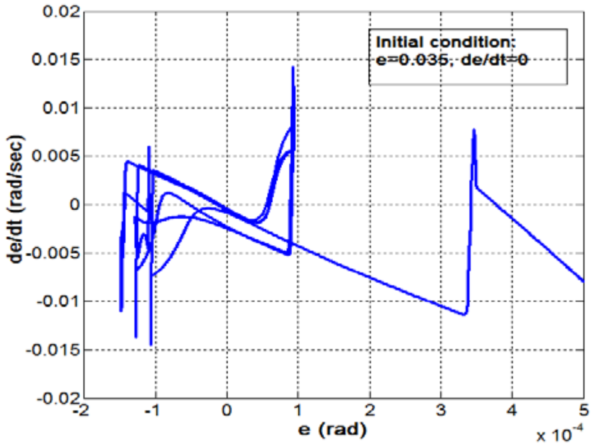
$$u = -\left(\frac{114}{\pi}\right) tan^{-1}(141 * s) \tag{35}$$

The sliding mode controller in Eq. (35) creates a positively invariant set equal to the set given in (34), but in this case the controller will not regulate the error to the origin. Indeed, the controller will regulate the error to the positively invariant set given in (32). Mathematically, the sets in (34) and (32) are two positively invariant sets created by the sliding mode controller in Eq. (35), but with a different set level (see reference [12] for the definition of set level), namely $\Delta_\delta \subset \Omega$.

As in Fig. 7, the phase plane plot for the initial condition $\left(e, \frac{de}{dt}\right) = (0.035,0)$ is plotted in Fig. 8 but without chattering around the switching manifold due to replacing the signum function in Eq. (33) by the approximate form in Eq. (35). Accordingly, the state will be regulated to a smaller positively invariant set and then reach the ω limit set as in case one.



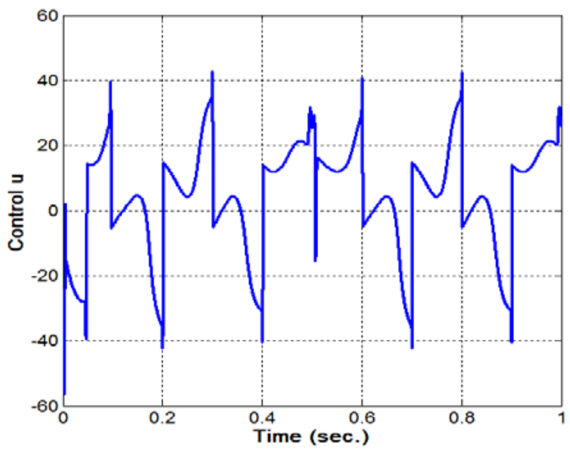
(a)



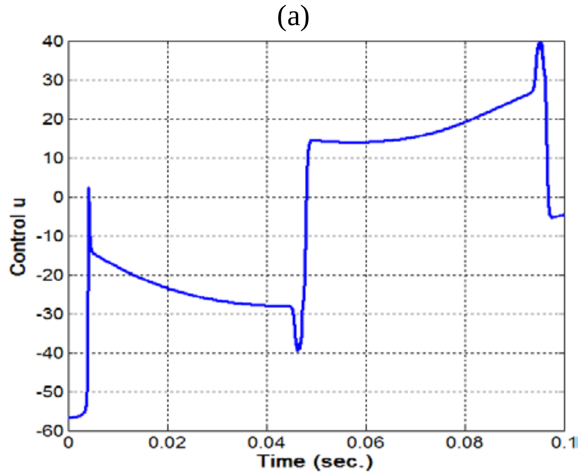
(b)

Fig. 2: the phase plane plot when using the controller in Eq. (35) a) full phase plot, b) small plot around the origin showing the oscillation behavior.

Finally, the chattering behavior is removed due to a continuous control action, where the continuity is shown in Fig. 9 with a magnitude lies between $\pm 42\ N.m$ after a period of time not exceeds 0.05 second.



(a)



(b)

Fig. 9: The control action vs. time a) plot for 1 second b) plot for 0.05 second.

7 Conclusions

A solution to the mismatched perturbation and the chattering problems in sliding mode control is suggested in this work. The solution is based on deriving the invariant set created by the sliding mode controller. Thus, the invariant sets for a second order affine system that uses a sliding mode controller are derived in this work. The size of the invariant sets is found functions to the initial condition, the controller gain and design parameters. The derived sets were used to calculate the sliding mode controller gain for the servo actuator and to attenuate the effects of the discontinuous perturbation by adjusting the controller parameters to control the size of the second positively invariant set. The simulation results prove the invariant property of the derived set and the effectiveness of using them in the sliding mode control design. The ability of the approximate sliding mode controller (a continuously and differentiable controller) has been verified when it used to attenuate the effect of a nonsmooth disturbances (the friction) that acts on the servo actuator system. The controller maintains the maximum error (the difference between the actual and the desired state) very close to zero and according to the permissible error value specified previously.

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Appendix (A)

To design the approximate sliding mode controller we need first to calculate h as it is given in Eq. (6):

$$h = \max\left|\frac{f(e)+\lambda e_2}{g(e)}\right| = \left\{\frac{\max\left|\left(\frac{-F-T_L}{J}\right)-\ddot{x}_d+\lambda e_2\right|}{\min\left(\frac{1}{J}\right)}\right\}$$
$$= \max|F| + \max|T_L| + (\max J)*\max|\ddot{x}_d| + \lambda*(\max J)*\max|e_2| \qquad \text{(A-1)}$$

From the set Δ_δ , the following bounds are estimated:

$$\max|e_2| = 2\delta$$

and,

$$\max|\dot{x}| = \max|e_2| + \max|\dot{x}_d| = 2\delta + 1$$

The term $\max|\dot{x}|$ enables the estimation of $\max|F|$ as follows:

$$\max|F| = 1.2\left\{F_{so}e^{-\left(\frac{2\delta+1}{\dot{x}_s}\right)^2} + F_{co}\left(1-e^{-\left(\frac{2\delta+1}{\dot{x}_s}\right)^2}\right) + \sigma_o(2\delta+1)\right\} \le 1.2\big(F_{co}+\sigma_o(2\delta+1)\big)$$

where F_{so},F_{co}, and σ_o are the nominal friction parameter values and we multiply by 1.2 to take into account the uncertainty in system parameters as assumed previously. In addition, we have

$$\max|J| = 1.2*J_o, \quad \text{and} \quad \min|J| = 0.8*J_o$$

where J_o is the nominal moment of inertia value and finally the load torque is bounded by

$$|T_L| \leq T_{L_{max}} = 1.2T_{L_0}$$

Therefore, h in Eq. (A-1) is function to the slope of the switching manifold λ and the boundary layer δ.

In sliding mode controller design, we mainly concern in calculating suitable value for the gain k after a proper selection to the switching function $s(x)$ (by proper we mean that the origin is an asymptotically stable after the state reaches the switching manifold $s(x) = 0$). Now, if we set the permissible error and λ as in the following

$$e_{per.} = 0.05 \text{ deg.} = \frac{\pi}{3600}\, rad, \qquad \lambda = 25$$

then from (24), we have

$$\delta = \lambda * e_{per.} = \frac{\pi}{144} \Rightarrow |e_1| \leq e_{per.}$$

Accordingly, to find the gain k, we first compute h as follows:

$$\max|F| \leq 1.2\big(F_{co} + \sigma_o(2\delta + 1)\big) = 20.84$$

$$\Rightarrow h = 20.84 + 2.4 + 0.24 * 12\pi + 0.24 * 25 * 2\frac{\pi}{144} = 32.55$$

and then for $\beta = 1.25$, we get

$$k = \alpha * 1.25 * 32.55 = 42\,, \quad \alpha > 1$$

Also, from Eq. (20), γ equal to

$$\gamma = \frac{144}{\pi} tan \frac{\pi}{2.5} = 141$$

Finally, the sliding mode controller to the servo actuator is

$$\left.\begin{aligned} u_{approx\,.} &= -\tfrac{84}{\pi} tan^{-1}(141 * s) \\ s &= (\dot{x} - \dot{x}_d) + 25 * (x - x_d) \end{aligned}\right\} \qquad \text{(A-2)}$$

The sliding mode controller will be able to prevent the state leaves the positively invariant set Δ_δ, which means that the error $(x - x_d)$ is less than the permissible limit that was specified earlier.

Appendix (B)

In this case we consider the same desired position and velocity as in Eq. (29) with the following initial condition

$$x = 0.035\, rad, \qquad \dot{x} = 0 \;\; rad/sec.$$

$$\Rightarrow e(0) = (e_1, e_2) = (0.035{,}0)$$

Also, consider the same switching function as in case one ($s = e_2 + 25e_1$). Then, the invariant set is given by

$$\Theta = \{x \in \mathcal{R}^2 : 0 \leq s(t) < 0.875\,, |e_1(t)| \leq 0.035\} \qquad \text{(B-1)}$$

In addition we have

$$|e_2(t)| \leq 1.75$$

$$\Rightarrow \max|\dot{x}| = \max|e_2| + \max|\dot{x}_d| = 2.75\, rad/sec.$$

Then $\max|F|$ can be estimated as

$$\max|F| \leq 1.2(F_{co} + 2.75 * \sigma_o) = 22.2$$

As in the first case, h is equal to

$$h = 22.2 + 2.4 + 0.24 * 12\pi + 0.24 * 25 * 1.75 = 44.15$$

The sliding mode controller gain k from Eq. (6) is taken equal to

$$k = 45 > h$$

Finally, the sliding mode controller for the second case is given by

$$\left.\begin{aligned} u &= -45 * \text{sgn}(s) \\ s &= (\dot{x} - \dot{x}_d) + 25 * (x - x_d) \end{aligned}\right\} \qquad \text{(B-2)}$$

If the state initiated inside the positively invariant set as given in (B-1), the sliding mode controller will regulate the error state to the origin irrespective to the uncertainty and the non-smooth components in the servo actuator model.