The optimal control problem with terminal condition and random intervention times

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Abstract: This paper is devoted to an impulse control problem where the control can only be exerted at the arrival times of the Poisson process N. We generalized control state process and cost function, under some assumptions, we obtained more general results. At the same time, we got the optimal control in special case.

Key–Words: Stochastic control, Random intervention times, Value function, Generalized Itô formula

1 Introduction

The impulse optimal control problem is an important research area in recent years. Baccarin [1] discussed the optimal control of a multidimensional cash management system where the cash balances fluctuated as a homogeneous diffusion process in R^n. They formulated the model as an impulse control problem on an unbounded domain with unbounded cost functions. Under general assumptions they characterized the value function as a weak solution of a quasi-variational inequality in a weighted Sobolev space and they showed the existence of an optimal policy. Meng and Siu [3] investigated an optimal reinsurance and dividend problem of an insurance company with the presence of reinvestments, or retained earnings, they considered the general situation that the company needed to pay both fixed and proportional costs as mixed classical and impulse control problems to get the value function and the optimal strategy. Yao, Yang and Wang [7] considered the dividend payments and capital injections control problem in a dual risk model. This leaded to an impulse control problem. Using the techniques of quasi-variational inequalities (QVI), this optimal control problem was solved. Numerical solutions were provided to illustrate the idea and methodologies, and some interesting economic insights were included.

The stochastic control problems with random intervention times were originally put forward by Rogers and Zane in 1998 [4]. They discussed a simple model of liquidity effects by the complexity of Log normal controlled state, there was no closed-form solution for corresponding cost problem, but they established certain qualitative features of the solution. Wang [5] simplified the controlled state, got explicit solutions about a quadratic deviation cost and a proportional control cost both discounted problem and the ergodic problem. Yao etc [7] introduced a diffusion in controlled state based on Wang's, discussed optimal control policies and value functions. The main feature of control problems with random intervention times is that the control is discrete, and control can only be exerted at the arrival times of the Poisson process N. This paper generalized state process and cost function, under some assumptions, we obtained more general results.

2 Mathematical Model

Consider a completed probability space (Ω, $\mathcal{F}$, P) with a filtration $\{ \mathcal{F}_t\}_{t\geq 0}$ generated by a one-dimensional Brownian motion W=$\{W_t : t \geq 0\}$, N=$\{N_t : t \geq 0\}$ is an $\mathcal{F}_t$-adapted Poisson process with intensity $\lambda > 0$, $\theta = \{\theta_t : t \geq 0\}$ is a non-negative $\mathcal{F}_t$ progressive measurable process. $\xi_t = \int_{[0,t)} \theta_s dN_s$ is a non-negative left continuous process, controlled state process evolves as follows

$$\begin{aligned} \mathrm{d}X_t &= \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t - d\xi_t \\ X_0 &= x \end{aligned}$$

where $\mu(\cdot)$, $\sigma(\cdot)$ satisfy conditions
for any $n \in \mathbf{N}$, there exists $K_n > 0$ such that

$$1)(|\mu(x)-\mu(y)|+|\sigma(x)-\sigma(y)|)I_{\{|x|\leq n,|y|\leq n\}} \leq K_n|x-y|$$

$$2)|\mu(x)| + |\sigma(x)| \leq C(1 + |x|)$$

Definition 1 *A set of admissible control π is that*

$$\pi = \{\xi_t : t \geq 0, X_t \geq 0\}$$

Definition 2 *A stopping time τ_0 is as follows*

$$\tau_0 = \inf\{\mathrm{t} \geq 0 : \mathrm{X_t} = 0\}$$

Definition 3 *Value function*

$$V_\xi(x) = E[\int_0^{\tau_0} e^{-\alpha t} f(X_t) d\xi_t + e^{-\alpha\tau_0} h(X_{\tau_0})]$$

where

- $\alpha > 0$ *is discount factor,*
- $h(\cdot)$ *is non-negative function and* $h(\cdot) \in C_b([0,\infty))$,
- $f(\cdot)$ *is non-negative function, and* $f(x)$ *is monotonic increasing, when* $x < m$, $f(x) = f(m)$, *when* $x \geq m$.

The objective is to seek optimal control ξ^*, in order to get

$$V(x) = \sup_{\xi\in\pi} V_\xi(x) = V_{\xi^*}(x).$$

Theorem 4 *Suppose that there exists a function $v(x)$ satisfies*

$$v(0) = h(0) \tag{1}$$

$$v(x) \in C([0,+\infty) \cap C_b^1[0,+\infty) \cap C^2[0,+\infty)) \tag{2}$$

$$v''(x) \leq 0, x \geq 0 \tag{3}$$

$$v'(m) = f(m), m > 0 \tag{4}$$

$$v'(x) > 0 \tag{5}$$

$$\max_{0\leq\theta\leq x}\{-\alpha v(x) + \mu(x)v'(x) + \frac{1}{2}\sigma^2(x)v''(x) + \lambda(v(x-\theta) - v(x) + f(m)\theta)\} \leq 0 \tag{6}$$

then

$$v(x) \geq V(x), \forall x \geq 0.$$

Furthermore, if $v(x)$ satisfies

$$\frac{1}{2}\sigma^2(x)v''(x) + \mu(x)v'(x) - (\alpha+\lambda)v(x) + \lambda(v(m) + f(m)(x-m)) = 0. \quad x > m \tag{7}$$

$$\frac{1}{2}\sigma^2(x)v''(x) + \mu(x)v'(x) - \alpha v(x) = 0, \quad 0 \leq x \leq m \tag{8}$$

then $\xi^* \in \pi$ is optimal, such that

$$v(x) = V_{\xi^*}(x) = V(x),$$

$v(x)$ is corresponding return function.

Proof:

Let $\tau = T \wedge \tau_0 \wedge \inf\{t > 0 : X_t^\xi \geq T\}, \forall T > 0,$

For $\{e^{-\alpha t}v(X_t) : t \geq 0\}$ using Itô formula on $[0,\tau]$, then we have

$$e^{-\alpha\tau}v(X(\tau)) - v(x)$$

$$= \int_0^\tau e^{-\alpha t}(\frac{1}{2}\sigma^2(X_t)v''(X_t) + \mu(X_t))v'(X_t) - \alpha v(X_t))dt + \int_0^\tau e^{-\alpha t}\sigma(X_t)v'(X_t)dW(t) + \sum_{0\leq t<\tau} e^{-\alpha t}[v(X_t - \Delta\xi_t) - v(X_t)]$$

also

$$v(x) = e^{-\alpha\tau}v(X(\tau)) - \int_0^\tau e^{-\alpha t}(\frac{1}{2}\sigma^2(X_t)v''(X_t) + \mu(X_t)v'(X_t) - \alpha v(X_t))dt - \int_0^\tau e^{-\alpha t}\sigma(X_t)v'(X_t)dW(t) - \sum_{0\leq t<\tau} e^{-\alpha t}[v(X_t - \Delta\xi_t) - v(X_t)].$$

by virtue of (6), we have

$$-(\frac{\sigma^2(x)}{2}v''(x) + \mu(x)v'(x) - \alpha v(x)) \geq \max_{0\leq\theta\leq x}\{\lambda(v(x-\theta) - v(x) + f(m)\theta)\}$$

let

$$g(x) = \max_{0\leq\theta\leq x}\{(v(x-\theta) - v(x) + f(m)\theta)\} = \begin{cases} 0, & 0 \leq x \leq m; \\ v(m) - v(x) + f(m)(x-m), & x > m. \end{cases} \tag{9}$$

then taking into account (3), (4), we have
$g(x) \in C[0, +\infty)$, and $0 \leq g(x) \leq f(m)x$.
when $\Delta\xi_t > 0$,

$$v(X_t - \Delta\xi_t) - v(X_t) + f(m))\Delta\xi_t \leq g(X_t)dN_t$$

and

$$\int_0^\tau e^{-\alpha t} f(m))d\xi_t = \sum_{0\leq t<\tau} e^{-\alpha t} f(m))\Delta\xi_t$$

so

$$v(x) \geq e^{-\alpha\tau} v(X(\tau)) + \int_0^\tau e^{-\alpha t}\lambda g(X_t)dt$$

$$- \int_0^\tau e^{-\alpha t}\sigma(X_t)v'(X_t)dW(t)$$

$$- \sum_{0\leq t<\tau} e^{-\alpha t}[v(X_t - \Delta\xi_t) - v(X_t) + f(m)\Delta\xi_t]$$

$$+ \int_0^\tau e^{-\alpha t} f(m))d\xi_t$$

$$\geq e^{-\alpha\tau} v(X(\tau)) + \int_0^\tau e^{-\alpha t}\lambda g(X_t)dt$$

$$- \int_0^\tau e^{-\alpha t}\sigma(X_t)v'(X_t)dW(t)$$

$$+ \int_0^\tau e^{-\alpha t}(-g(X_t)dN(t) + \int_0^\tau e^{-\alpha t} f(X_t)d\xi_t$$

$$= e^{-\alpha\tau} v(X(\tau)) - \int_0^\tau e^{-\alpha t} g(X_t)d\hat{N}_t$$

$$- \int_0^\tau e^{-\alpha t}\sigma(X_t)v'(X_t)dW(t) + \int_0^\tau e^{-\alpha t} f(X_t)d\xi_t$$

$$\geq e^{-\alpha\tau} h(0) - \int_0^\tau e^{-\alpha t} g(X_t)d\hat{N}_t$$

$$- \int_0^\tau e^{-\alpha t}\sigma(X_t)v'(X_t)dW(t) + \int_0^\tau e^{-\alpha t} f(X_t)d\xi_t$$

where $\hat{N}_t = \{N_t - \lambda t : t \geq 0\}$ is compensation Poisson process, it is also a martingale.

Take expectation on both sides of inequality,we get

$$v(x) \geq Ee^{-\alpha\tau} h(0) + E\int_0^\tau e^{-\alpha t} f(X_t)d\xi_t$$

$$- E\int_0^\tau e^{-\alpha t} g(X_t)d\hat{N}_t - E\int_0^\tau e^{-\alpha t}\sigma X_t v'(X_t)dW(t)$$

Let

$$Z_\tau = \int_0^\tau e^{-\alpha t} g(X_t)d\hat{N}_t$$

$$M_\tau = \int_0^\tau e^{-\alpha t}\sigma(X_t)v'(X_t)dW(t)$$

we can prove $-E(Z_\tau + M_\tau) \geq 0$.
then we have

$$v(x) \geq Ee^{-\alpha\tau} h(0) + E\int_0^\tau e^{-\alpha t} f(X_t)d\xi_t$$

when $T \to +\infty$,$\tau \to \tau_0$, by the monotone convergence theorem [2]

$$\lim_{T\to\infty} E\int_0^\tau e^{-\alpha t} f(X_t)d\xi_t \to E\int_0^{\tau_0} e^{-\alpha t} f(X_t)d\xi_t$$

$$v(x) \geq E\int_0^{\tau_0} e^{-\alpha t} f(X_t)d\xi_t + Ee^{-\alpha\tau_0} h(X_{\tau_0})$$

which implies that $v(x) \geq V(x)$.
Next we prove the existence of ξ^*, such that

$$v(x) = V(x) = V_{\xi^*}(x)$$

Let

$$\tau^* = T \wedge \tau_0^* \wedge \inf\{t > 0 : X_t^{\xi^*} \geq T\}$$

$$\xi^* = \{\xi_t^* : t \geq 0\}, \xi_t^* = \int_{[0,t)} \theta_s^* \mathrm{d}N_s$$

$$\theta_s^* = \begin{cases} 0, & 0 \leq X_s^* \leq m; \\ X_s^* - m, & X_s^* \geq m. \end{cases} \tag{10}$$

we have

$$e^{-\alpha\tau^*} v(X^*(\tau^*)) - v(x)$$

$$= \int_0^{\tau^*} e^{-\alpha t}(\frac{1}{2}\sigma^2(X_t^*)v''(X_t^*) + \mu(X_t^*)v'(X_t^*)$$

$$- \alpha v(X_t^*))dt + \int_0^{\tau^*} e^{-\alpha t}\sigma(X_t^*)v'(X_t^*)dW(t)$$

$$+\sum_{0\le t<\tau^*} e^{-\alpha t}[v(X_t^*-\Delta\xi_t^*)-f(m)v(X_t^*)]$$

then

$$v(x)=e^{-\alpha\tau^*}v(X^*(\tau^*))-\int_0^{\tau^*}e^{-\alpha t}(\frac{1}{2}\sigma^2(X_t^*)v''(X_t^*)$$

$$+\mu(X_t^*)v'(X_t^*)-\alpha v(X_t^*))dt$$

$$-\int_0^{\tau^*}e^{-\alpha t}\sigma(X_t^*)v'(X_t^*)dW(t)+\int_0^{\tau^*}e^{-\alpha t}f(m)d\xi_t^*$$

$$-\sum_{0\le t<\tau^*}e^{-\alpha t}[v(X_t^*-\Delta\xi_t^*)-v(X_t^*)+f(m)\Delta\xi_t^*]$$

on account of

$$\int_0^{\tau^*}e^{-\alpha t}f(X_t^*)d\xi_t^*=\sum_{0\le t<\tau^*}e^{-\alpha t}f(m))\Delta\xi_t^*$$

and

$$v(X_t^*-\Delta\xi_t^*)-v(X_t^*)+f(m)\Delta\xi_t^*=g(X_t^*)\Delta N_t^*.$$

therefore

$$v(x)=e^{-\alpha\tau^*}v(X^*(\tau^*))+\int_0^{\tau^*}e^{-\alpha t}f(m)d\xi_t^*$$

$$-\int_0^{\tau^*}e^{-\alpha t}(\frac{1}{2}\sigma^2(X_t^*)v''(X_t^*)+\mu(X_t^*)v'(X_t^*)$$

$$-\alpha v(X_t^*))dt-\int_0^{\tau^*}e^{-\alpha t}\sigma(X_t^*)v'(X_t^*)dW(t)$$

$$-\int_0^{\tau^*}e^{-\alpha t}g(X_t^*)dN_t^*$$

$$=e^{-\alpha\tau^*}v(X^*(\tau^*))+\int_0^{\tau^*}e^{-\alpha t}f(m)d\xi_t^*$$

$$-\int_0^{\tau^*}e^{-\alpha t}(\frac{1}{2}\sigma^2(X_t^*)v''(X_t^*)+\mu(X_t^*)v'(X_t^*)$$

$$-\alpha v(X_t^*)+\lambda g(X_t^*))dt$$

$$-\int_0^{\tau^*}e^{-\alpha t}\sigma(X_t^*)v'(X_t^*)dW(t)$$

$$-\int_0^{\tau^*}e^{-\alpha t}g(X_t^*)d\hat{N}_t^*$$

as well

$$g(X_t^*)$$

$$=\begin{cases}0,\quad 0\le X_t^*\le m;\\ v(m)-v(X_t^*)+f(X_t^*)(X_t^*-m),\ X_t^*>m.\end{cases}\tag{11}$$

substitution

$$v(x)=e^{-\alpha\tau^*}v(X^*(\tau^*))+\int_0^{\tau^*}e^{-\alpha t}f(X_t^*)d\xi_t^*$$
$$-(Z_{\tau^*}^*+M_{\tau^*}^*)$$

where

$$Z_{\tau^*}^*=\int_0^{\tau^*}e^{-\alpha t}g(X_t^*)d\hat{N}^*{}_t$$

$$M_{\tau^*}^*=\int_0^{\tau^*}e^{-\alpha t}\sigma(X_t^*)v'(X_t^*)dW(t)$$

$$Z_t^*+M_t^*=\int_0^t e^{-\alpha t}g(X_t^*)I_{]]0,\tau^*]]}d\hat{N}^*{}_t$$
$$+\int_0^t e^{-\alpha t}\sigma(X_t^*)v'(X_t^*)I_{]]0,\tau^*]]}dW_t)$$

then $Z_t^*+M_t^*$ is local martingale.
As a result there exists a sequence of stopping time $\tau_n'\uparrow\infty$, $n\in$ N such that $(Z_t^*)^{\tau_n'}$ is a uniformly integrable martingale with 0 initial value.

$$(Z_t^*+M_t^*)^{\tau_n'}=\int_0^{t\wedge\tau^*\wedge\tau_n'}e^{-\alpha t}g(X_t^*)d\hat{N}^*{}_t$$

$$+\int_0^{t\wedge\tau^*\wedge\tau_n'}e^{-\alpha t}\sigma(X_t^*)v'(X_t^*)dW(t)$$

$$E[(Z_t^*+M_t^*)^{\tau_n'}|\mathcal{F}_{\tau^*}]=((Z_{\tau^*}^*)^{\tau_n'}+(M_{\tau^*}^*)^{\tau_n'})$$
$$E((Z_{\tau^*}^*)^{\tau_n'}+(M_{\tau^*}^*)^{\tau_n'})=E((Z_0^*)^{\tau_n'}+(M_0^*)^{\tau_n'})=0$$

$$v(x)=e^{-\alpha(\tau^*\wedge\tau_n')}v(X^*(\tau^*\wedge\tau_n'))$$

$$+\int_0^{\tau^*\wedge\tau_n'}e^{-\alpha t}f(X_t^*)d\xi_t^*-((Z_{\tau^*}^*)^{\tau_n'}+(M_{\tau^*}^*)^{\tau_n'}))$$

Take expectation on both sides

$v(x) = Ee^{-\alpha(\tau^* \wedge \tau_n')} v(X^*(\tau^* \wedge \tau_n'))$

$+ E \int_0^{\tau^* \wedge \tau_n'} e^{-\alpha t} f(X_t^*) d\xi_t^*$

Let $\tau_n' \uparrow \infty (n \to \infty)$, where $T \to \infty$, $\tau^* \uparrow \tau_0^*$, at this moment,

$$\begin{aligned} v(x) &= Ee^{-\alpha\tau_0^*} v(X^*(\tau_0^*)) + E\int_0^{\tau_0^*} e^{-\alpha t} d\xi_t^* \\ &= Ee^{-\alpha\tau_0^*} v(0)) + E\int_0^{\tau_0^*} e^{-\alpha t} d\xi_t^* \\ &= Ee^{-\alpha\tau_0^*} h(X_{\tau_0^*}) + E\int_0^{\tau_0^*} e^{-\alpha t} f(X_t^*) d\xi_t^* \\ &= V_{\xi^*}(x) \end{aligned}$$

□

From the above theorem, we know that the optimal solution meet differential equation (7). But it is a pity that this kind of equation no specific general solution expression, so for as to no analytical solution. Different forms of $\sigma(x)$ and $\mu(x)$ have the corresponding optimal solution in different forms, we must according to the specific function to get the optimal solution.

3 Example of optimal solutions

Model of linear stochastic growth

$$\begin{aligned} \mathrm{d}X_t &= \mu \mathrm{d}t + \sigma \mathrm{d}W_t - d\xi_t \\ X_0 &= x \end{aligned}$$

where parameters satisfy $\mu > 0, \sigma > 0$, $h(\cdot) \in C_b([0, \infty))$, and non-negative. $f(\cdot)$ is non-negative function, $f(x)$ is monotone increasing, when $x < m$, $f(x) = f(m)$, when $x \geq m$. and meet the following assumptions, we can get the optimal solution of the analytical form.

$(I).2\mu^2\lambda > \alpha^2\sigma^2$

$(II).f(0) > max(\frac{2(\alpha+\lambda)\alpha}{2\mu(\alpha+\lambda)+\alpha\sigma^2\bar{r_2}}, \frac{2r_2^4}{r_1(r_2^2-r_1^2)})h(0)$

$(III).x \in (0, m), 0 \leq f'(x) < r_1 f(0)$

where $r_1 > 0 > r_2$ are two roots of equation

$$\frac{\sigma^2}{2} z^2 + \mu z - \alpha = 0$$

$\bar{r_1} > 0 > \bar{r_2}$ are two roots of equation

$$\frac{\sigma^2}{2} z^2 + \mu z - (\alpha + \lambda) = 0$$

In linear stochastic growth model, theorem 4 changes into

Theorem 5 *Assume that there exists a function $v(x)$, satisfies*

$$v(0) = h(0) \tag{12}$$

$$v(x) \in (C[0, +\infty) \cap C_b^1[0, +\infty) \cap C^2[0, +\infty)) \tag{13}$$

$$v''(x) \leq 0, x \geq 0 \tag{14}$$

$$v'(m) = f(m), m > 0 \tag{15}$$

$$v'(x) > 0 \tag{16}$$

$$\begin{aligned} \max_{0 \leq \theta \leq x} \{ -\alpha v(x) + \mu(x) v'(x) + \frac{1}{2}\sigma^2(x) v''(x) + \\ \lambda(v(x-\theta) - v(x) + f(m)\theta)\} \leq 0 \end{aligned} \tag{17}$$

then $v(x) \geq V(x), \forall x \geq 0$.
furthermore, If $v(x)$ satisfies

$$\begin{aligned} \frac{1}{2}\sigma^2(x) v''(x) + \mu(x) v'(x) - (\alpha + \lambda) v(x) + \\ \lambda(v(m) + f(m)(x - m)) = 0. \\ x > m \end{aligned} \tag{18}$$

$$\frac{1}{2}\sigma^2 v''(x) + \mu v'(x) - \alpha v(x) = 0 \tag{19}$$

when $0 \leq x \leq m$, then $\xi^* \in \pi$ is optimal control, such that $v(x) = V_{\xi^*}(x) = V(x)$, $v(x)$ is corresponding return function.

Lemma 6 *Construct a function*

$$F(x) = Ar_1^2 e^{r_1 x} + Br_2^2 e^{r_2 x}$$

where $A + B = h(0)$,

$$A = \frac{r_2 e^{r_2 m} h(0) - f(m)}{r_2 e^{r_2 m} - r_1 e^{r_1 m}},$$

m is positive constant to be confirmed. In given conditions, there exists unique constant $\hat{m} > 0$, that $F(\hat{m}) = 0$.

Proof: For

$$F(0) < 0, F'(x) = Ar_1^3 e^{r_1 x} + Br_2^3 e^{r_2 x} > 0,$$

and

$$\lim_{x->\infty} F(x) = +\infty$$

then there exists unique constant $\hat{m} > 0$, such that $F(\hat{m}) = 0$, and

$$\hat{m} = \frac{1}{r_1 - r_2} \ln \frac{A - h(0)}{A} (\frac{r_2}{r_1})^2$$

□

Let

$$G(x) = a + f(x)(be^{-r_1 x} + ce^{-r_2 x})$$

where

$$a = r_1 r_2 (r_1 - r_2) h(0)$$

$$b = r_2^2 - \frac{\alpha r_2 \bar{r_2}}{\alpha + \lambda}$$

$$c = \frac{\alpha}{\alpha + \lambda} \bar{r_2} r_1 - r_1^2$$

Lemma 7 *Under assumption, there exists unique constant $m \in (0, \hat{m})$, such that $G(m) = 0$.*

Proof:

$$\begin{aligned} G(0) &= a + f(0)(b + c) \\ &= r_1 r_2 (r_1 - r_2) h(0) + f(0)(r_2^2 - r_1^2 \\ &+ \frac{\alpha}{\alpha + \lambda} \bar{r_2} (r_1 - r_2)) \\ &> h(0)(r_1 - r_2)(r_1 r_2 + \frac{2\alpha}{\sigma^2}) = 0 \end{aligned}$$

$$\begin{aligned} G(\hat{m}) \frac{e^{(r_1 + r_2)\hat{m}}}{\bar{r_2}(r_2 e^{r_2 \hat{m}} - r_1 e^{r_1 \hat{m}})} &= \frac{1}{\bar{r_2}} F(\hat{m}) - \frac{\alpha}{\alpha + \lambda} f(\hat{m}) \\ &= -\frac{\alpha}{\alpha + \lambda} f(\hat{m}) < 0 \end{aligned}$$

For

$$\frac{e^{(r_1 + r_2)\hat{m}}}{\bar{r_2}(r_2 e^{r_2 \hat{m}} - r_1 e^{r_1 \hat{m}})} > 0$$

hence $G(\hat{m}) < 0$.
and

$$G'(x) = f'(x)(be^{-r_1 x} + ce^{-r_2 x})$$
$$+ f(x)(-br_1 e^{-r_1 x} - cr_2 e^{-r_2 x})$$

since

$$b > 0, -br_1 < 0, c < 0, -cr_2 < 0, f(x) > 0,$$

then

$$f(x)(-br_1 e^{-r_1 x} - cr_2 e^{-r_2 x}) < 0$$

when $0 \leq f'(x) < r_1 f(0)$

$$\begin{aligned} G'(x) &< r_1 f(0)(be^{-r_1 x} + ce^{-r_2 x}) \\ &+ f(x)(-br_1 e^{-r_1 x} - cr_2 e^{-r_2 x}) \\ &= br_1 e^{-r_1 x}(f(0) - f(x)) \\ &+ c(r_1 f(0) - f(x) r_2) e^{-r_2 x} < 0 \end{aligned}$$

thus $G(x)$ is monotone decreasing on $(0, \hat{m})$, then there exists unique constant $m \in (0, \hat{m})$, such that $G(m) = 0$. □

Lemma 8 *Under assumption,there exists function $v_1(x), x \in [m, \infty)$ satisfies*

$$\frac{1}{2}\sigma^2 v_1''(x) + \mu v_1'(x) - (\alpha + \lambda) v_1(x)$$

$$+\lambda(v_1(m) + f(m)(x - m)) = 0 \quad (20)$$

$$v_1''(x) \leq 0, x \in [m, +\infty) \quad (21)$$

$$v_1'(m) = f(m) \quad (22)$$

Proof: Let

$$v_1(x) = c_2 e^{\bar{r_2} x} + px + q, x \in [m, \infty)$$

where

$$c_2 = \frac{Ar_1^2 e^{r_1 m} + Br_2^2 e^{r_2 m}}{\bar{r_2}^2 e^{\bar{r_2} m}} < 0$$

$$p = \frac{\lambda}{\alpha + \lambda} f(m)$$

$$q = \frac{(\mu - m\alpha)\lambda}{\alpha(\alpha + \lambda)} f(m)$$

easy to get $v_1(x)$ meet (20).
By calculation we have

$$v_1''(x) = c_2 \bar{r_2}^2 e^{\bar{r_2} x} \leq 0$$

(21) hold.

$$v_1'(m) - f(m) = G(m) \frac{e^{(r_1 + r_2)m}}{\bar{r_2}(r_2 e^{r_2 m} - r_1 e^{r_1 m})} = 0$$

i.e.

$$v_1'(m) = f(m)$$

(22) hold. □

Theorem 9 *There exists $v(x)$satisfies all conditions of theorem 5. At same time value function can be obtained i.e.$v(x)$, corresponding optimal control $\xi_t^* = \int_{[0,t)} \theta_s^* dN_s$, θ_t^* can be given by (10).*

Proof: Define a function $v(x)$

$$v(x) = \begin{cases} Ae^{r_1 x} + Be^{r_2 x}, & 0 \le x \le m; \\ v_1(x), & x > m. \end{cases} \quad (23)$$

where

$$A = \frac{r_2 e^{r_2 m} h(0) - f(m)}{r_2 e^{r_2 m} - r_1 e^{r_1 m}},$$
$$B = h(0) - A,$$

$v_1(x)$ is similar to lemma 8 .
from the definition of $v(x)$, (15) hold, so $v(x) \in C_b^1[0, +\infty)$ for

$$v''_-(m) = Ar_1^2 e^{r_1 m} + Br_2^2 e^{r_2 m}, v''_+(m) = c_2 \bar{r_2}^2 e^{\bar{r_2} m}$$

by the simple calculation we have

$$v''_-(m) = v''_+(m)$$

so $v(x) \in C^2[0, +\infty)$
Due to (18), (19) we get $v_-(m) = v_+(m)$,
consequently $v(x) \in C[0, +\infty)$, (13) hold.
From the definition of $v(x)$, we have $v(0) = h(0)$, (12) hold.
As a result of

$$v'(x) = \begin{cases} Ar_1 e^{r_1 x} + Br_2 e^{r_2 x}, & 0 \le x \le m; \\ c_2 \bar{r_2} e^{\bar{r_2} x} + p, & x > m. \end{cases} \quad (24)$$

we have $v'(x) > 0$, (16) hold.

$$v''(x) = \begin{cases} Ar_1^2 e^{r_1 x} + Br_2^2 e^{r_2 x}, & 0 \le x \le m; \\ c_2 \bar{r_2}^2 e^{\bar{r_2} x}, & x > m. \end{cases} \quad (25)$$

$v''(x) < 0$, (14) hold.
We can easy to deduce $Ae^{r_1 x} + Be^{r_2 x}$ and $v_1(x)$ are solutions of differential equation (18), (19) separately. Hence $v(x)$ satisfies all conditions of theorem 5. i.e. $v(x)$ is optimal value function,corresponding optimal control $\xi_t^* = \int_{[0,t)} \theta_s^* dN_s$, θ_t^* can be given by (10). □

4 Conclusion

In this paper, by introducing a random insertion into the controlled state governed by a Poisson process, we extend originally function and terminal condition for function satisfied some conditions, and discuss sufficient condition about the existence of optimal control. In addition, we find the optimal solution about linear stochastic growth.

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