A Flow-Based Approximation of Observer Error Linearization for Non-Linear Systems

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Abstract: This paper deals with observer design for non-linear single output systems. When a classical observer error linearization by output injection is not possible due to restrictive existence conditions, a simple approximation scheme is proposed. The method can be applied to a wider class of non-linear systems. Our approach is constructive and can be implemented using computer algebra packages. The design procedure is carried out for two example systems.

Key–Words: Differential geometry, non-linear differential equations, observers

1 Introduction

We consider the observer design problem for a non-linear single output system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad y = h(\boldsymbol{x}) \tag{1}$$

with smooth maps $\boldsymbol{f} : \mathcal{X} \to \mathbb{R}^n$ and $h : \mathcal{X} \to \mathbb{R}$ defined on an open subset $\mathcal{X} \subseteq \mathbb{R}^n$. During the last decades, several different observer design methods have been developed and analyzed [1–3].

Observer design is often carried out in appropriate coordinates. A very popular approach has been suggested in [4, 5], where high gain observer design is carried out using the observability canonical form [6, 7]. Similarly, the Byrnes-Isisodri normal form [8, 9] can be employed to design unknown input observers [10, 11].

An especially suitable choice of the new coordinates yields a system with linear output and linear dynamics driven by a non-linear output injection. This form is called observer canonical form [12, 13]. The associated observer called normal form observer has linear error dynamics. Starting with single output systems [12, 13], this approach has been extended into several directions, e.g. to multi output systems [14–16] and to adaptive observers [17–19].

However, the existence conditions of the transformation into observer canonical form are quite restrictive due to a non-generic involutivity condition. Several publications are aimed at relaxing these existence conditions, e.g. by system immersion [20, 21], dynamic error linearization [22–24] and by time-transformation [25, 26]. Recent approaches do not assume that the output map of the transformed system is linear, which results in significantly weaker existence conditons [27–30].

When exact error linearization by output injection is not possible, many approaches have been developed to achieve an approximate observer error linearization, e.g. by an extended Taylor linearization of the error dynamics [12, 31–33], higher order series expansions [34–37], and with splines [38, 39]. Other methods are presented in [40, 41].

In Section 2 we remind the reader of the exact observer error linearization. Our approach to an approximate error linearization is explained in Section 3. Computational issues are addressed in Section 4. The developed observer design procedure is illustrated on two example systems in Section 5.

2 Exact Observer Error Linearization

First we recall observer design by exact error linearization [12, 13]. The design methodology is straightforward if there exists a diffeomorphism

$$\boldsymbol{z} = T(\boldsymbol{x}) \quad \text{with} \quad \boldsymbol{x} = S(\boldsymbol{z}) \tag{2}$$

that transforms (1) into the form

$$\begin{array}{rcl} \dot{\boldsymbol{z}} &=& A\boldsymbol{z} + \boldsymbol{\alpha}(\boldsymbol{c}^T\boldsymbol{z}) \\ y &=& \gamma(\boldsymbol{c}^T\boldsymbol{z}) \end{array} \tag{3}$$

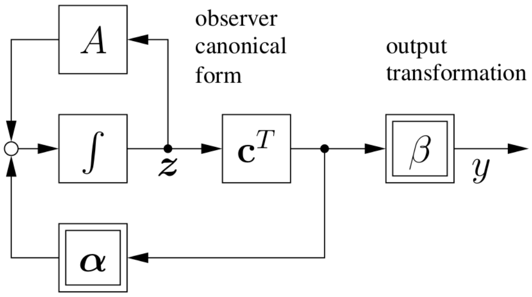

Figure 1: Observer canonical form

with

$$
\begin{array}{rcl}
A & = & \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \\
\boldsymbol{c}^T & = & \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.
\end{array} \tag{4}
$$

The linear part described by (4) is in dual Brunovsky form [42]. The map $\boldsymbol{\alpha}$ is an output injection and the map γ describes an output transformation. The form (3) is called *observer canonical form* [6, 7] and depicted in Fig. 1.

For a system in form (3), the observer

$$
\begin{array}{rcl}
\dot{\hat{\boldsymbol{z}}} & = & A\hat{\boldsymbol{z}} + \boldsymbol{\alpha}(\gamma^{-1}(y)) + \boldsymbol{k}(\gamma^{-1}(y) - \boldsymbol{c}^T\hat{\boldsymbol{z}}) \\
\hat{\boldsymbol{x}} & = & S(\hat{\boldsymbol{z}})
\end{array} \tag{5}
$$

yields an observation error $\tilde{\boldsymbol{z}} = \boldsymbol{z} - \hat{\boldsymbol{z}}$ governed by the linear time-invariant error dynamics

$$\dot{\tilde{\boldsymbol{z}}} = (A - \boldsymbol{k}\boldsymbol{c}^T)\,\tilde{\boldsymbol{z}}. \tag{6}$$

Since the pair (4) is observable, the eigenvalues of the linear error system (6) can be placed arbitrarily via the observer gain $\boldsymbol{k} \in \mathbb{R}^n$. More precisely, for a desired characteristic polynomial

$$\det(sI - (A - \boldsymbol{k}\boldsymbol{c}^T)) = p_0 + p_1 s + \cdots + p_{n-1}s^{n-1} + s^n$$

of the error dynamcis (6) we have to use the gain vector $\boldsymbol{k} = (p_0, \ldots, p_{n-1})^T$.

The difficulty of this approach is the transformation of system (1) into (3). The existence conditions of (3) are given by the next theorem [13]. For Lie derivatives and Lie brackets we use the same notation as in [9, 43].

Theorem 1 *In a neighbourhood of $\boldsymbol{x}_0 \in \mathcal{X}$ there exists a diffeomorphism* (2) *transforming* (1) *into* (3) *if and only if*

1. $\dim\left(dh(\boldsymbol{x}_0), L_{\boldsymbol{f}}dh(\boldsymbol{x}_0), \ldots, L_{\boldsymbol{f}}^{n-1}dh(\boldsymbol{x}_0)\right) = n$,
2. $[ad_{\boldsymbol{f}}^i\boldsymbol{v}, ad_{\boldsymbol{f}}^j\boldsymbol{v}](\boldsymbol{x}) = \boldsymbol{0}$ *for* $0 \le i, j \le n-1$ *and all* $\boldsymbol{x}$ *in a neighbourhood of* $\boldsymbol{x}_0$,

where the vector field $\boldsymbol{v}$ is the solution of

$$L_{\boldsymbol{v}}L_{\boldsymbol{f}}^k h(\boldsymbol{x}) = \begin{cases} 0 & \textit{for} \quad k = 0, \ldots, n-2 \\ \beta(h(\boldsymbol{x})) & \textit{for} \quad k = n-1 \end{cases} \tag{7}$$

for an appropriate scalar-valued continuous function β with $\beta(h(\boldsymbol{x}_0)) \neq 0$.

The first condition is the well-known observability rank condition. The vector field $\boldsymbol{v}$ is the last column of the inverse observability matrix scaled by $1/\beta$. In other words, $\boldsymbol{v}$ is the solution of $Q(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x}) = \beta(h(\boldsymbol{x}))\frac{\partial}{\partial x_n}$ with the observability matrix

$$Q(\boldsymbol{x}) = \begin{pmatrix} dh(\boldsymbol{x}) \\ \\ dL_{\boldsymbol{f}}^{n-1}h(\boldsymbol{x}) \end{pmatrix}.$$

The construction of the transformation (2) is based on the Simultaneous Rectification Theorem [43, Theorem 2.36]:

Theorem 2 *Consider r vector fields $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r : \mathcal{X} \to \mathbb{R}^n$. Assume that these vector fields are linearly independent at $\boldsymbol{x}_0 \in \mathcal{X}$ and that $[\boldsymbol{v}_i, \boldsymbol{v}_j](\boldsymbol{x}) = \boldsymbol{0}$ holds for $1 \le i, j \le r$ and all $\boldsymbol{x}$ in a neighbourhood of $\boldsymbol{x}_0$. Then there exists a change of coordinates* (2) *such that*

$$T'(\boldsymbol{x})\boldsymbol{v}_i(\boldsymbol{x}) = \frac{\partial}{\partial z_i} \quad \textit{and} \quad \boldsymbol{v}_i(\boldsymbol{x}) = S'(\boldsymbol{z})\frac{\partial}{\partial z_i} \tag{8}$$

holds for $1 \le i \le r$ in a neighbourhood of $\boldsymbol{x}_0$.

Proof. There exists $n - r$ vector fields $\boldsymbol{v}_{r+1}, \ldots, \boldsymbol{v}_n$ such that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent at $\boldsymbol{x}_0$. We construct the inverse transformation as follows

$$\boldsymbol{x} = S(\boldsymbol{z}) = \boldsymbol{\varphi}_{z_1}^{\boldsymbol{v}_1} \circ \boldsymbol{\varphi}_{z_2}^{\boldsymbol{v}_2} \circ \cdots \circ \boldsymbol{\varphi}_{z_n}^{\boldsymbol{v}_n}(\boldsymbol{x}_0), \tag{9}$$

where $\boldsymbol{\varphi}_t^{\boldsymbol{v}_i}$ denotes the flow[1] of vector field $\boldsymbol{v}_i$ at time t. By construction we have $S(\boldsymbol{0}) = \boldsymbol{x}_0$. The first order series expansion of (9) has the form

$$
\begin{array}{rcl}
\boldsymbol{x} & = & S(\boldsymbol{z}) \\
& = & \boldsymbol{v}_1(\boldsymbol{x}_0)z_1 + \cdots + \boldsymbol{v}_n(\boldsymbol{x}_0)z_n + \mathcal{O}(\|\boldsymbol{z}\|^2).
\end{array} \tag{10}
$$

[1] Recall that the flow $\boldsymbol{\varphi}_t^{\boldsymbol{v}}$ of a vector field $\boldsymbol{v} : \mathcal{X} \to \mathbb{R}^n$ is the general solution of the associated ordinary differential equation $\dot{\boldsymbol{x}}(t) = \boldsymbol{v}(\boldsymbol{x}(t))$.

The associated Jacobian matrix

$$S'(\mathbf{0}) = (\boldsymbol{v}_1(\boldsymbol{x}_0), \boldsymbol{v}_2(\boldsymbol{x}_0), \ldots, \boldsymbol{v}_n(\boldsymbol{x}_0)) \in \mathbb{R}^{n\times n}$$

is regular since the vector fields occurring there are linearly independent. Therefore, S is a local diffeomorphism.

Condition $[\boldsymbol{v}_i, \boldsymbol{v}_j] \equiv \mathbf{0}$ implies that

$$\boldsymbol{\varphi}_{z_i}^{\boldsymbol{v}_i} \circ \boldsymbol{\varphi}_{z_j}^{\boldsymbol{v}_j} \equiv \boldsymbol{\varphi}_{z_j}^{\boldsymbol{v}_j} \circ \boldsymbol{\varphi}_{z_i}^{\boldsymbol{v}_i}$$

for $1 \leq i, j \leq r$, see [43, Lemma 2.3.4]. Then, we have

$$\begin{aligned}
\frac{\partial}{\partial z_i} S(\boldsymbol{z}) &= \frac{\partial}{\partial z_i} \boldsymbol{\varphi}_{z_1}^{\boldsymbol{v}_1} \circ \cdots \circ \boldsymbol{\varphi}_{z_i}^{\boldsymbol{v}_i} \circ \cdots \circ \boldsymbol{\varphi}_{z_n}^{\boldsymbol{v}_n}(\boldsymbol{x}_0) \\
&= \frac{\partial}{\partial z_i} \boldsymbol{\varphi}_{z_i}^{\boldsymbol{v}_i} \circ \boldsymbol{\varphi}_{z_1}^{\boldsymbol{v}_1} \circ \cdots \circ \boldsymbol{\varphi}_{z_{i-1}}^{\boldsymbol{v}_{i-1}} \\
&\quad \circ \boldsymbol{\varphi}_{z_{i+1}}^{\boldsymbol{v}_{i+1}} \circ \cdots \circ \boldsymbol{\varphi}_{z_n}^{\boldsymbol{v}_n}(\boldsymbol{x}_0) \\
&= \boldsymbol{v}_i(\boldsymbol{\varphi}_{z_1}^{\boldsymbol{v}_1} \circ \cdots \circ \boldsymbol{\varphi}_{z_i}^{\boldsymbol{v}_i} \circ \cdots \circ \boldsymbol{\varphi}_{z_n}^{\boldsymbol{v}_n}(\boldsymbol{x}_0)) \\
&= \boldsymbol{v}_i(S(\boldsymbol{z})) = \boldsymbol{v}_i(\boldsymbol{x})
\end{aligned}$$

for $1 \leq i \leq r$. This implies (8). □

The proof of Theorem 1 is based on Theorem 2 with $r = n$ and $\boldsymbol{v}_i = ad_{-\boldsymbol{f}}^{i-1} \boldsymbol{v}$ for $i = 1, \ldots, n$. Many systems of practical relevance fulfill the observability rank condition of Theorem 1, but violate the integrability condition 2. If the existence conditions in Theorem 1 are fulfilled, there exists several different ways for the symbolic computation of the normal form observer (5), see [41, 44–47].

3 Approximate Observer Error Linearization

Our approximation of the observer form (5) is based on a slightly modified version of Theorem 2:

Corollary 3 *Consider n vector fields $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n : \mathcal{X} \to \mathbb{R}^n$. Assume that these vector fields are linearly independent at $\boldsymbol{x}_0 \in \mathcal{X}$ and that $[\boldsymbol{v}_i, \boldsymbol{v}_j](\boldsymbol{x}) = \mathbf{0}$ holds for $1 \leq i, j \leq r$ and all $\boldsymbol{x}$ in a neighbourhood of $\boldsymbol{x}_0$. Then there exists a change of coordinates* (2) *such that*

1. *Eq.* (8) *holds for $1 \leq i \leq r$ in a neighbourhood of $\boldsymbol{x}_0$, and*
2. *Eq.* (8) *holds for $1 \leq i \leq n$ at the point $\boldsymbol{x}_0$.*

Proof. Consider the map (9). The locally exact rectification of the first r vector fields follows immediately from the proof of Th. 2. The point-wise rectification follows directly from the linearization (10). □

The rectification according to Theorem 2 and Corollary 3 is sketchted in Fig. 2 for $n = 2$ as well as $r = 2$ and $r = 1$, respectively.

Now, we will discuss the transformation of system (1) into an *approximate observer form*

$$\begin{aligned}
\dot{\boldsymbol{z}} &= \bar{\boldsymbol{f}}(\boldsymbol{z}) = A\boldsymbol{z} + \hat{\boldsymbol{\alpha}}(z_r, \ldots, z_n) \\
y &= \bar{h}(\boldsymbol{z}) = \hat{\gamma}(z_n).
\end{aligned} \tag{11}$$

Due to the usage of the weaker form (11) instead of (3) our approach belongs to the class of methods called approximate observer error linearization [38, 40, 41]. Because the system linearized in the origin is in observer canonical form, our design methodology is also related to local approximations via series expansions [12, 31, 35, 36].

Sufficient existence conditions for (11) are given in the following theorem:

Theorem 4 *Assume that for a point $\boldsymbol{x}_0 \in \mathcal{X}$ there is an integer $r \in \{1, \ldots, n\}$ such that*

1. $\dim \left(dh(\boldsymbol{x}_0), L_{\boldsymbol{f}} dh(\boldsymbol{x}_0), \ldots, L_{\boldsymbol{f}}^{n-1} dh(\boldsymbol{x}_0) \right) = n$,
2. $[ad_{\boldsymbol{f}}^i \boldsymbol{v}, ad_{\boldsymbol{f}}^j \boldsymbol{v}](\boldsymbol{x}) = \mathbf{0}$ *for* $0 \leq i, j \leq r-1$ *and all $\boldsymbol{x}$ in a neighbourhood of $\boldsymbol{x}_0$,*

where the vector field $\boldsymbol{v}$ is the solution of (7) *for an appropriate scalar-valued continuous function β with $\beta(h(\boldsymbol{x}_0)) \neq 0$. Then, the map*

$$\boldsymbol{x} = S(\boldsymbol{z}) = \boldsymbol{\varphi}_{z_1}^{\boldsymbol{v}} \circ \boldsymbol{\varphi}_{z_2}^{ad_{-\boldsymbol{f}} \boldsymbol{v}} \circ \cdots \circ \boldsymbol{\varphi}_{z_n}^{ad_{-\boldsymbol{f}}^{n-1} \boldsymbol{v}}(\boldsymbol{x}_0) \tag{12}$$

is a local diffeomorphism having the inverse $\boldsymbol{z} = T(\boldsymbol{x})$ with $T(\boldsymbol{x}_0) = \mathbf{0}$. Under this change of coordinates system (1) *is transformed into the approximate observer form* (11)*. Moreover, the linearization of* (11) *at $\boldsymbol{z} = \mathbf{0}$ is in observer canonical form.*

The existence conditions in Theorem 4 are weaker than these in Theorem 1 because the involutivity condition must only hold for Lie brackets up to order $r-1$ instead of $n-1$.

Proof. From the observability rank condition we conclude that (7) has a unique solution $\boldsymbol{v}$ for a given map β. Moreover, Eq. (7) implies that matrix product

$$\begin{pmatrix} dh(\boldsymbol{x}_0) \\ \vdots \\ dL_{\boldsymbol{f}}^{n-1} h(\boldsymbol{x}_0) \end{pmatrix} \left(\boldsymbol{v}(\boldsymbol{x}_0), \ldots, ad_{-\boldsymbol{f}}^{n-1} \boldsymbol{v}(\boldsymbol{x}_0) \right) = \begin{pmatrix} 0 & & \beta(\boldsymbol{x}_0) \\ & \ddots & \\ \beta(\boldsymbol{x}_0) & & * \end{pmatrix} \tag{13}$$

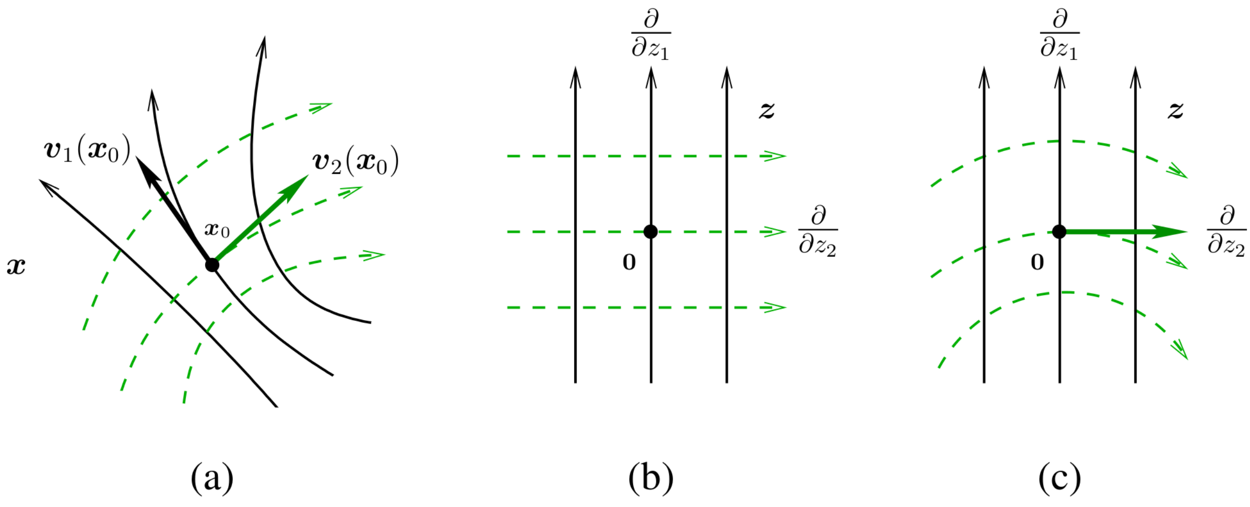

Figure 2: Rectification of $n = 2$ vector fields: (a) originial flows, (b) rectification according to Theorem 2 for $r = 2$, (c) partial rectification according to Corollary 3 for $r = 1$

is an lower trianglular matrix, see [9, Lemma 4.1.2]. The matrix is regular since $\beta(\boldsymbol{x}_0) \neq 0$. Therefore, the vectors $\boldsymbol{v}(\boldsymbol{x}_0), \ldots, ad_{-\boldsymbol{f}}^{n-1}\boldsymbol{v}(\boldsymbol{x}_0)$ are linearly independent. Now we apply Corollary 3. The diffeomorphism (2) with (12) results in

$$T_* ad_{-\boldsymbol{f}}^i \boldsymbol{v} = \frac{\partial}{\partial z_{i+1}} \quad \text{for} \quad i = 0, \ldots, r-1 \tag{14}$$

in a *neighbourhood* of $\boldsymbol{x}_0$, where T_* denotes the differential of the diffeomorphism T. The diffeomorphism transforms the maps of system (1) into

$$\begin{aligned} \bar{\boldsymbol{f}}(\boldsymbol{z}) &= T_* \boldsymbol{f}(\boldsymbol{x})|_{\boldsymbol{x}=S(\boldsymbol{z})} \\ \bar{h}(\boldsymbol{z}) &= h(\boldsymbol{x})|_{\boldsymbol{x}=S(\boldsymbol{z})} . \end{aligned}$$

Due to (13) we have

$$\begin{aligned} \tfrac{\partial}{\partial z_{i+1}} \bar{h} &= \left\langle dh, ad_{-\boldsymbol{f}}^i \boldsymbol{v} \right\rangle \\ &= \begin{cases} 0 & \text{for} \quad i = 0, \ldots, n-2, \\ \beta(\boldsymbol{x}) & \text{for} \quad i = n-1. \end{cases} \end{aligned} \tag{15}$$

Therefore, the output map $\bar{h}$ has the form given in (11). Next we consider the vector field

$$\bar{\boldsymbol{f}} = \bar{f}_1 \tfrac{\partial}{\partial z_1} + \cdots + \bar{f}_n \tfrac{\partial}{\partial z_n}.$$

Because of (14) we have

$$\begin{aligned} \tfrac{\partial}{\partial z_{i+1}} &= T_* ad_{-\boldsymbol{f}}^i \boldsymbol{v} \\ &= T_*[-\boldsymbol{f}, ad_{-\boldsymbol{f}}^{i-1} \boldsymbol{v}] \\ &= [-T_* \boldsymbol{f}, T_* ad_{-\boldsymbol{f}}^{i-1} \boldsymbol{v}] \\ &= [-\bar{\boldsymbol{f}}, \tfrac{\partial}{\partial z_i}] \\ &= \sum_{j=1}^n \left(\left(\tfrac{\partial}{\partial z_i} \bar{f}_j \right) \tfrac{\partial}{\partial z_j} \right) \end{aligned} \tag{16}$$

for $i = 1, \ldots, r-1$. Comparing both sides of (16) we get $\frac{\partial}{\partial z_i} \bar{f}_j = 0$ for $1 \leq j \leq n, j \neq i+1, 1 \leq i \leq r-1$ and $\frac{\partial}{\partial z_i} \bar{f}_{i+1} = 1$ for $1 \leq i \leq r-1$. This implies that the transformed vector field $\bar{\boldsymbol{f}}$ has the form given in (11) in a neighbourhood of $\boldsymbol{z} = \boldsymbol{0}$ (see Fig 3(a)).

Finally, we consider the linearization of the transformed system. The output map is already in the appropriate form (cf. (15)). The diffeomorphism (2) with (12) from Corollary 3 is such that

$$T_* ad_{-\boldsymbol{f}}^i \boldsymbol{v} = \frac{\partial}{\partial z_{i+1}} \quad \text{for} \quad i = 0, \ldots, n-1$$

holds in the *point* $\boldsymbol{x}_0$. Then, Eq. (16) is also valid in the point $\boldsymbol{x} = \boldsymbol{x}_0$ (and $\boldsymbol{z} = \boldsymbol{0}$, resp.) for $i = 0, \ldots, n-1$. Therefore, we have $\frac{\partial}{\partial z_i} \bar{f}_j(\boldsymbol{0}) = 0$ for $1 \leq j \leq n, j \neq i+1, 1 \leq i \leq n-1$ and $\frac{\partial}{\partial z_i} \bar{f}_{i+1}(\boldsymbol{0}) = 1$ for $1 \leq i \leq n-1$. Hence, the Jacobian matrix has the form sketched in Fig. 3(b), i.e., the system linearized in the origin $\boldsymbol{z} = \boldsymbol{0}$ is in observer canonical form. □

Assume that system (1) can be transformed into the approximate observer form (11). We suggest an observer of the form

$$\begin{aligned} \dot{\hat{\boldsymbol{z}}} &= A\hat{\boldsymbol{z}} + \hat{\boldsymbol{\alpha}}(\hat{z}_r, \ldots, \hat{z}_{n-1}, \hat{\gamma}^{-1}(y)) \\ &\quad + \boldsymbol{k}(\hat{\gamma}^{-1}(y) - \hat{z}_n) \\ \hat{\boldsymbol{x}} &= S(\hat{\boldsymbol{z}}) \end{aligned} \tag{17}$$

with the constant gain vector $\boldsymbol{k} \in \mathbb{R}^n$. The error dynamics is given by

$$\begin{aligned} \dot{\tilde{\boldsymbol{z}}} = (A - \boldsymbol{k}\boldsymbol{c}^T)\tilde{\boldsymbol{z}} \; &+ \; \hat{\boldsymbol{\alpha}}(z_r, \ldots, z_{n-1}, z_n) \\ &- \; \hat{\boldsymbol{\alpha}}(\hat{z}_r, \ldots, \hat{z}_{n-1}, z_n). \end{aligned} \tag{18}$$

The last argument z_n of both non-linearities in (18) is the same and obtained from the measured output:

$$
\boldsymbol{f}'(\boldsymbol{z}) = \left(\begin{array}{ccc|ccc}
0 & \cdots & 0 & * & \cdots & * \\
1 & \ddots & \vdots & * & \cdots & * \\
0 & \ddots & 0 & \vdots & \ddots & \vdots \\
\vdots & \ddots & 1 & \vdots & \ddots & \vdots \\
\vdots & & \vdots & * & \cdots & * \\
0 & \cdots & 0 & * & \cdots & *
\end{array}\right)
\qquad
\boldsymbol{f}'(\boldsymbol{0}) = \left(\begin{array}{ccccc|c}
0 & 0 & \cdots & \cdots & 0 & * \\
1 & 0 & \ddots & & \vdots & * \\
0 & 1 & \ddots & \ddots & \vdots & \\
\vdots & 0 & \ddots & 0 & 0 & \vdots \\
\vdots & & \ddots & 1 & 0 & * \\
0 & \cdots & \cdots & 0 & 1 & *
\end{array}\right)
$$

(a) (b)

Figure 3: Jacobians of $\bar{\boldsymbol{f}}$: (a) in a neighbourhood of $\boldsymbol{z} = \boldsymbol{0}$, (b) in the point $\boldsymbol{z} = \boldsymbol{0}$

$z_n = \hat{\gamma}^{-1}(y)$. Otherwise, the error dynamics (18) has the typical structure occurring in high gain design. Since $\boldsymbol{f}$ and h are assumed to be smooth, the map $\hat{\boldsymbol{\alpha}}$ is also smooth and at least locally Lipschitz. To determine the observer gain vector $\boldsymbol{k}$ we can employ several high gain design techniques [48–51]. Modern approaches are also based on the concept of passivity [52,53].

4 Computational Issues

For the implementation of (17) we need the transformation (2) with (12). Using the chain rule we get

$$
\begin{aligned}
\dot{\boldsymbol{z}} &= T'(\boldsymbol{x})\dot{\boldsymbol{x}}|_{\boldsymbol{x}=S(\boldsymbol{z})} \\
&= T'(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})|_{\boldsymbol{x}=S(\boldsymbol{z})} \\
&= \left(S'(\boldsymbol{z})\right)^{-1} \boldsymbol{f}(\boldsymbol{x})|_{\boldsymbol{x}=S(\boldsymbol{z})} \\
&= \bar{\boldsymbol{f}}(\boldsymbol{z}).
\end{aligned}
\tag{19}
$$

In particular, to compute the form (3) we need the map S, but we do not have to compute T symbolically. The diffeomorphism S results from a concatenation of flows according to (12). To obtain the flows we have to find the general solution of ordinary differential equations. Computer algebra packages such as Mathematica and Maple have build-in functions as well as additional libraries to solve ordinary differential equations (e.g., see [54]). Unfortunately, many differential equations cannot be solved symbolically. In this cases, one could basically use any approximate solution. In the following, we recall two approximation methodes. We explain these methods for a given vector field $\boldsymbol{v} : \mathcal{X} \to \mathbb{R}^n$. To compute (12), we also have to apply these procedures to the vector fields $ad_{-\boldsymbol{f}}\boldsymbol{v}, \ldots, ad_{-\boldsymbol{f}}^{n-1}\boldsymbol{v}$ from Section 3.

Taylor expansion The initial value problem for the vector field $\boldsymbol{v}$ reads as

$$
\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathcal{X} \tag{20}
$$

with $\boldsymbol{x}_0 \in \mathcal{X}$. If $\boldsymbol{v}$ is analytic, the solution of (20) can be expanded into an infinite series

$$
\boldsymbol{x}(t) = \boldsymbol{\varphi}_t^{\boldsymbol{v}}(\boldsymbol{x}_0) = \sum_{k=0}^{\infty} \boldsymbol{\zeta}_k(\boldsymbol{x}_0)\frac{t^k}{k!} \tag{21}
$$

with vector fields $\boldsymbol{\zeta}_k : \mathcal{X} \to \mathbb{R}^n$, $k \geq 0$. These vector fields can be computed recursively by

$$
\boldsymbol{\zeta}_{k+1}(\boldsymbol{x}) = \boldsymbol{\zeta}_k'(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x}) \quad \text{with} \quad \boldsymbol{\zeta}_0(\boldsymbol{x}) = \boldsymbol{x}.
$$

The series (21) is called *Lie series* [55]. The function values $\boldsymbol{\zeta}_k(\boldsymbol{x}_0)$ of the vector fields can efficiently be calculated with algorithmic differentiation [56–61].

Picard Iteration The initial value problem (20) is equivalent to the integral equation

$$
\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{v}(\boldsymbol{x}(\tau))d\tau.
$$

In the proof of the existence and uniqueness theorem of Picard-Lindelöff for ordinary differential equations, this integral equation is solved by the recursion

$$
\boldsymbol{\xi}_{k+1}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{v}(\boldsymbol{\xi}_k(\tau))d\tau \ \text{ with } \ \boldsymbol{\xi}_0(t) = \boldsymbol{x}_0.
$$

If the vector field $\boldsymbol{v}$ is Lipschitz continuous, we have $\lim_{k\to\infty} \boldsymbol{\xi}_k(t) = \boldsymbol{x}(t)$, see [62].

5 Example

We will illustrate our approach on two example systems. In both cases the transformation (12) can be computed symbolically.

5.1 Synchronous Motor

In [63], a synchronous motor is modelled by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= B_1 - A_1 x_2 - A_2 x_3 \sin x_1 - 0.5 B_2 \sin(2x_1)\\ \dot{x}_3 &= u - D_1 x_3 + D_2 \cos x_1\\ y &= x_1,\end{aligned} \tag{22}$$

where x_1 denotes the rotor angle, x_2 the speed deviation and x_3 the field flux linkage. The observability matrix reads as

$$Q(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ -A_2 x_3 \cos x_1 - B_2 \cos(2x_1) & -A_1 & -A_2 \sin x_1\end{pmatrix}.$$

Clearly, the observability matrix is singular for integral multiplicities of π.

We compute the starting vector field $\boldsymbol{v}$ symbolically without specifying the map β. We obtain $[\boldsymbol{v}, ad_{-\boldsymbol{f}}\boldsymbol{v}] = [\boldsymbol{v}, ad^2_{-\boldsymbol{f}}\boldsymbol{v}] = \boldsymbol{0}$ and

$$[ad_{-\boldsymbol{f}}\boldsymbol{v}, ad^2_{-\boldsymbol{f}}\boldsymbol{v}] = \mu(x_1)\frac{\partial}{\partial x_2} + \varrho(\boldsymbol{x})\frac{\partial}{\partial x_3} \tag{23}$$

with

$$\mu(x_1) = \frac{\beta(x_1)}{\sin(x_1)}(\cos(x_1)\beta(x_1) - 3\sin(x_1)\beta'(x_1))$$

and with a complicated function ϱ. We will try to choose β in such a way that the integrability condition of Theorem 1 is fulfilled. The second component of (23) is identically zero if β satisfies the ordinary differential equation

$$\cos(x_1)\beta(x_1) - 3\sin(x_1)\beta'(x_1) = 0.$$

The general solution is $\beta(x_1) = C_1(\sin(x_1))^{1/3}$ with the constant C_1. Even with this choise of β, the third component of (23) is not the zero function. Hence, system (22) cannot be transformed into the normal form (3).

The aforementioned choice of β results in complicated expressions for $ad_{-\boldsymbol{f}}\boldsymbol{v}$ and $ad^2_{-\boldsymbol{f}}\boldsymbol{v}$. As the integrability condition cannot be fulfilled anyway, we will use the degree of freedom in order to simplify the vector fields $\boldsymbol{v}, ad_{-\boldsymbol{f}}\boldsymbol{v}, ad^2_{-\boldsymbol{f}}\boldsymbol{v}$. With $\beta(x_1) = -A_2 \sin x_1$ we obtain the simplest possible vector field $\boldsymbol{v}(\boldsymbol{x}) = \frac{\partial}{\partial x_3}$ in form of the 3rd unit vector. The next vector fields are

$$\begin{aligned} ad_{-\boldsymbol{f}}\boldsymbol{v}(\boldsymbol{x}) &= -A_2 \sin x_1 \tfrac{\partial}{\partial x_2} - D_1 \tfrac{\partial}{\partial x_3},\\ ad^2_{-\boldsymbol{f}}\boldsymbol{v}(\boldsymbol{x}) &= -A_2 \sin x_1 \tfrac{\partial}{\partial x_1} - A_2(x_2 \cos x_1\\ &\quad + (A_1 + D_1)\sin x_1)\tfrac{\partial}{\partial x_2}\\ &\quad + D_1^2 \tfrac{\partial}{\partial x_3}.\end{aligned}$$

We have $[\boldsymbol{v}, ad_{-\boldsymbol{f}}\boldsymbol{v}] = [\boldsymbol{v}, ad^2_{-\boldsymbol{f}}\boldsymbol{v}] = \boldsymbol{0}$ but $[ad_{-f}\boldsymbol{v}, ad^2_{-\boldsymbol{f}}\boldsymbol{v}] \neq \boldsymbol{0}$. Hence, the conditions of Theorem 4 hold for $r = 2$ and $x_1 \neq \ell\pi$ with $\ell \in \mathbb{Z}$. Using $\boldsymbol{x}_0 = (\pi/2, 0, 0)^T$, the transformation (12) becomes

$$\begin{aligned}\boldsymbol{x} &= S(\boldsymbol{z})\\ &= \begin{pmatrix} 2\arccos\left(\frac{\exp(A_2 z_3)}{\sqrt{1+\exp(2A_2 z_3)}}\right)\\ (A_1 + D_1)\sinh(A_2 z_3)\\ D_1^2 z_3\end{pmatrix}.\end{aligned} \tag{24}$$

With (24) we were able to compute the form (11) of (22) symbolically. The vector field $\bar{\boldsymbol{f}}$ is too complicate to be shown here, but its linearization

$$\bar{\boldsymbol{f}}'(\boldsymbol{0}) = \begin{pmatrix} 0 & 0 & B_2 D_1 + A_2 D_2\\ 1 & 0 & B_2 - A_1 D_1\\ 0 & 1 & -A_1 - D_1\end{pmatrix}$$

is in observer canonical form. In $\boldsymbol{z}$-coordinates, the output becomes

$$\begin{aligned} y &= \bar{h}(\boldsymbol{z})\\ &= \hat{\gamma}(z_3) = 2\arccos\left(\frac{\exp(A_2 z_3)}{\sqrt{1+\exp(2A_2 z_3)}}\right).\end{aligned}$$

For the simulation we used the parameter values $A_1 = 0.2703$, $A_2 = 12.01$, $B_1 = 39.19$, $B_2 = -48.04$, $D_1 = 0.3222$ and $D_2 = 1.9$. Moreover, we used the initial values $\boldsymbol{x}(0) = (0.8, 0.1, 10)^T$ and $\hat{\boldsymbol{x}}(0) = (0.8, 0, 0)^T$ and the constant input $u = 1.933$. The eigenvalues of the linear part of the error dynamics were placed at -10 by $\boldsymbol{k} = (1000, 300, 30)^T$. Fig. 4 shows the simulated trajectories. We used black solid lines for the original system (22), blue dashed lines for the observer (17) and green dashed-dotted lines for the extended Luenberger observer [12, 32]. Indeed, the new observer shows a good performance. For the discussion of observer design for similar applications we refer to [64–66].

5.2 Lorenz System

As a further example we consider the Lorenz system

$$\begin{aligned}\dot{x}_1 &= s(x_2 - x_1)\\ \dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3\\ \dot{x}_3 &= x_1 x_2 - b x_3\\ y &= x_1\end{aligned} \tag{25}$$

with the parameters $s, \rho, b > 0$, see [67]. The observability matrix has the form

$$Q(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 & 0\\ -s & s & 0\\ s + s(\rho - x_3) & -s - s & -s x_1\end{pmatrix}.$$

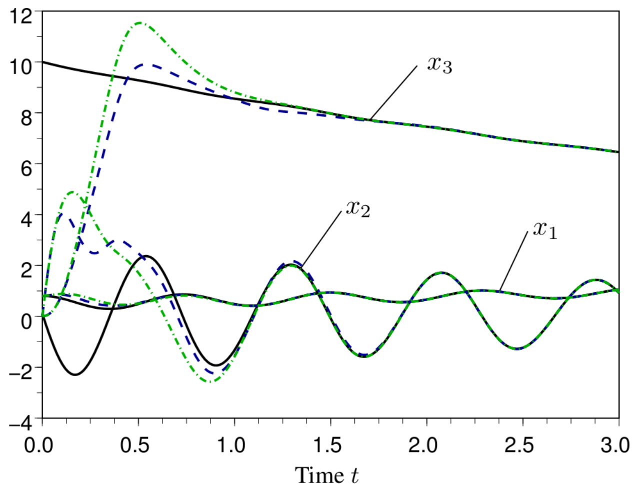

Figure 4: Trajectories of the motor model (22) with the observer (17) and an extended Luenberger observer

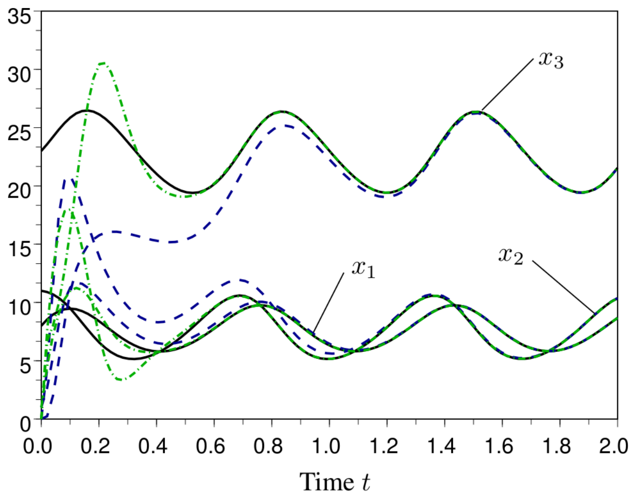

Figure 5: Trajectories of the Lorenz system (25) with the observer (17) and a high gain observer

Clearly, the observability rank condition is violated for $x_1 = 0$. Using $\beta(x_1) = -sx_1$ we get $\boldsymbol{v}(\boldsymbol{x}) = \frac{\partial}{\partial x_3}$, $ad_{-\boldsymbol{f}}\boldsymbol{v}(\boldsymbol{x}) = -x_1 \frac{\partial}{\partial x_2} - b\frac{\partial}{\partial x_3}$, and $ad_{-\boldsymbol{f}}\boldsymbol{v}(\boldsymbol{x}) = -sx_1 \frac{\partial}{\partial x_1} + ((1+b-s)x_1 + sx_2)\frac{\partial}{\partial x_2} - (b-x_1)\frac{\partial}{\partial x_3}$. With $\boldsymbol{x}_0 = (0,0,1)^T$, the transformation (12) becomes

$$\begin{aligned} \boldsymbol{x} &= S(\boldsymbol{z}) \\ &= \begin{pmatrix} \exp(-sz_3) \\ -\exp(sz_3)z_2 + \frac{1+b-s}{s}\sinh(sz_3) \\ z_1 + bz_3 - bz_2 - \frac{1}{2s}(1-\exp(-2sz_3)) \end{pmatrix}. \end{aligned}$$

Again, the equations of the transformed system are to complicated to be shown here, but can easily be computed by (19) with computer algebra systems. Because of

$$[ad_{-\boldsymbol{f}}\boldsymbol{v}, ad^2_{-\boldsymbol{f}}\boldsymbol{v}] = -2sx_1 \frac{\partial}{\partial x_2} \neq \boldsymbol{0}$$

the system cannot be transformed into observer canonical form (3). Since $[\boldsymbol{v}, ad_{-\boldsymbol{f}}\boldsymbol{v}] = \boldsymbol{0}$ we have $r = 2$, i.e., the map $\hat{\boldsymbol{\alpha}}$ of (11) depends on z_2 and z_3, but not on z_1.

The simulation was carried out with the parameter values $s = 10$, $b = 8/3$, $\rho = 24$, and the initial value $\boldsymbol{x}(0) = (8, 11, 23)^T$. In this setting, the plane $x_1 = 0$ is not crossed, i.e., the observability rank condition holds along the system's trajectory. For the observer we used the initial value $\hat{\boldsymbol{x}}(0) = (1,0,0)^T$ and placed all three eigenvalues at -10. Fig. 5 shows the simulation result. We used black solid lines for the original system (25), blue dashed lines for the observer (17) and green dashed-dotted lines for the high gain observer suggested in [4, 5]. Although we also designed an extended Luenberger observer, the simulation could not be carried out because the observer trajectory went into a singularity of the observer gain. However, the new observer (17) converges.

6 Conclusions

This paper outlined an observer design method for single output systems. The procedure yields approximately linear error dynamics. The design method is constructive and can easily be implemented with computer algebra packages such as Maple or Mathematica. Extensions to systems with inputs and multi-output systems are straightforward.

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References:

[1] B. L. Walcott, M. J. Corless, S. H. Żak, Comparative study of non-linear state-observation techniques, *International Journal of Control*, 1987, Vol. 45, No. 6, pp. 2109–2132.

[2] P. Misra, P. van Dooren, A. Varga, Computation of structural invariants of generalized state-space systems, *Automatica*, 1994, Vol. 30, No. 12, pp. 1921–1936.

[3] H. Nijmeijer, T. I. Fossen (Editors), *New Directions in Nonlinear Observer Design*, *Lecture Notes in Control and Information Science*, Vol. 244, Springer-Verlag, London, 1999.

[4] J. P. Gauthier, H. Hammouri, S. Othman, A simple observer for nonlinear systems — application to bioreactors, *IEEE Trans. on Automatic Control*, 1992, Vol. 37, No. 6, pp. 875–880.

[5] G. Ciccarella, M. D. Mora, A. Germani, A Luenberger-like observer for nonlinear systems, *International Journal of Control*, 1993, Vol. 57, No. 3, pp. 537–556.

[6] M. Zeitz, Canonical forms for nonlinear systems, in: *Geometric Theory of Nonlinear Control Systems*, (Editors) B. Jakubczyk, W. Respondek, K. Tchoń, Wroclaw Technical University Press, 1985, pp. 255–278.

[7] M. Zeitz, Canonical forms for nonlinear systems, in: *Proc. IFAC-Symposium Nonlinear Control System Design, Capri (Italy), 1989*.

[8] C. I. Byrnes, A. Isidori, Asymptotic stabilization of minimum phase nonlinear systems, *IEEE Trans. on Automatic Control*, 1991, Vol. 36, No. 10, pp. 1122–1137.

[9] A. Isidori, *Nonlinear Control Systems: An Introduction*, Springer-Verlag, London, 3rd edition, 1995.

[10] J. Moreno, Unknown input observers for SISO nonlinear systems, in: *Proc. IEEE Conference on Decision and Control (CDC)*, 2000, Vol. 1, pp. 790–801.

[11] J. Moreno, E. Rocha-Cózatl, Pasivizaión y existencia de observadores con entradas desconocidas para sistemas no lineales SISO, in: *Proc. Conferencia de Ingenieéa Eléctrica*, Mexico, 2000, pp. 334–341.

[12] D. Bestle, M. Zeitz, Canonical form observer design for non-linear time-variable systems, *International Journal of Control*, 1983, Vol. 38, No. 2, pp. 419–431.

[13] A. J. Krener, A. Isidori, Linearization by output injection and nonlinear observers, *Systems & Control Letters*, 1983, Vol. 3, pp. 47–52.

[14] A. J. Krener, W. Respondek, Nonlinear observers with linearizable error dynamics, *SIAM J. Control and Optimization*, 1985, Vol. 23, No. 2, pp. 197–216.

[15] X.-H. Xia, W.-B. Gao, Non-linear observer design by observer canonical form, *International Journal of Control*, 1988, Vol. 47, No. 4, pp. 1081–1100.

[16] X. H. Xia, W. B. Gao, Nonlinear observer design by observer error linearization, *SIAM J. Control and Optimization*, 1989, Vol. 27, No. 1, pp. 199–216.

[17] R. Marino, Adaptive observers for single output nonlinear systems, *IEEE Trans. on Automatic Control*, 1990, Vol. 35, No. 9, pp. 1054–1058.

[18] R. Marino, P. Tomei, Global adaptive observers for nonlinear systems via filtered transformations, *IEEE Trans. on Automatic Control*, 1992, Vol. 37, No. 8, pp. 1239–1245.

[19] R. Marino, P. Tomei, Adaptive observers with arbitrary exponential rate of convergence for nonlinear systems, *IEEE Trans. on Automatic Control*, 1995, Vol. 40, No. 7, pp. 1300–1304.

[20] J. Back, J. H. Seo, Immersion technique for nonlinear observer design, in: *Proc. American Control Conference (ACC)*, 2002, pp. 2645–2646.

[21] J. Back, H. Shim, J. H. Seo, An algorithm for system immersion into nonlinear observer form: forced systems, in: *Preprints of the 16th IFAC World Congress, Prague, Czech Republic, 2005, July 3-8*.

[22] D. Noh, N. H. Jo, J. H. Seo, Nonlinear observer design by dynamic observer error linearization, *IEEE Trans. on Automatic Control*, 2004, Vol. 49, No. 10, pp. 1746–1750.

[23] J. Back, K. T. Yu, J. H. Seo, Dynamic observer error linearization, *Automatica*, 2006, Vol. 42, No. 12, pp. 2195–2200.

[24] K. T. Yu, J. Back, J. H. Seo, Constructive algorithm for dynamic observer error linearization via integrators: single output case, *International Journal of Robust and Nonlinear Control*, 2007, Vol. 17, No. 1, pp. 25–49.

[25] M. Guay, Observer linearization by output-dependent time-scale transformations, *IEEE Trans. on Automatic Control*, 2002, Vol. 47, No. 10, pp. 1730–1735.

[26] W. Respondek, A. Pogromsky, H. Nijmeijer, Time scaling for observer design with linearizable error dynamics, *Automatica*, 2004, Vol. 40, No. 2, pp. 277–285.

[27] N. Kazantzis, C. Kravaris, Nonlinear observer design using Lyapunov's auxiliary theorem, *Systems & Control Letters*, 1998, Vol. 34, pp. 241–247.

[28] A. Krener, M. Xiao, Nonlinear observer design in the Siegel domain, *SIAM J. Control and Optimization*, 2002, Vol. 41, No. 3, pp. 932–953.

[29] M. Xiao, The global existence of nonlinear observers with linear error dynamics: A topological point of view, *Systems & Control Letters*, 2006, Vol. 55, pp. 849–858.

[30] J. Deutscher, M. Bäuml, Design of nonlinear observers with approximately linear error dynamics using multivariable Legendre polynomials, *Int. J. of Robust and Nonlinear Control*, 2006.

[31] M. Zeitz, The extended Luenberger observer for nonlinear systems, *Systems & Control Letters*, 1987, Vol. 9, pp. 149–156.

[32] J. Birk, M. Zeitz, Extended Luenberger observer for non-linear multivariable systems, *International Journal of Control*, 1988, Vol. 47, No. 6, pp. 1823–1836.

[33] J. Schaffner, M. Zeitz, Variants of nonlinear normal form observer design, in: Hijmeijer and Fossen [3], 1999, pp. 161–180.

[34] S. Karahan, *Higher degree linear approximations of nonlinear systems*, Ph.D. thesis, University of California, Davis, 1989.

[35] A. Krener, M. Hubbard, S. Karaham, A. Phelps, B. Maag, Poincaré's linearization method applied to the design of nonlinear compensators, in: *Algebraic computing in control*, *Lecture Notes in Control and Information Science*, Vol. 165, Springer, 1991, pp. 76–114.

[36] K. Röbenack, A. F. Lynch, An efficient method for observer design with approximately linear error dynamics, *International Journal of Control*, 2004, Vol. 77, No. 7, pp. 607–612.

[37] K. Röbenack, Direct approximation of observer error linearization for nonlinear forced systems, *IMA Journal of Mathematical Control and Information*, 2007, Vol. 24, No. 4, pp. 551–566.

[38] A. F. Lynch, S. A. Bortoff, Nonlinear observer design by approximate error linearization, *Systems & Control Letters*, 1997, Vol. 32, No. 3, pp. 161–172.

[39] A. F. Lynch, S. A. Bortoff, Nonlinear observers with approximately linear error dynamics: The multivariable case, *IEEE Trans. on Automatic Control*, 2001, Vol. 46, No. 7, pp. 927–932.

[40] A. Banaszuk, W. Sluis, On nonlinear observers with approximately linear error dynamics, in: *Proc. American Control Conf. (ACC)*, Albuquerque, 1997, pp. 3460–3464.

[41] K. Nam, An approximate nonlinear observer with polynomial coordinate transformation maps, *IEEE Trans. on Automatic Control*, 1997, Vol. 42, No. 4, pp. 522–527.

[42] P. Brunovsky, A classification of linear controllable systems, *Kybernetica*, 1970, Vol. 6, No. 3, pp. 173–188.

[43] H. Nijmeijer, A. J. van der Schaft, *Nonlinear Dynamical Control systems*, Springer, New York, 1990.

[44] H. Keller, Entwurf nichtlinearer, zeitvarianter Beobachter durch Polvorgabe mit Hilfe einer Zwei-Schritt-Transformation, *Automatisierungstechnik*, 1986, Vol. 34, no. 7, 8, pp. 271–274, 326–331.

[45] H. Keller, Non-linear observer design by transformation into a generalized observer canonical form, *International Journal of Control*, 1987, Vol. 46, No. 6, pp. 1915–1930.

[46] C. W. Li, L. W. Tao, Observing non-linear time-variable systems through a canonical form observer, *International Journal of Control*, 1986, Vol. 44, No. 6, pp. 1703–1713.

[47] K. Röbenack, Entwurf nichtlinearer Beobachter mit linearer und näherungsweise linearer Fehlerdynamik, *Automatisierungstechnik*, 2010, Vol. 58, No. 9, pp. 489–497.

[48] S. Raghavan, J. K. Hedrick, Observer design for a class of nonlinear systems, *International Journal of Control*, 1994, Vol. 59, No. 2, pp. 515–528.

[49] R. Rajamani, Observers for Lipschitz nonlinear systems, *IEEE Trans. on Automatic Control*, 1998, Vol. 43, No. 3, pp. 397–401.

[50] K. Röbenack, A. F. Lynch, High-gain nonlinear observer design using the observer canonical form, *IET Control Theory & Applications*, 2007, Vol. 1, No. 6, pp. 1574–1579.

[51] K. Röbenack, Structure matters – some notes on high gain observer design for nonlinear systems, in: *Proc. of the 9th International Multi-Conference on Systems, Signals and Devices*, Chemnitz, 2012.

[52] J. A. Moreno, Design of nonlienar observers using Popov's criterion, in: *Preprints of the 16th IFAC World Concress*, Prague, 2005, Section 2.3.

[53] J. A. Moreno, Approximate observer error linearization by dissipativity, in: *Control and Observer Design for Nonlinear Finite and Infinite Dimensional Systems*, (Editors) T. Meurer, K. Graichen, E. D. Gilles, *Lecture Notes in Control and Information Science*, Vol. 322, Springer, 2006, pp. 35–51.

[54] A. Marasco, A. Romano, *Scientific Computing with Mathematica, Mathematical Problems for Ordinary Differential Equations*, Modeling and Simulation in Science, Engineering and Technology, Birkhäuser, 2001.

[55] W. Gröbner, *Die Lie-Reihen und ihre Anwendung*, Deutscher Verlag der Wissenschaften, Berlin, 1967.

[56] A. Griewank, ODE solving via automatic differentiation and rational prediction, in: *Numerical Analysis 1995*, (Editors) D. F. Griffiths, G. A. Watson, *Pitman Research Notes in Mathematics Series*, Vol. 344, Addison-Wesley, Longman, 1995, pp. 37–56.

[57] A. Griewank, G. Corliss, P. Henneberger, G. Kirlinger, F. Potra, H. Stetter, High-order stiff ODE solvers via automatic differentiation and rational prediction, in: *Numerical Analysis and Its Applications*, *Lecture Notes in Computer Science*, Vol. 1196, Springer, 1997, pp. 114–125.

[58] A. Griewank, *Evaluating Derivatives — Principles and Techniques of Algorithmic Differentiation*, *Frontiers in Applied Mathematics*, Vol. 19, SIAM, Philadelphia, 2000.

[59] K. Röbenack, K. J. Reinschke, The computation of Lie derivatives and Lie brackets based on automatic differentiation, *Z. Angew. Math. Mech. (ZAMM)*, 2004, Vol. 84, No. 2, pp. 114–123.

[60] K. Röbenack, Computation of Lie derivatives of tensor fields required for nonlinear controller and observer design employing automatic differentiation, *Proc. in Applied Mathematics and Mechanics*, 2005, Vol. 5, No. 1, pp. 181–184.

[61] K. Röbenack, J. Winkler, S. Wang, Structure matters – some notes on high gain observer design for nonlinear systems, in: *Proc. of the 4th International Workshop on Equation-Based Object-Oriented Modeling Languages and Tools (EOOLT)*, Zurich, 2011.

[62] V. I. Arnold, *Ordinary Differential Equations*, Springer, 1992.

[63] B. K. Mukhopadhyay, O. P. Malik, Optimal control of synchronous-machine excitation by quasilinearisation techniques, *Proc. IEE*, 1972, Vol. 119, No. 1, pp. 91–98.

[64] H. R. Karimi, A. Babazadeh, Modelling and output tracking of transverse flux permanent magnet machines using high gain observer and RBF neural network, *ISA Transactions*, 2005, Vol. 44, No. 4, pp. 445–456.

[65] I. I. Siller-Alcala, M. Abderrahim, J. Jaimes-Ponce, R. Alcantara-Ramirez, Speed-Sensorless Nonlinear Predictive Control of a Squirrel Cage Motor, *WSEAS Transactions on Systems and Control*, 2008, Vol. 3, No. 2, pp. 99–104.

[66] D. Zaltni, M. N. Abdelkrim, Synchronous Motor Observability Study and a New Robust MRAS Speed Observer with an Improved Zero-speed Position Estimation Design for Surface PMSM, *WSEAS Transactions on Systems and Control*, 2010, Vol. 7, No. 7, pp. 567–580.

[67] E. N. Lorenz, Deterministic non-periodic flow, *J. Atmos. Sci.*, 1963, Vol. 20, pp. 130–141.