Co-ordinated Average Consensus of Multi-agent Systems via Sampled Control

YONGLI CHENG
Tianjin University
Department of Mathematics, School of Science
Weijin Street 92, Nankai district, Tianjin
CHINA
chengyongli8275@sina.com

DONGMEI XIE*
Tianjin University
Department of Mathematics
Weijin Street 92, Nankai district, Tianjin
CHINA
dongmeixie@tju.edu.cn

Abstract: This paper studies the co-ordinated average consensus of multi-agent system under Cartesian co-ordinate coupling in a discrete-time setting with switching sampling interval. Two cases are considered: 1) network without sampling delay; 2) network with sampling delay. For both cases, by algebraic graph theory and matrix theory, we analyze the convergence of the system and not only prove the existence of the sampling period, the Euler angle of the rotation matrix and sampling delay, but also give the approach of how to choose them. Our study shows that the eigenvalues of the corresponding Laplacian matrix, the size of the Euler angle, sampling period and delay have an important effect on consensus. Finally, some numerical simulations are presented to illustrate our theoretical results.

Key–Words: Multi-agent systems, Cartesian co-ordinated coupling, average consensus, sampled control, switching sampling interval.

1 Introduction

In recent years, consensus problem of multi-agent systems (MASs) has appeared as a new challenging area of research due to its broad applications in many areas such as computer science, vehicle systems, unmanned air vehicles, scheduling of automated highway systems, formation control of satellite clusters (see [1-12] and the references therein), etc. The main objective in the consensus problems is to design appropriate protocols and algorithms such that a group of agents can converge to a consistent value.

Different from the traditional consensus algorithms, collective motion from consensus with Cartesian coordinate coupling [13-17] has received much attention. [14-16] investigated a team of vehicles in 3D by introducing a rotation matrix to an existing consensus algorithm for single-integrator kinematics. It was shown that both the network topology and the value of the Euler angle affected the resulting collective motion. [16] further extended the results in [15] from single-integrator kinematics to double-integrator dynamics and proved that the network topology, the damping gain and the value of the Euler angle affected the resulting collective motion. A common assumption in [15-17] is that information transmission among agents is also continuous. However, information may not be transmitted continuously due to the unreliability of communication channels, the limitations of sensing ability of agents, and the constraints of total cost. Thus, it is more practical to consider the case of sample-based information transmission for its extensive applications.

Recently, some results for multi-agent systems via sampled control have been obtained in [18-29]. In [24], Ren et al. studied the collective motion patterns for a group of autonomous vehicles in a discrete-time setting with time delay and proposed sufficient conditions on the network topology, the sampling period, the time delay and the Euler angle such that different collective motion pattern can be achieved. However, the sampling interval investigated in [24] was assumed to be constant and time-invariant. Assuming the sampling interval is time-varying, in [11, 12], Gao et al. used the Lyapunov method to solve consensus problems of second-order multi-agent sampled control system under undirected networks and only sufficient consensus criteria were established. Furthermore, in [25], Wang et al. studied the average-consensus problem of second-order multi-agent sampled control system with switching sampling interval and not only proved the existence of the scaling parameter and switching sampling interval but also gave the approach of how to choose them. But it didn't investigate the Cartesian co-ordinate coupling protocol with time delay or give us a specific range of sampling interval either. Nevertheless, as pointed out in [14], Cartesian co-ordinate coupling can be regarded as a rather significant part of coordinated motions playing an important role in applications involving multi-agent networks repetitive movements such as cooperative patrol including cooperating mobile robots mentioned in [2], mapping, sampling or surveillance. This motivates us to write this paper.

In this paper, we will investigate the consensus problems of multi-agent switched system via sampled control protocol with Cartesian coordinate coupling. Consensus problems with switching sampled data and sampling delay based on a rotation matrix are considered. Different from the discussion in [24] investigated co-ordinated collective motion pattern under fixed sampling period, this paper focuses on the consensus problems for multi-agent system via the switching sampling interval. Besides, compared with [25], in absence of time delay and specific range of system parameters, we further consider Cartesian co-ordinate coupling and consensus criteria are established for multi-agent system with time delay. Combining these two aspects, the contributions of this paper are t-wofold: (I) We consider the switching sampling period rather than fixed and general non-period case since fixed case is special and it is difficult to establish better results for general non-period sampling(like [11, 12]). (II)We not only prove the existence of the sampling interval, Euler angle of the rotation matrix and sampling delay via Cartesian co-ordinate coupling protocol, but also give a specific range of the system parameters to guarantee consensus of multi-agent system.

The remainder of this paper is organized as follows. Some basic definitions and supporting results are presented in the next section. Our main results are given in Section 3 and Section 4. Numerical examples are given in Section 5 to illustrate our results and present the relationship among consensus, topology graph, sampling period, Euler angle and sampling delay. In Section 6, some concluding remarks and future research topics are discussed.

Notations: We use standard notations throughout this paper. M^T is the transpose of the matrix M. M^{-1} is the inverse of the matrix M. $rank(M)$ denotes the rank of the matrix M.$\otimes$ is the Kronecker product. $\mathbf{1}_n = [1, \cdots, 1]^T \in \mathbb{R}^n$ and **0** represents any zero matrix with appropriate dimensions. Let $\ell = \{1, 2, \cdots, n\}$, $\ell_1 =: \{0, 1, \cdots\}$, $\ell_2 =: \{1, 2, \cdots, 3n\}$. For any square matrix H, $\Lambda(H)$ denotes the set of all eigenvalues of H. I_n denotes the $n \times n$ identity matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. ι and $\bar{\cdot}$ denote the imaginary unit and the complex conjugate of a number, respectively.

2 Preliminaries

In this section, some basic knowledge on graph theory, problem formulations, some definitions and lemmas are given as the preliminaries of this paper.

2.1 Graph Theorem

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be an undirected graph with the set of nodes $\mathcal{V}$=$\{v_1, v_2, \cdots, v_n\}$, set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ with nonnegative adjacency elements a_{ij}. An edge of $\mathcal{G}$ is denoted by $e_{ij} = (v_j, v_i)$. The adjacency elements associated with the edges of the graph are positive, i.e., $e_{ij} \in \mathcal{E}$ if and only if $a_{ij} > 0$. Moreover, we assume $a_{ii} = 0$ for all $i \in \ell$. The set of neighbors of node v_i is denoted by N_i=$\{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. Since the graph is undirected, it means once $e_{ij} = (v_j, v_i)$ is an edge of $\mathcal{G}$, $e_{ji} = (v_i, v_j)$ is an edge of $\mathcal{G}$ as well. As a result, the adjacency matrix $\mathcal{A}$ is a symmetric nonnegative matrix. Correspondingly, the Laplacian matrix of the undirected graph is defined as $L = [\ell_{ij}] \in \mathbb{R}^{n\times n}$,where

$$\ell_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum\limits_{k=1,k\neq i}^{n} a_{ik}, & i = j. \end{cases}$$

An important property of L is that all the row sums and column sums of L are zero and thus $\mathbf{1}_n$ is either a right or a left eigenvector of L associated with zero eigenvalue. $\mu_i \in \Lambda(L), i \in \ell$, denotes the ith eigenvalue of Laplacian matrix.

A path between each distinct nodes i and j is a finite ordered sequence of distinct edges $(i, k_1), (k_1, k_2), \cdots, (k_l, j)$. A graph is said to be connected if there exists a path between any two distinct nodes of the graph.

2.2 Problem Formulations

Assume that each agent has the dynamics given by

$$\dot{r}_i(t) = u_i(t), \forall i \in \ell, \tag{1}$$

where $r_i(t) := \begin{bmatrix} x_i(t) \\ y_i(t) \\ z_i(t) \end{bmatrix} \in \mathbb{R}^3, u_i(t) \in \mathbb{R}^3$ are, respectively, the state and control input associated with the ith agents.

A consensus algorithm investigated in [14] for system (1) is given as

$$u_i(t) = \sum_{v_j\in N_i} a_{ij}C(r_j(t) - r_i(t)), \forall i \in \ell, \tag{2}$$

where $a_{ij} \geq 0, \forall i, j \in \ell$, and the matrix $C \in \mathbb{R}^{3\times3}$ denotes a rotation matrix. This paper assumes that each agent can receive the information of positions from its neighbors at sampling times. Without loss of generality, we let $t_0, t_1, \cdots$ denote the sampling times. A sampled-data control protocol induced from (2) is given as follows:

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}C(r_j(t_k) - r_i(t_k)), t \in [t_k, t_{k+1}), \quad (3)$$

where $i \in \ell, k \in \ell_1 =: \{0, 1, \cdots\}$. It follows from (3) that the control input of each agent during $[t_k, t_{k+1})$ is time-invariant. Then, we can obtain

$$r_i(t_{k+1}) = r_i(t_k) + (t_{k+1} - t_k)u_i(t_k), \forall i \in \ell, k \in \ell_1. \quad (4)$$

Let $T_k = t_{k+1} - t_k, k \in \ell_1$, then

$$r_i(t_{k+1}) = r_i(t_k) + T_k u_i(t_k), \forall i \in \ell, k \in \ell_1, \quad (5)$$

where $r_i(t_k)$ represents the position of the ith agent at kth sampling time.

Assumption 1. T_k is selected in turn from the set of $\{h, 2h\}$, where $h > 0$, i.e., $T_0 = h, T_1 = 2h, T_2 = h, T_3 = 2h, \cdots$.

Remark 1. In this paper, we only consider the case that all the intervals are selected in turn from the set $\{h, 2h\}$. For the set $\{l_1h, l_2h\}$ and $\{l_1h, l_2h, \cdots, l_mh\}$, similar analysis can be obtained.

Define $r(k) = [r_1^T(t_k), r_1^T(t_k), \cdots, r_n^T(t_k)]^T$. Then the network dynamics system is summarized as follows:

$$r(k+1) = \underbrace{(I_{3n} - T_k(L \otimes C))}_{\Phi_k} r(k), \quad (6)$$

where $\Phi_k \in \{\Phi_0, \Phi_1\}$,

$$\Phi_0 = I_{3n} - h(L \otimes C),$$

$$\Phi_1 = I_{3n} - 2h(L \otimes C).$$

Remark 2. System (6) is a periodic switched system with two subsystems matrices Φ_0, Φ_1. By Remark 2 in [12], system (1) achieves consensus if and only if system (6) reaches consensus.

From system (6), we can get

$$\begin{aligned}
&r(1) = \Phi_0 r(0),\\
&r(2) = \Phi_1 r(1) = (\Phi_1\Phi_0)r(0),\\
&r(4) = (\Phi_1\Phi_0)r(2),\\
&\vdots\\
&r(2k) = (\Phi_1\Phi_0)r(2(k-1)), k \in \ell_1.
\end{aligned}$$

Let $\xi(k) = r(2k)$, then

$$\xi(k+1) = (\Phi_1\Phi_0)\xi(k), k \in \ell_1. \quad (7)$$

Remark 3. By Assumption 1, we know that $t_0 = 0, t_1 = h, t_2 = 3h, t_3 = 4h, t_4 = 6h, t_5 = 7h, t_6 = 9h, \cdots$. Then, it is easily obtained that $t_{2k} = 3hk$. Hence,

$$\xi(k) = r(2k) = \begin{bmatrix} r_1(t_{2k}) \\ \vdots \\ r_n(t_{2k}) \end{bmatrix} = \begin{bmatrix} r_1(3hk) \\ \vdots \\ r_n(3hk) \end{bmatrix}.$$

Remark 4. Cartesian co-ordinate coupling has wide applications in engineering fields. Most existing literatures concerning this topic often assume that the sampling period is fixed. However, in real applications, the ideal assumption cannot be achieved. On the other hand, it is really difficult to deal with purely time-varying sampling issue and the results obtained are far from satisfaction [11, 12]. Therefore, we are inspired to investigate the average co-ordinate consensus via switching sampling interval.

Before moving on, we need the following lemmas.

2.3 Basic Lemmas

In this subsection, we introduce some lemmas, which will play an important role in the proof of our main theorems.

Lemma 1. *(Lemma 1.1 in [14]) The Laplacian matrix L has at least one zero eigenvalue and all the other eigenvalues are positive. The topology graph $\mathcal{G}$ is connected if and only if L has a simple zero eigenvalue and all the nonzero eigenvalues are positive. $\Phi_1\Phi_0$ has at least three 1 eigenvalues.*

Lemma 2. *(Lemma 1.20 in [14]) Given a rotation matrix $C \in \mathbb{R}^{3\times3}$, let $a = [a_1, a_2, a_3]^T$ and θ denote, respectively, the Euler axis(i.e., the unit vector in the direction of rotational) and the Euler angle(i.e., the rotational angle). The eigenvalues of the matrix C are $1, e^{\iota\theta}, e^{-\iota\theta}$ with associated right eigenvectors given by, respectively, $\zeta_1, \zeta_2, \zeta_3 = \bar{\zeta_2}$, and associated left eigenvectors given by, respectively, $\varpi_1 = \zeta_1, \varpi_2 = \bar{\zeta_2}, \varpi_3 = \bar{\zeta_3}$, where $\zeta_1 = a$, $\zeta_2 = [(a_2^2 + a_3^2)sin^2(\frac{\theta}{2}), -a_1a_2\sin^2(\frac{\theta}{2}) + \iota a_3\sin(\frac{\theta}{2})|\sin(\frac{\theta}{2})|, - a_1a_3\sin^2(\frac{\theta}{2}) - \iota a_2\sin(\frac{\theta}{2})|\sin(\frac{\theta}{2})|]^T$.*

Lemma 3. *([32]) Suppose that $U \in \mathbb{R}^{p\times p}$, $V \in \mathbb{R}^{q\times q}$, $X \in \mathbb{R}^{p\times p}$, and $Y \in \mathbb{R}^{q\times q}$. The following statements are true.*

(i)$(U + X) \otimes V = U \otimes V + X \otimes V$;
(ii)$(U \otimes V)(X \otimes Y) = (UX) \otimes (VY)$;
(iii)If U and V are symmetric, so is $U \otimes V$.

Moreover, suppose that U has the eigenvalues β_i with associated eigenvectors $f_i \in \mathbb{C}^p, i = 1, 2, \cdots, p$, *and V has eigenvalues ϱ_j with associated eigenvectors $g_j \in \mathbb{C}^q$, $j = 1, 2, \cdots, q$. Then the pq eigenvalues of $U \otimes V$ are $\beta_i \varrho_j$ with associated eigenvectors $f_i \otimes g_j$, $i = 1, 2, \cdots, p, j = 1, 2, \cdots, q$.*

Lemma 4. *System (1) under sampled control protocol (3) reaches consensus if and only if system (7) reaches consensus.*

Proof: Since the proof of Lemma 4 is obvious, we delete it for space saving. □

Remark 5. By Lemma 4, to solve the consensus problem of system (1), we only need analyze the consensus of system (7).

3 Consensus Analysis for Switched Sampled Control System

This section will establish two theorems to solve the consensus problem of switched multi-agent system via sampled control without delay.

Assumption 2. $3-2h\mu_i \neq 0$, where $\mu_i, i \in \ell$, denotes eigenvalue of Laplacian matrix L corresponding to the undirected graph.

Proposition 5. *The multiplicity of 1 eigenvalue in the matrix $\Phi_1\Phi_0$ is three times as many as that of zero eigenvalue in Laplacian matrix L if and only if Assumption 2 holds.*

Proof: It follows from Lemmas 2-3, that the eigenvalues of $L \otimes C$ are $\mu_i, \mu_i e^{\iota\theta}, \mu_i e^{-\iota\theta}, i \in \ell$,

$$\Phi_1\Phi_0 = I_{3n} - 3h(L \otimes C) + 2h^2(L \otimes C)^2.$$

Let $\lambda_j, j = 1, 2, \cdots, 3n$ denote the eigenvalue of $\Phi_1\Phi_0$. We can easily get

$$\begin{cases} \lambda_{3i-2} = 1 - 3h\mu_i + 2h^2{\mu_i}^2, \\ \lambda_{3i-1} = 1 - 3h\mu_i e^{\iota\theta} + 2h^2{\mu_i}^2 e^{\iota 2\theta}, \\ \lambda_{3i} = 1 - 3h\mu_i e^{-\iota\theta} + 2h^2{\mu_i}^2 e^{-\iota 2\theta}, i \in \ell. \end{cases}$$

By Lemma 1, we obtain that when $\mu_i = 0$ or $\mu_i = \frac{3}{2h}$, $\Phi_1\Phi_0$ has 1 eigenvalue.

(Sufficiency.) Since $\mu_i \neq \frac{3}{2h}$. One knows if once $\mu_i = 0$, then $\lambda_{3i-2} = \lambda_{3i-1} = \lambda_{3i} = 1$. So it is easy to prove that Laplacian matrix L has a zero eigenvalue with algebraic multiplicity m if and only if the matrix $\Phi_1\Phi_0$ has a 1 eigenvalue with algebraic multiplicity 3m.

(Necessity.) Now we prove the necessity by contradiction. If the condition $\mu_i \neq \frac{3}{2h}$ is not satisfied, there at least exists $l \in \{2, 3, \cdots, n\}$, such that $\mu_l = \mathbf{\frac{3}{2h}}$. Obviously, we have $\lambda_{3l-2} = 1$, which implies that the multiplicity of 1 eigenvalue of the matrix $\Phi_1\Phi_0$ is not triple as many as that of zero eigenvalue of the matrix L, which results in contradiction. □

Theorem 6. *Consider an undirected graph $\mathcal{G}$. The co-ordinated average consensus of system (1) under sampled control protocol (3) can be achieved if and only if the matrix $\Phi_1\Phi_0$ has exactly three 1 eigenvalues and all the other eigenvalues lie in the unit circle. In particular, $x_i(t) \to \frac{1}{n}\sum\limits_{i=1}^{n} x_i(0), y_i(t) \to \frac{1}{n}\sum\limits_{i=1}^{n} y_i(0), z_i(t) \to \frac{1}{n}\sum\limits_{i=1}^{n} z_i(0)$, as $t \to \infty$.*

Proof: The matrix $\Phi_1\Phi_0$ can be written in Jordan canonical form, i.e., $\Phi_1\Phi_0 = MJM^{-1}$, where J is the Jordan canonical form of $\Phi_1\Phi_0$ with $\lambda_l, l \in \ell_2$ as the diagonal element. $M := [m_1, m_2, \cdots, m_{3n}]$, where m_k is the right or generalized right eigenvector of the matrix $\Phi_1\Phi_0$. $M^{-1} := [p_1^T, p_2^T, \cdots, p_{3n}^T]^T$, where p_k is the left or generalized left eigenvector of the matrix $\Phi_1\Phi_0$. Moreover, $p_k^T m_k = 1, p_k^T m_l = 0, l \neq k, k \in \ell_2$.

(Sufficiency.) Since the matrix $\Phi_1\Phi_0$ has exactly three 1 eigenvalues and all the other eigenvalues lie in the unit circle, then Jordan canonical form J can be written as $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \bar{J} \end{bmatrix}$, where $\bar{J}$ is Jordan block corresponding to those eigenvalues of $\Phi_1\Phi_0$ which lie in the unit circle. i.e., $\lim\limits_{k\to+\infty} (\bar{J})^k = O_{(3n-3)\times(3n-3)}$. It follows from Lemma 2 and Lemma 4 that $\mathbf{1}_n \otimes \zeta_i$, $\frac{1}{n}\mathbf{1}_n \otimes \frac{\varpi_i}{\varpi_i^T \zeta_i}, i = 1, 2, 3$ are right and left eigenvectors associated with 1 eigenvalue of $\Phi_1\Phi_0$, respectively. So

$$M = \begin{bmatrix} \mathbf{1}_n \otimes \zeta_1, & \mathbf{1}_n \otimes \zeta_2, & \mathbf{1}_n \otimes \zeta_3, & * & \cdots & * \end{bmatrix},$$

$$M^{-1} = \begin{bmatrix} \frac{1}{n}\mathbf{1}_n^T \otimes \frac{\varpi_1^T}{\varpi_1^T \zeta_1} \\ \frac{1}{n}\mathbf{1}_n^T \otimes \frac{\varpi_2^T}{\varpi_2^T \zeta_2} \\ \frac{1}{n}\mathbf{1}_n^T \otimes \frac{\varpi_2^T}{\varpi_2^T \zeta_2} \\ * \\ \vdots \\ * \end{bmatrix}.$$

By (7), we get

$$\xi(k) = (\Phi_1\Phi_0)\xi(k-1) = \cdots = (\Phi_1\Phi_0)^k \xi(0). \quad (8)$$

Then,

$$\lim_{k\to+\infty}\xi(k) = M\begin{bmatrix}1&&&\\&1&&\\&&1&\\&&&\mathbf{0}\end{bmatrix}M^{-1}\xi(0) = (\frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T\otimes I_3)\xi(0).$$

Obviously,

$$\lim_{k\to+\infty}\|x_i(3hk)-\tfrac{1}{n}\mathbf{1}_n^T x(0)\|=0,$$
$$\lim_{k\to+\infty}\|y_i(3hk)-\tfrac{1}{n}\mathbf{1}_n^T y(0)\|=0,$$
$$\lim_{k\to+\infty}\|z_i(3hk)-\tfrac{1}{n}\mathbf{1}_n^T z(0)\|=0.$$

For any $t\geq 0$, there exists a nonnegative integer k, such that $t\in[t_k,t_{k+1})$. Obviously, $t\to+\infty$ is equivalent to $k\to+\infty$. Hence, when $t\to+\infty$, $\|x_i(t)-\frac{1}{n}\mathbf{1}_n^T x(0)\|\to 0, \|y_i(t)-\frac{1}{n}\mathbf{1}_n^T y(0)\|\to 0, \|z_i(t)-\frac{1}{n}\mathbf{1}_n^T z(0)\|\to 0$. Therefore, the co-ordinated average consensus of system (1) under sampled control protocol (3) is achieved.

(Necessity.) Now we prove the necessity by contradiction.

If the condition that matrix $\Phi_1\Phi_0$ has exactly three 1 eigenvalues and all the other eigenvalues are in the unit circle is not satisfied, then by Proposition 5, the multiplicity of 1 eigenvalue in $\Phi_1\Phi_0$ is at least three. Hence, there are three cases needed to be discussed.

Case *I:* $\Phi_1\Phi_0$ has exactly three 1 eigenvalues, and there exists at least an eigenvalue which is not in the unit circle;

Case *II:* $\Phi_1\Phi_0$ has more than three 1 eigenvalues , and all the other eigenvalues lie in the unit circle;

Case *III:* $\Phi_1\Phi_0$ has more than three 1 eigenvalues, and there exists at least an eigenvalue which is not in the unit circle.

For these three cases, we get $\lim_{k\to+\infty}(\Phi_1\Phi_0)^k$ doesn't exist or has a rank more than three. Thus, system (1) under protocol (3) can not reach consensus, which results in a contradiction. □

Theorem 7. *For an undirected graph $\mathcal{G}$, the co-ordinated average consensus of system (1) under sampled control protocol (3) can be achieved if and only if the graph $\mathcal{G}$ is connected and h, Euler angle θ satisfy the following conditions:*

$$\begin{array}{l}(1)\ 0<h<\min_{\mu\in\Lambda^+(L)}\{\frac{3}{2\mu}\},\\(2)\ \max_{\mu\in\Lambda^+(L)}\{-1,H_1\}<cos\theta<\min_{\mu\in\Lambda^+(L)}\{1,H_2\},\end{array}\tag{9}$$

where

$$H_1=\frac{3+6h\mu-\sqrt{4(h\mu)^4-4(h\mu)^2+9}}{8h\mu},$$

$$H_2=\frac{3+6h\mu+\sqrt{4(h\mu)^4-4(h\mu)^2+9}}{8h\mu},$$

$\Lambda^+(L)$ represents the set of the positive eigenvalues of Laplacian matrix L.

Proof: By Lemma 1 and Proposition 5, the topology graph is connected $\Leftrightarrow$ L has only one zero eigenvalue$\Leftrightarrow\Phi_1\Phi_0$ has only three 1 eigenvalues.There exists $h>0$ and Euler angle θ such that all the other eigenvalues of the matrix $\Phi_1\Phi_0$ except three 1 eigenvalues lie in the unit circle if and only if the following inequalities have a solution:

$$\begin{cases}|\lambda_{3i-2}|<1,\\|\lambda_{3i-1}|<1,\\|\lambda_{3i}|<1, i=2,3,\cdots,n,\end{cases}$$

where $\lambda_{3i-2},\lambda_{3i-1},\lambda_{3i}$ are defined as in the proof of Proposition 5, i.e.,

$$\begin{cases}|1-3h\mu_i+2h^2{\mu_i}^2|<1,\\|1-3h\mu_i e^{\iota\theta}+2h^2{\mu_i}^2e^{\iota 2\theta}|<1,\\|1-3h\mu_i e^{-\iota\theta}+2h^2{\mu_i}^2e^{-\iota 2\theta}|<1, i=2,3,\cdots,n,\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}-2<2h^2{\mu_i}^2-3h\mu_i<0,\\8h^2{\mu_i}^2(cos\theta)^2+(-6h\mu_i-12h^2\mu_i^2)cos\theta\\+3h^4\mu_i^4+5h^2\mu_i^2<0, i=2,3,\cdots,n,\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}0<h<\dfrac{3}{2\mu},\\\frac{3+6h\mu-\sqrt{4(h\mu)^4-4(h\mu)^2+9}}{8h\mu}<cos\theta,\\\frac{3+6h\mu+\sqrt{4(h\mu)^4-4(h\mu)^2+9}}{8h\mu}>cos\theta, i=2,3,\cdots,n.\end{cases}$$

Summarizing the above discussions, we can easily obtain (9). Then we complete the proof. □

4 Consensus under Switching Sampling Delay

If sampling induced time delay is concerned, the situation becomes more complicated. We assume that the sampling delay τ is fixed and less than the sampling period, i.e., $0<\tau<h$.

In this situation, the protocol becomes

$$u_i(t)=\begin{cases}\sum_{j\in N_i}a_{ij}C(r_j(t_{k-1})-r_i(t_{k-1})),\ t\in[t_k,t_k+\tau),\\\sum_{j\in N_i}a_{ij}C(r_j(t_k)-r_i(t_k)),\ t\in[t_k+\tau,t_{k+1}),\end{cases}\tag{10}$$

where C is a symmetric rotation matrix.

Similar to previous analysis, then we can obtain

$$r_i(t_{k+1}) = r_i(t_k)+\tau u_i(t_{k-1})+(T_k-\tau)u_i(t_k), \forall i \in \ell, k \in \ell_1. \quad (11)$$

Denote $\xi(k) =: [\xi_1^T(k), \xi_2^T(k), \cdots, \xi_n^T(k)]^T$, $\xi_i(k) =: [\xi_{i1}^T(k), \xi_{i2}^T(k), \xi_{i3}^T(k)]^T$, $\xi_{i1} =: [x_i(t_{k-1}), x_i(t_k)]^T, \xi_{i2} =: [y_i(t_{k-1}), y_i(t_k)]^T$, $\xi_{i3} =: [z_i(t_{k-1}), z_i(t_k)]^T, i \in \ell$, then the network dynamics is summarized as follows:

$$\xi(k+1) = \Psi(k)\xi(k), \quad k \in \ell_1, \quad (12)$$

where $\Psi(k) \in \{\Psi_0, \Psi_1\}$,

$\Psi_0 = I_{3n} \otimes A_0 + (L \otimes C) \otimes B_0, A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$,

$B_0 = \begin{bmatrix} 0 & 0 \\ -\tau & -(h-\tau) \end{bmatrix}$,

$\Psi_1 = I_{3n} \otimes A_0 + (L \otimes C) \otimes B_1$,

$B_1 = \begin{bmatrix} 0 & 0 \\ -\tau & -(2h-\tau) \end{bmatrix}$.

From system (12), we can easily get

$$\xi(2k) = (\Psi_1\Psi_0)\xi(2(k-1)), k \in \ell_1.$$

Let $\chi(k) = \xi(2k)$, then

$$\chi(k+1) = (\Psi_1\Psi_0)\chi(k), k \in \ell_1. \quad (13)$$

Remark 6. Similar to Remark 3, we get

$$\chi(k) = \xi(2k) = \begin{bmatrix} \xi_1(2k) \\ \vdots \\ \xi_n(2k) \end{bmatrix} = \begin{bmatrix} \xi_{11}^T(2k) \\ \xi_{12}^T(2k) \\ \xi_{13}^T(2k) \\ \vdots \\ \xi_{n1}^T(2k) \\ \xi_{n2}^T(2k) \\ \xi_{n3}^T(2k) \end{bmatrix},$$

where

$\xi_{i1}^T(2k) = \begin{bmatrix} x_i(t_{2k-1}) \\ x_i(3hk) \end{bmatrix}, \xi_{i2}^T(2k) = \begin{bmatrix} y_i(t_{2k-1}) \\ y_i(3hk) \end{bmatrix}$,

$\xi_{i3}^T(2k) = \begin{bmatrix} z_i(t_{2k-1}) \\ z_i(3hk) \end{bmatrix}, i \in \ell, k \in \ell_1$.

Assumption 3. $3 - (2h - 3\tau)\nu_i \neq 0$, where ν_i, $i = 1, 2, \cdots, 3n$, denotes the eigenvalue of matrix $L \otimes C$.

Proposition 8. *The multiplicity of 1 eigenvalue in the matrix $\Psi_1\Psi_0$ is three times as many as that of zero eigenvalue in Laplacian matrix L if and only if Assumption 3 holds.*

Proof: Denote $r(k) = [r_1^T(t_k), r_2^T(t_k), \cdots, r_n^T(t_k)]^T$, then system (12) can be rewritten as follows:

$$\begin{bmatrix} r(k) \\ r(k+1) \end{bmatrix} = \Gamma(k) \begin{bmatrix} r(k-1) \\ r(k) \end{bmatrix}, \quad (14)$$

where $\Gamma(k) \in \{\Gamma_0, \Gamma_1\}$,

$$\Gamma_0 = \begin{bmatrix} 0 & I_{3n} \\ -\tau(L \otimes C)L & I_{3n} - (h-\tau)(L \otimes C) \end{bmatrix},$$

$$\Gamma_1 = \begin{bmatrix} 0 & I_{3n} \\ -\tau(L \otimes C)L & I_{3n} - (2h-\tau)(L \otimes C) \end{bmatrix}.$$

It's easy to verify that $E_p^{-1}(\Psi_1\Psi_0)E_p = \Gamma_1\Gamma_0$, i.e., $\Gamma_1\Gamma_0 \sim \Psi_1\Psi_0$. $E_p \in \mathbb{R}^{6n\times 6n}$ is a permutation matrix. Let s denote eigenvalue of matrix $\Gamma_1\Gamma_0$, then we get

$$\begin{aligned} & det(sI_{6n} - \Gamma_1\Gamma_0) \\ = \; & det[s^2 I_{3n} + (3h\hat{L} - h_1 h_2 \hat{L}^2 - I_{3n})s + \tau^2\hat{L}^2], \end{aligned}$$

i.e.,

$$\begin{aligned} & det(sI_{6n} - \Gamma_1\Gamma_0) \\ = \; & \prod_{i=1}^{3n}[s^2 + (3h\nu_i - h_1 h_2 \nu_i^2 - 1)s + \tau^2\nu_i^2], \end{aligned}$$

where $h_1 = h-\tau, h_2 = 2h-\tau, \hat{L} = L\otimes C, \nu_i, i \in \ell_2$ is defined as in Assumption 3.

It's easy to obtain that when $\nu_i = 0$ or $\nu_i = \frac{3}{2h-3\tau}$, $\Psi_1\Psi_0$ has 1 eigenvalue.Without loss of generality, by Lemma 3, suppose $\nu_{3j-2} = \mu_j, \nu_{3j-1} = \mu_j e^{\iota\theta}, \nu_{3j} = \mu_j e^{-\iota\theta}, \mu_j \in \Lambda(L), j \in \ell$.

(Sufficiency.) Since $\nu_i \neq \frac{3}{2h-3\tau}$, we have $\nu_1 = \nu_2 = \nu_3 = 0$, when $\mu_1 = 0$, which obviously implies the matrix $\Psi_1\Psi_0$ has a 1 eigenvalue with algebraic multiplicity 3 by solving the above characteristic polynomial of $\Psi_1\Psi_0$. Therefore, the multiplicity of 1 eigenvalue in the matrix $\Psi_1\Psi_0$ is triple as many as that of zero eigenvalue in Laplacian matrix L.

(Necessity.) Since the proof is similar to that of the necessity in Proposition 5, we omit it here for space saving. □

4.1 Consensus Analysis for Switched Sampled Control System

In this subsection, we mainly establish two theorems to solve the consensus problem of switched systems via sampled-data control with time delay.

Theorem 9. *Consider an undirected graph $\mathcal{G}$. The co-ordinated average consensus of system (1) under sampled control protocol (10) can be achieved if and only if the matrix $\Psi_1\Psi_0$ has exactly three 1 eigenvalues and all the other eigenvalues lie in the unit circle. In particular, $x_i(t) \to \frac{1}{n}\sum_{i=1}^{n} x_i(0), y_i(t) \to \frac{1}{n}\sum_{i=1}^{n} y_i(0), z_i(t) \to \frac{1}{n}\sum_{i=1}^{n} z_i(0)$, as $t \to \infty$.*

Proof: (Sufficiency.) By Proposition 8, since the matrix $\Psi_1\Psi_0$ has exactly three 1 eigenvalues, then the matrix L has a simple 0 eigenvalue. Suppose the eigenvalues of L, $L \otimes C$ be $0, \mu_2, \cdots, \mu_n$, and $0, 0, 0, \nu_4, \cdots, \nu_{3n}$, respectively. Since the topology graph is an undirected graph and C is a symmetric matrix, by Lemma 3, one knows that $L \otimes C$ is a symmetric matrix and can be diagonalizable. i.e., there exists a non-singular matrix M such that

$$M^{-1}(L \otimes C)M = diag\{0, 0, 0, \nu_4, \cdots, \nu_{3n}\}.$$

Hence,

$$\begin{aligned} & (M^{-1} \otimes I_2)\Psi_0(M \otimes I_2) \\ = \; & \hat{M}^{-1}\left[I_{3n} \otimes A_0 + (L \otimes C) \otimes B_0\right]\hat{M} \\ = \; & I_{3n} \otimes A_0 + diag\{0, 0, 0, \nu_4, \cdots, \nu_{3n}\} \otimes B_0 \\ = \; & diag\{A_0, A_0, A_0, Q_4, \cdots, Q_{3n}\} \\ \triangleq \; & \Lambda_0. \end{aligned}$$

where

$$\hat{M} = M \otimes I_2, \hat{M}^{-1} = M^{-1} \otimes I_2.$$

Similarly,

$$\begin{aligned} & (M^{-1} \otimes I_2)\Psi_1(M \otimes I_2) \\ = \; & diag\{A_0, A_0, A_0, D_4, \cdots, D_{3n}\} \\ \triangleq \; & \Lambda_1, \end{aligned}$$

where

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, B_0 = \begin{bmatrix} 0 & 0 \\ -\tau & -(h-\tau) \end{bmatrix},$$

$$Q_i = \begin{bmatrix} 0 & 1 \\ -\tau\nu_i & 1-(h-\tau)\nu_i h \end{bmatrix}, i = 4, \cdots, 3n.$$

$$D_i = \begin{bmatrix} 0 & 1 \\ -\tau\nu_i & 1-(2h-\tau)\nu_i h \end{bmatrix}, i = 4, \cdots, 3n.$$

$$\begin{aligned} & \Psi_1\Psi_0 \\ = \; & \hat{M}\Lambda_1\hat{M}^{-1}\hat{M}\Lambda_0\hat{M}^{-1} \\ = \; & \hat{M}(\Lambda_1\Lambda_0)\hat{M}^{-1} \\ = \; & \hat{M} diag\{A_0, A_0, A_0, D_4Q_4, \cdots, D_{3n}Q_{3n}\}\hat{M}^{-1}, \end{aligned}$$

Hence, $\Psi_1\Psi_0$ and $diag\{A_0, A_0, A_0, D_4Q_4, \cdots, D_{3n}Q_{3n}\}$ have the same eigenvalues due to their similarities. From (13), we get

$$\chi(k) = (\Psi_1\Psi_0)\chi(k-1) = \cdots = (\Psi_1\Psi_0)^k\chi(0). \quad (15)$$

Furthermore, by Lemma 3, we have

$$\begin{aligned} & (\Psi_1\Psi_0)^k \\ = \; & \hat{M} \underbrace{diag\{A_0, A_0, A_0, (D_4Q_4)^k, \cdots, (D_{3n}Q_{3n})^k\}}_{\Xi} \hat{M}^{-1}. \end{aligned}$$

Since $\Psi_1\Psi_0$ has exactly three eigenvalues and all the other eigenvalues lie in the unit circle, then all the eigenvalues of $diag\{A_0, A_0, A_0, D_4Q_4, \cdots, D_{3n}Q_{3n}\}$ have the same characters. Hence, the eigenvalues of $D_iQ_i, i = 4, 5, \cdots, 3n$, lie in the unit circle. Furthermore,

$$\lim_{k \to +\infty}(D_iQ_i)^k = \mathbf{O}_{2\times 2}, i = 4, 5, \cdots, 3n.$$

Then, we can obtain

$$\begin{aligned} & \lim_{k \to +\infty}\chi(k) \\ = \; & \lim_{k \to +\infty}(\Psi_1\Psi_0)^k\chi(0) \\ = \; & \lim_{k \to +\infty}(M \otimes I_2)\Xi(M^{-1} \otimes I_2)\chi(0) \\ = \; & (M \otimes I_2)\Xi_0(M^{-1} \otimes I_2)\chi(0) \\ = \; & [\mathbf{1}_n(\frac{1}{n}\mathbf{1}_n)^T \otimes I_3] \otimes \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\chi(0), \end{aligned}$$

where M, M^{-1} are defined as in Theorem 6,

$$\Xi_0 = diag\{A_0, A_0, A_0, O_{2\times 2}, \cdots, O_{2\times 2}\},$$

$$\chi(0) = \begin{bmatrix} \xi_{11}(0) \\ \xi_{12}(0) \\ \xi_{13}(0) \\ \vdots \\ \xi_{n1}(0) \\ \xi_{n2}(0) \\ \xi_{n3}(0) \end{bmatrix},$$

$\xi_{i1}(0) = \begin{bmatrix} x_i(t_{-1}), x_i(0) \end{bmatrix}^T, \xi_{i2}(0) = \begin{bmatrix} y_i(t_{-1}), y_i(0) \end{bmatrix}^T,$

$\xi_{i3}(0) = \begin{bmatrix} z_i(t_{-1}), z_i(0) \end{bmatrix}^T.$

It is easy to get $\|x_i(3hk) - \frac{1}{n}\sum_{i=1}^{n} x_i(0)\| \to 0, \|x_i(t_{2k-1}) - \frac{1}{n}\sum_{i=1}^{n} x_i(0)\| \to 0, \|y_i(3hk) - \frac{1}{n}\sum_{i=1}^{n} y_i(0)\| \to 0, \|y_i(t_{2k-1}) - \frac{1}{n}\sum_{i=1}^{n} y_i(0)\| \to 0, \|z_i(3hk) - \frac{1}{n}\sum_{i=1}^{n} z_i(0)\| \to 0, \|z_i(t_{2k-1}) - \frac{1}{n}\sum_{i=1}^{n} z_i(0)\| \to 0$, as $k \to +\infty$. For any $t \geq 0$, there exists a non-negative integer k such that $t \in [t_k, t_{k+1})$. Obviously, $t \to +\infty$ is equivalent to $k \to +\infty$. Therefore, $\|x_i(t) - \frac{1}{n}\mathbf{1}_n^T x(0)\| \to 0, \|y_i(t) -$

$\frac{1}{n}\mathbf{1}_n^T y(0)\| \to 0, \|z_i(t) - \frac{1}{n}\mathbf{1}_n^T z(0)\| \to 0$, as $t \to +\infty$. Hence, the average-consensus of system (1) under sampled delay control protocol (10) is achieved.

(Necessity.) Now, we prove the necessity by contradiction. If the condition that matrix $\Psi_1\Psi_0$ has exactly 1 eigenvalue with multiplicity of three and all the other eigenvalues are in the unit circle is not satisfied, then by Proposition 8, the multiplicity of 1 eigenvalue in $\Psi_1\Psi_0$ is at least three. Hence, there are three cases needed to be discussed.

Case *I:* $\Psi_1\Psi_0$ has exactly three 1 eigenvalues, and there exists at least an eigenvalue which is not in the unit circle;

Case *II:* $\Psi_1\Psi_0$ has more than three 1 eigenvalues, , and all the other eigenvalues lie in the unit circle;

Case *III:* $\Psi_1\Psi_0$ has more than three 1 eigenvalues, and there exists at least an eigenvalue which is not in the unit circle.

In either case, $\lim_{k\to+\infty}(D_iQ_i)^k \neq \mathbf{O}_{2\times 2}, i = 4, 5, \cdots, 3n$. We get $\lim_{k\to+\infty}(\Psi_1\Psi_0)^k$ doesn't exist or has a rank more than three. Thus, system (1) under protocol (10) can not reach consensus, which results in contradiction. □

It's difficult to obtain the solutions of sampling period h, Euler angle θ and sampling delay τ for the general rotation matrix C. To give a specific range of sampling period h and sampling delay τ, we will investigate a special case of the rotation matrix, i.e., $C = I_3$ in the sequel.

Theorem 10. *For an undirected graph $\mathcal{G}$, the co-ordinated average consensus of system (1) under sampling delay control protocol (10) can be achieved if and only if the graph $\mathcal{G}$ is connected and h, sampling delay τ should satisfy the following conditions:*

$$\begin{array}{l}(1)\ 2h - 3\tau < \min\limits_{\mu\in\Lambda^+(L)} \dfrac{3}{\mu},\\ (2)\ If \quad 0 < h < \min\limits_{\mu\in\Lambda^+(L)} \dfrac{12-4\sqrt{2}}{7\mu}, or\\ \qquad h > \max\limits_{\mu\in\Lambda^+(L)} \dfrac{12+4\sqrt{2}}{7\mu},\\ \quad then\ 0 < \tau < \min\limits_{\mu\in\Lambda^+(L)}\{h, \frac{1}{\mu}\},\\ \quad If \max\limits_{\mu\in\Lambda^+(L)} \dfrac{12-4\sqrt{2}}{7\mu} < h < \min\limits_{\mu\in\Lambda^+(L)} \dfrac{12+4\sqrt{2}}{7\mu},\\ \quad then\ 0 < \tau < \min\limits_{\mu\in\Lambda^+(L)}\{H_3, \frac{1}{\mu}\},\end{array} \tag{16}$$

where

$$H_3 = \frac{3h\mu - \sqrt{-7(h\mu)^2 + 24(h\mu) - 16}}{4\mu}.$$

Proof: (Necessity.) By Theorem 9, if the co-ordinated average-consensus of system (1) under sampling delay control protocol (10) can be achieved, one knows that $\Psi_1\Psi_0$ has exactly three 1 eigenvalues and all the other eigenvalues lie in the unit circle. Moreover, by Proposition 8, L has a simple 0 eigenvalue. Then, the graph $\mathcal{G}$ is connected by Lemma 1. Since $\Psi_1\Psi_0$ and $diag\{A_0, A_0, A_0, D_4Q_4, \cdots, D_{3n}Q_{3n}\}$ have the same eigenvalues, the eigenvalues of $\Psi_1\Psi_0$ can be obtained by solving the equation

$$det\begin{bmatrix} s & -1 \\ 0 & s-1 \end{bmatrix}^3 \prod_{i=4}^{3n} a_i(s) = 0,$$

where

$$\begin{aligned} & a_i(s) \\ = \ & det(sI_2 - D_iQ_i) \\ = \ & det\begin{bmatrix} s+\tau\nu_i & -1+h_1\nu_i \\ \tau\nu_i - \tau h_2\nu_i^2 & s-1+h_3\nu_i - h_1h_2\nu_i^2 \end{bmatrix}, \\ & i = 4, \cdots, 3n, \end{aligned}$$

where $h_1 = h - \tau, h_2 = 2h - \tau, h_3 = 3h - \tau$.

It is obvious that $\Psi_1\Psi_0$ has three 1 eigenvalues and three 0 eigenvalues. Noting that these three 0 eigenvalues lie in the unit circle without explanations. The other eigenvalues can be obtained by solving $a_i(s) = 0, i = 4, 5, \cdots, 3n$. Noticing that they have the same form, we can analyze them uniformly. Since $\nu_{3j-2} = \nu_{3j-1} = \nu_{3j} = \mu_j, j = 2, \cdots, n$, then we only need to determine the Schur stability of the following polynomial:

$$\begin{aligned} & a(s) \\ = \ & det\begin{bmatrix} s+\tau\nu & -1+h_1\nu \\ \tau\nu - \tau h_2\nu^2 & s-1+h_3\nu - h_1h_2\nu^2 \end{bmatrix} \\ = \ & s^2 + (3h\nu - h_1h_2\nu^2 - 1)s + \tau^2\nu^2 \\ = \ & s^2 + (3h\mu - h_1h_2\mu^2 - 1)s + \tau^2\mu^2, \end{aligned}$$

where $h_1 = h - \tau, h_2 = 2h - \tau$,$\mu \in \Lambda^+(L)$ denotes the positive eigenvalue of Laplacian matrix L.

By Jury criterion [33], a(s) is Schur stable if and only if the following conditions hold:
(a) $a(1) > 0$;
(b) $a(-1) > 0$;
(c) $\tau^2\mu^2 < 1$.
Then,

$$\begin{aligned} a(1) &= 1 + 3h\mu - 2h^2\mu^2 + 3h\tau\mu^2 - \tau^2\mu^2 - 1 + \tau^2\mu^2 \\ &= 3h\mu - 2h^2\mu^2 + 3h\tau\mu^2 \\ &= h\mu(3 - 2h\mu + 3\tau\mu). \end{aligned}$$

From (a), we have

$$a(1) > 0 \ \Leftrightarrow \ 3 - 2h\mu + 3\tau\mu > 0 \tag{17}$$

$$\Leftrightarrow \ 2h - 3\tau < \frac{3}{\mu}. \tag{18}$$

$$\begin{aligned} a(-1) &= 1 - 3h\mu + 2h^2\mu^2 - 3h\tau\mu^2 + \tau^2\mu^2 + 1 + \tau^2\mu^2 \\ &= 2\tau^2\mu^2 - 3h\mu^2\tau + 2h^2\mu^2 - 3h\mu + 2. \end{aligned}$$

From (b), we have

$$a(-1) > 0 \Leftrightarrow 2\tau^2\mu^2 - 3h\mu^2\tau + 2h^2\mu^2 - 3h\mu + 2 > 0. \quad (19)$$

For the inequality (19), if

$$\Delta := -\mu^2(7h^2\mu^2 - 24h\mu + 16) > 0,$$

we have

$$\frac{12 - 4\sqrt{2}}{7\mu} < h < \frac{12 + 4\sqrt{2}}{7\mu}.$$

Then, we can easily obtain

$$\tau < \frac{3h\mu - \sqrt{-7h^2\mu^2 + 24h\mu - 16}}{4\mu}.$$

If

$$\Delta = -\mu^2(7h^2\mu^2 - 24h\mu + 16) < 0,$$

we get

$$h < \frac{12 - 4\sqrt{2}}{7\mu}, or\ h > \frac{12 + 4\sqrt{2}}{7\mu},$$

which implies that (19) holds for any $0 < \tau < h$.
From (c), we have

$$\tau^2\mu^2 < 1 \Leftrightarrow 0 < \tau < \frac{1}{\mu}. \quad (20)$$

Therefore, summarizing the above discussions, $a_i(s), i = 4, 5, \cdots, 3n$, is Schur stable if and only if the following conditions hold:

$$\begin{cases} 2h - 3\tau < \min\limits_{\mu\in\Lambda^+(L)}\{\frac{3}{\mu}\}, \\ If\ 0 < h < \min\limits_{\mu\in\Lambda^+(L)} \frac{12 - 4\sqrt{2}}{7\mu}, or\ h > \max\limits_{\mu\in\Lambda^+(L)} \frac{12 + 4\sqrt{2}}{7\mu}, \\ \quad then\ 0 < \tau < \min\limits_{\mu\in\Lambda^+(L)}\{h, \frac{1}{\mu}\}, \\ If\ \max\limits_{\mu\in\Lambda^+(L)} \frac{12 - 4\sqrt{2}}{7\mu} < h < \min\limits_{\mu\in\Lambda^+(L)} \frac{12 + 4\sqrt{2}}{7\mu}, \\ \quad then\ 0 < \tau < \min\limits_{\mu\in\Lambda^+(L)}\{H_3, \frac{1}{\mu}\}, \end{cases}$$

where

$$H_3 = \frac{3h\mu - \sqrt{-7(h\mu)^2 + 24(h\mu) - 16}}{4\mu}.$$

(Sufficiency.) Because the graph is connected, 0 is a simple eigenvalue of L. Then, by Proposition 8, $\Psi_1\Psi_0$ has exactly three 1 eigenvalues. From the proof of necessity, one obtains that if h, τ satisfy (16), then $a_i(s), i = 4, \cdots, 3n$, is Schur stable, i.e., all the other eigenvalues of $\Psi_1\Psi_0$ lie in the unit circle except three 1 eigenvalues. Hence, by Theorem 10, the average-consensus of system (1) under sampling delay control protocol (10) can be achieved. □

5 Numerical Simulations

In this section, an example is given to illustrate our theoretical results.

The topology graph in our example has 0-1 weight. Consider the system (1) in 2-D with 4 agents (see Figure 1 below), where $r_i(t) = [x_i(t), y_i(t)]^T \in \mathbb{R}^2, i = 1, 2, 3, 4$.

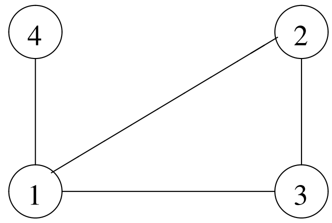

Figure 1. The topology graph of agents.

The four eigenvalues of the corresponding Laplacian matrix

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix},$$

are $\mu_1 = 0, \mu_2 = 1, \mu_3 = 3, \mu_4 = 4$. $C \in R^{2\times 2}$ is given by

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

By Figure 1 and Lemma 1, we can see that the topology graph is connected.

Case I. Network without sampling delay.

By Theorem 7, we can easily get that $0 < h < 0.375, 0 < \theta < \arccos 0.7087$ can guarantee the consensus of system (1) under the sampled control protocol (3). By choosing h=0.3, $\theta = \frac{\pi}{6}$, Figure 2 shows that the system (1) can achieve consensus with the initial values $r_1(0) = [1, 2]^T, r_2(0) = [1, 0]^T, r_3(0) = [1, 1]^T, r_4(0) = [0, 1]^T$. Whereas, Figure 3 shows that system (1) under the sampled control protocol (3) cannot reach consensus with the same initial values by choosing $h = 0.3, \theta = \frac{\pi}{4}$.

Case II. Network with sampling delay.

For simplicity, let $C = I_2$. By Theorem 10, system (1) under the sampling delay control protocol (10) can achieve consensus with $0 < h < 0.2265, 0 < \tau < h$ or $0.3020 < h < 0.6306$, then $0 < \tau < 0.2728$. By choosing $h = 0.6, \tau = 0.2$, Figure 4 shows that system (1) can reach consensus with initial values $r_1(0) = [-1, -1]^T$, $r_2(0) = [0, -1]^T, r_3(0) = [0, -1]^T, r_4(0) = [-1, -2]^T$. However, Figure 5 shows that system (1) cannot achieve consensus by choosing $h = 0.6, \tau = 0.3$ with the same initial values.

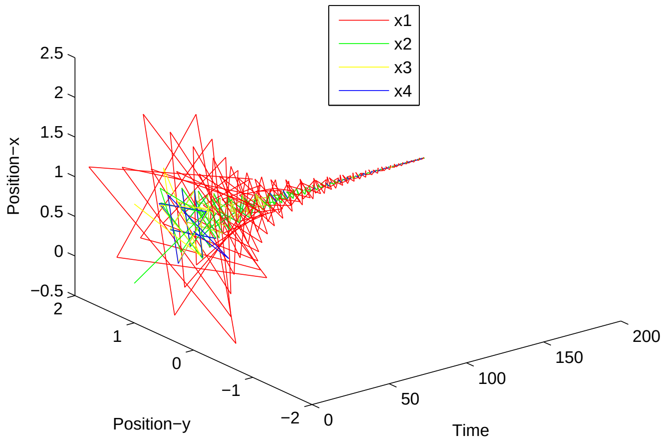

Figure 2. State trajectories of the four agents with $h = 0.3, \theta = \frac{\pi}{6}$

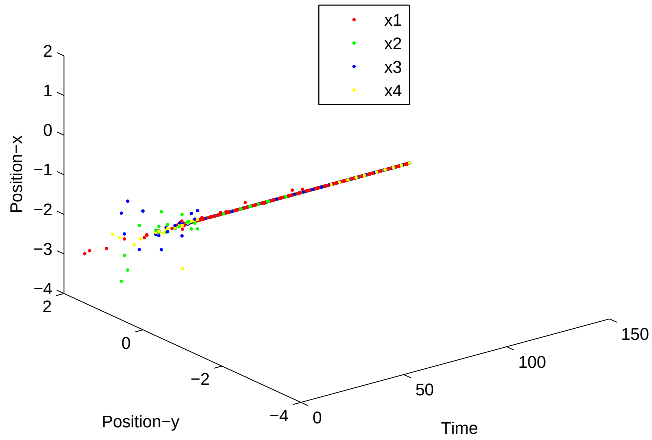

Figure 4. State trajectories of the four agents with $h = 0.6, \tau = 0.2$

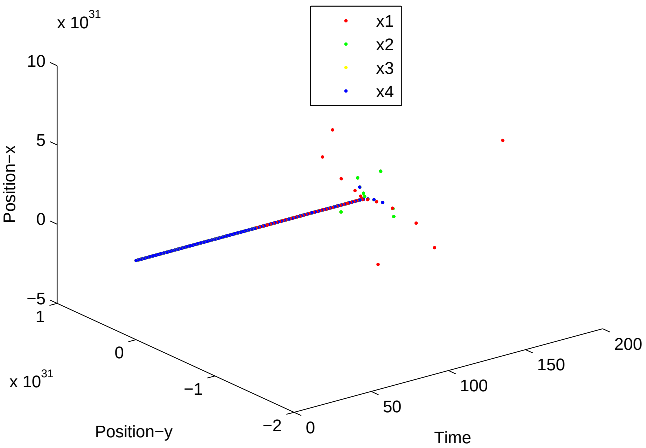

Figure 3. State trajectories of the four agents with $h = 0.3, \theta = \frac{\pi}{4}$

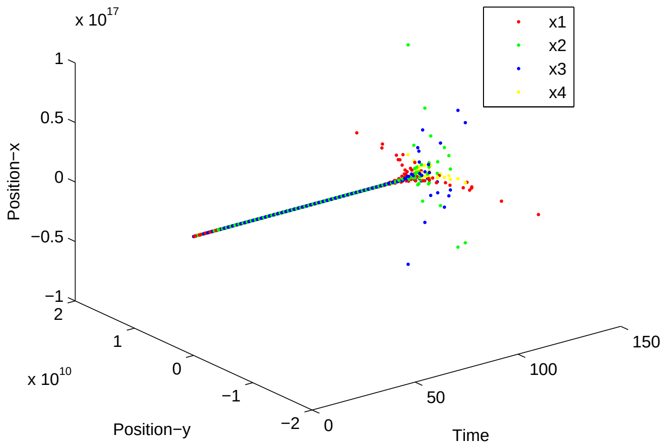

Figure 5. State trajectories of the four agents with $h = 0.6, \tau = 0.3$

6 Conclusion

In this paper, the co-ordinated average-consensus of single-integrator multi-agent coupling sampled control system with switching sampling interval has been investigated. By the iterative product method, the continuous-time multi-agent system can be equivalently transformed into a linear discrete-time system. Through analyzing the eigenvalues of system matrix, sufficient and necessary conditions for reaching co-ordinated average-consensus under undirected fixed topology are established. Then, we generalize our results to the case of discrete-time system with sampling delay. Finally, an example is provided to illustrate the effectiveness of the theoretical results. Our future work will focus on studying the consensus of the double-order multi-agent dynamic system under Cartesian co-ordinated coupling sampled control protocol, switching topology and sampling delay, respectively.

Acknowledgements: This work is supported by Natural Science Foundation of Tianjin (No.11JCYBJC06400).

References:

[1] R. Olfati-Saber and R.M. Murray, Consensus Problems in Networks of Agents with Switching Topology and Time-delays. *IEEE Transactions on Automatic Control*, 2004, 49(9): 1520–1533.

[2] M. A. Khawaldah, S. Livatino, L. Meng, Exploration with Two Cooperating Mobile Robots, WSEAS Transactions on Systems and Control, ISSN:1991-8763, Vol.5, No.5, 2010, pp.243-252.

[3] W. Ren and R.W. Beard, Consensus Seeking in Multi-agent Systems under Dynamically Changing Interaction Topologies. *IEEE Transactions on Automatic Control*, 2005, 50(5): 655–661.

[4] W. Ren and E. Atkins, Distributed Multi-vehicle Coordinated Control via Local Information Exchange. *International Journal of Robust and Nonlinear Control*, 2007, 17(10-11): 1002–1033.

[5] W. Ren, G. Chen and M. Cao, Some Necessary and Sufficient Conditions for Second-order Consensus in Multi-agent Dynamical Systems. *Automatica*, 2010, 46: 1089–1095.

[6] P. Lin and Y. Jia, Further Results on Decentralized Coordination in Networks of Agents with Second-order Dynamics. *IET Control Theory and Applications*, 2009, 3(7): 957–970.

[7] Wang L., Xiao F., A New Approach to Consensus Problems in Discretetime Multi-agent Systems with Time-delays. *Science in China Series: FInformation Science*, 2007, 50(4): 625–635.

[8] Y. Gao and L. Wang, Consensus of Multiple Second-order Agents Without Velocity Measurements. *Proceedings of the 2009 American Control Conference*, 2009, 4464–4469.

[9] W. Yu, G. Chen and M. Cao, Some Necessary and Sufficient Conditions for Second-order Consensus in Multi-agent Dynamical Systems. *Automatica*, 2010, 46: 1089–1095.

[10] L. Moreau, Stability of Multi-agent Systems With Time-dependent Communication Links. *IEEE Transactions on Automatic Control*, 2005, 50(2): 169–182.

[11] Y. Gao and L. Wang, Consensus of Multiple Dynamic Agents with Sampled Information. *IET Control Theory and Applications*, 2010, 4(6): 945–956.

[12] Y. Gao and L. Wang, Consensus of Multiple Double-integrator Agents with Intermittent Measurement.Int. J. Robust Nonlinear Control 20(10), 1140–1155. (2009).

[13] R. W. Beard and V. Stepanyan, Synchronization of Information in Distributed Multiple Vehicle Coordination Control, *IEEE Conf. Decision and Control,* Maui, 2003, 12: 2029–2034.

[14] W. Ren and Y. Cao, Distributed Coordnation of Multi-agent Networks: Emergent Problems, Models and Issues, Springer, 2011.

[15] W. Ren, Collective Motion from consensus with Cartesian Coordinate Coupling -Part I: Single-integrator Kinematics, 47th IEEE Conf. Decision and Control, 2008: 1006–1011.

[16] W. Ren, Collective Motion from Consensus with Cartesian Coordinate Coupling, IEEE Trans. Autom. Control, 54(6): 1330-1335, 2009.

[17] W. Ren, Consensus Strategies for Cooperative Control of Vehicle Formations. *IET Control Theory and Applications*, 2007, 1(2): 505–512.

[18] G. Xie, H. Liu and L. Wang, Jia Y. Consensus in Networked Multi-agent Systems via Sampled Control: Fixed Topology Case. *2009 American Control Conference*, 2009: 3902–3907.

[19] G. Xie, H. Liu, L. Wang and Y. Jia, Consensus in Networked Multi-agent Systems via Sampled Control: Switching Topology Case. *2009 American Control Conference,* 2009:4525–4530.

[20] Y. Cao and W. Ren, Sampled-data Formation Control under Dynamic Directed Interaction. In:Proceedings of the American Control Conference, pp. 5186–5191, St. Louis, MO, June 2009.

[21] P. Lin, Y. Jia and J. Du, Distributed Consensus Control for Second-order Agents with Fixed Topology and Time-delay. *Proc of the 26th Chinese Control Conf, Zhangjiajie, China*, 2007, 577–581.

[22] J. Hu and Y. Hong, Leader-following Coordination of Multi-agent Systems with Coupling Time Delays. *Physica A*, 2007, 374(2): 853–863.

[23] H. Liu, G. Xie and L. Wang, Necessary and Sufficient Conditions for Solving Consensus Problems of Double-integrator Dynamics via Sampled Control. *International Journal of Robust and Nonlinear Control*, 2010, 20: 1706–1722.

[24] Y. Cao, D. Stuart and W. Ren, Co-ordianted Collective Motion Patterns in a Discrete-time Setting with Experiments.*IET Control Theory Appl.*, 2010, Vol. 4, lss. 11, pp. 2579–2591.

[25] SK. Wang and DM. Xie, Consensus of Second-order Multi-agent Systems via Sampled Control: undirected fixed topology case.*IET Control Theory Appl.*, 2012, Vol. 6, Iss. 7, pp. 893–899.

[26] Y. Cao and W. Ren, Sampled-data Discrete-time Coordination Algorithms for Double-integrator Dynamics under Dynamic Directed Interaction. Int. J. Control 83(3), 506–515 (2010).

[27] T. S. Tsay, Automatic Regulation Time Series for Sampled-data Feedback Control Systems with Limited Control Inputs, WSEAS Transactions on Systems and Control, ISSN:1991-8763, Vol.5,No.5, 2010, pp.259-268.

[28] Y. Gao, L. Wang, G. Xie and B. Wu, Consensus of Multi-agent Systems Based on Sampled-data Control. Int. J. Control 82(12), 2193–2205 (2009).

[29] Y. Cao and W. Ren, Multi-vehicle Coordination for Double-integrator Dynamics in a Sampled Data Setting. Int. J. Robust Nonlinear Control 20(9), 987–1000 (2010).

[30] Y. Gao and L. Wang, Sampled-data Based Consensus of Continuous-time Multi-agent Systems with Time-varying Topology. *IEEE Transactions on Automatic Control*, 2011, 56(5): 1226–1231.

[31] P. Lin and Y. Jia, Average Consensus in Networks of Multiagents with Both Switching Topology and Coupling Time Delay. *Physica A*, 2008, 387(1): 303–313.

[32] A. Graham, Kronecker Products and Matrix Calculus With Applications. Halsted, New York (1981).

[33] EI. Jury, A Note on the Modified Stability Table for Linear Discrete Time Systems. IEEE Transactions on circuits and systems, 1991, 38(2): 221–223.